Photonic Quantum Information Processing
OPTI 647: Lecture 18

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Announcements

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- Final presentations schedule:
  - December 17 [OSC 307 at 9.00 (TBC)].
  - We'll have an external examiner (it might be some visitor).
  - Keep calm.

Plan for today: Second order non-linear optics.

Quantum analysis of non-linear optics

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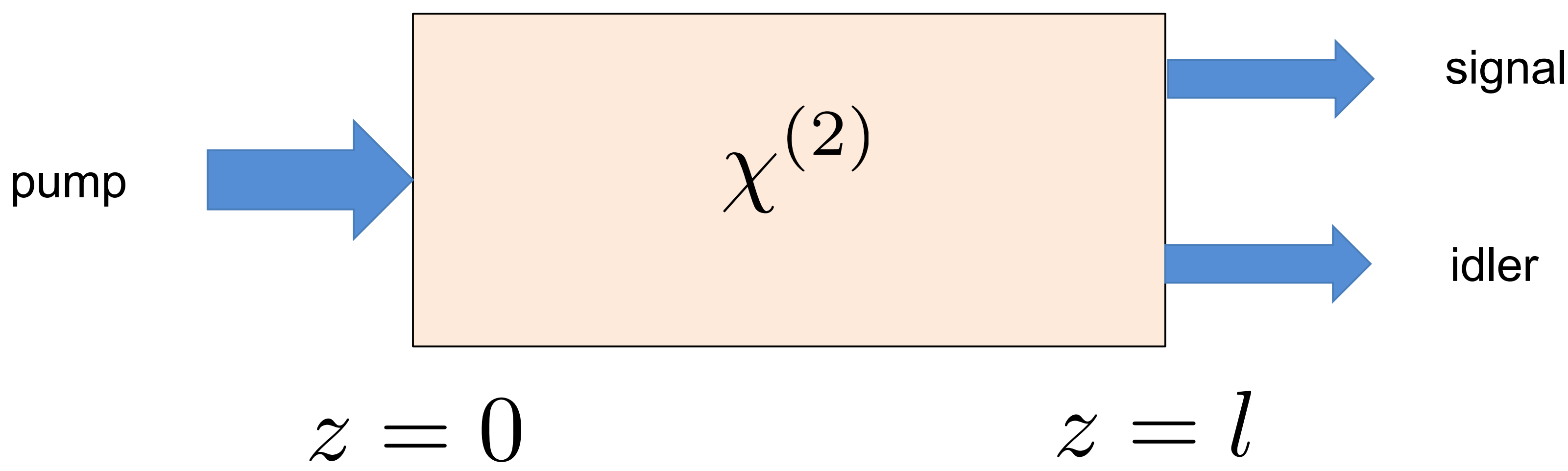
- How can we create non-classical light beams that exhibit the signatures we've discussed in our one-mode and multi-mode analyses?
- In particular, we will study spontaneous parametric downconversion (SPDC) and optical parametric amplification (OPA) in second-order nonlinear crystals
- These closely-related processes have been and continue to be the primary vehicles for generating non-classical light beams in the lab

Plan for today and Thursday

- Given our interest in the system-theoretic aspects of photonic quantum information processing and lack of a serious EM theory pre-requisite, we will not get into too much detail
  - focus on the coupled-mode equations characterization of collinear configurations, i.e., we shall suppress transverse spatial effects
  - Nevertheless, we will be able to get to the basic physics of these interactions and provide continuous-time versions of the non-classical signatures that we discussed in single-mode, two-mode and n-mode forms
- Today, we will begin with a classical analysis. Thereafter we will extend to a quantum analysis.

Second-order nonlinear optics

- Spontaneous parametric down conversion (SPDC)



  - Strong pump at frequency, $\omega_P = \omega_S + \omega_I$
  - No input at signal frequency, ω_S
  - No input at idler frequency, ω_I
  - Nonlinear mixing in $\chi^{(2)}$ crystal produces signal and idler outputs

Spontaneous parametric down conversion (SPDC)

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- *Downconversion*—because, the signal and idler light arises from a higher-frequency pump beam.
- *Parametric*—because the downconversion is due to the presence of the pump modifying the effective material parameters encountered by the fields propagating at the signal and idler frequencies.
- *Spontaneous*—because there is no illumination of the crystal’s input facet at the signal and idler frequencies.

Classical analysis of SPDC

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- In SPDC, the z = 0 signal and idler frequencies are unexcited, i.e., in their vacuum states. Action of pump in conjunction with the crystal's nonlinearity is responsible for the excitation at these frequencies at z = l. Need quantum analysis to understand SPDC
- We will get a hint of the quantum interpretation because the signal and idler frequencies, in the classical theory, will obey $\omega_P = \omega_S + \omega_I$
  - Rewriting it as, $\hbar\omega_P = \hbar\omega_S + \hbar\omega_I$, *suggests* that a **photon fission** process—in which a single pump photon spontaneously downconverts into a signal photon plus an idler photon such that energy is conserved—is what is happening in SPDC. In fact, such is the case.

Classical electromagnetic theory

- Maxwell's Equations in free space

Faraday's law of induction

$$\nabla \times \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\vec{H}(\vec{r},t)$$

Gauss' laws

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t)$$

$$\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(\vec{r},t) + \vec{J}(\vec{r},t)$$

$$\nabla \cdot \mu_0 \vec{H}(\vec{r},t) = 0$$

Ampere's law with Maxwell's correction

- flux densities D, B vs. field intensities E, H

$$\vec{B}(\vec{r},t) = \mu_0 \vec{H}(\vec{r},t) \qquad \vec{D}(\vec{r},t) = \epsilon_0 \vec{E}(\vec{r},t)$$

- Speed of light in vacuum, $c = 1/\sqrt{\epsilon_0 \mu_0}$

Constitutive relations in dielectric

- The electric and magnetic flux densities D, B are related to the field intensities E, H via the so-called **constitutive relations**
- For simple homogeneous isotropic dielectrics

$$\boldsymbol{D} = \epsilon \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

$$\epsilon = \epsilon_0 (1 + \chi)$$

$$\mu = \mu_0 (1 + \chi_m)$$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ farad/m}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/m}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/sec}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ ohm}$$

- Susceptibilities χ, χ_m are measures of electric and magnetic polarization properties of the material

$$\vec{D}(\vec{r}, t) = \epsilon_0 \vec{E}(\vec{r}, t) + \epsilon_0 \chi \vec{E}(\vec{r}, t)$$
$$= \epsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t)$$

P: dielectric polarization (average electric dipole moment per m^3)

Susceptibilities and refractive index

- Speed of light in dielectric, $c = \frac{1}{\sqrt{\mu\epsilon}}$

$$\epsilon_{\mathrm{rel}} = \frac{\epsilon}{\epsilon_0} = 1 + \chi\,, \qquad \mu_{\mathrm{rel}} = \frac{\mu}{\mu_0} = 1 + \chi_m$$

- Refractive index, $n = \sqrt{\epsilon_{\mathrm{rel}}\mu_{\mathrm{rel}}}$
  - For $\mu_{\mathrm{rel}} = 1$,

$$n = \sqrt{1 + \chi}$$

Reduce to: +z-propagating plane wave

- Taking curl of Faraday's law, using the vector identity, $\nabla \times [\nabla \times \vec{F}(\vec{r},t)] = \nabla[\nabla \cdot \vec{F}(\vec{r},t)] - \nabla^2 \vec{F}(\vec{r},t)$,

$$\nabla[\nabla \cdot \vec{E}(\vec{r},t)] - \nabla^2 \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}[\nabla \times \vec{H}(\vec{r},t)] = -\mu_0 \frac{\partial^2}{\partial t^2}\vec{D}(\vec{r},t)$$

- For +z propagating field, it simplifies to:

$$\frac{\partial^2}{\partial z^2}\vec{E}(z,t) - \mu_0 \frac{\partial^2}{\partial t^2}\vec{D}(z,t) = \vec{0}.$$

  – For free-space, $\vec{D}(z,t) = \epsilon_0 \vec{E}(\zeta,t)$

  – Hence, $\frac{\partial^2}{\partial z^2}\vec{E}(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(z,t) = \vec{0}$

Solution to: $\frac{\partial^2}{\partial z^2}\vec{E}(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(z,t) = \vec{0}$

- +z-propagating plane wave

$$\vec{E}(z,t) = f(t - z/c)\vec{i}_f$$

is a solution to: $\frac{\partial^2}{\partial z^2}\vec{E}(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(z,t) = \vec{0}$

- For arbitrary time function f(t) and unit vector $\vec{i}_f$ in the x-y plane

EM theory in a linear dielectric medium

- We will use the temporal frequency domain:

$$\vec{\mathcal{F}}(\vec{r},\omega) = \int \mathrm{d}t\, \vec{F}(\vec{r},t)e^{j\omega t} \quad \leftrightarrow \quad \vec{F}(\vec{r},t) = \int \frac{\mathrm{d}\omega}{2\pi}\,\vec{\mathcal{F}}(\vec{r},\omega)e^{-j\omega t}$$

- The constitutive law for a linear dielectric is:

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0[1+\boldsymbol{\chi}^{(1)}(\omega)]\vec{\mathcal{E}}(\vec{r},\omega)$$

$$\vec{\mathcal{P}}(\vec{r},\omega) = \epsilon_0\boldsymbol{\chi}^{(1)}(\omega)\vec{\mathcal{E}}(\vec{r},\omega)$$ need not be parallel to the electric field

$\boldsymbol{\chi}^{(1)}(\omega)$, is a frequency-dependent tensor **Linear susceptibility**

If E is polarized along a principal axis of the crystal, $\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0 n^2(\omega)\vec{\mathcal{E}}(\vec{r},\omega)$

Helmholtz equation

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- If we take the Fourier transform, and presume fields with no (x,y) dependence with an electric field polarized along a principal axis, we obtain the **Helmholtz equation**

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z,\omega) = \vec{0}.$$

The $+z$-going plane-wave solution to this equation is

$$\vec{\mathcal{E}}(z,\omega) = \mathrm{Re}[\vec{E}e^{-j(\omega t - kz)}].$$

where $k \equiv \omega n(\omega)/c$ and $\vec{E}$ is a constant vector in the x-y plane.

Non-linear dielectric

- For a nonlinear dielectric, constitutive relation:

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0[1 + \boldsymbol{\chi}^{(1)}(\omega)]\vec{\mathcal{E}}(\vec{r},\omega) + \vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$$

where $\boldsymbol{\chi}^{(1)}(\omega)$ is the medium's *linear* susceptibility tensor at frequency ω and $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is the *nonlinear* polarization, i.e., $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is a nonlinear function of the electric field.

- Assuming +z-going plane wave whose electric field is polarized along a principal axis of the $\chi^{(1)}$ tensor,

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z,\omega) = -\mu_0\omega^2\vec{\mathcal{P}}_{\rm NL}(z,\omega)$$

  – LHS includes medium's linear behavior, nonlinear character appearing as a source term on RHS. General solutions for arbitrary nonlinearities are beyond our reach

Coupled mode theory for second-order non-linearity

- Material's nonlinear polarization arises from a second-order nonlinearity
- Assume E field propagating from z = 0 to z = l in the nonlinear crystal consists of three +z-going monochromatic plane waves: frequency-ω_P pump beam; frequency-ω_S signal beam; and frequency-ω_I idler beam. We will assume that:
  - $\omega_P = \omega_S + \omega_I$
  - pump is very strong while the signal and idler are very weak
  - Allowing—as will be necessary to account for the tensor properties of the second-order susceptibility—the pump, signal, and idler to have different linear polarizations along the crystal's principal axes, we will take the E field to be:

Pump, signal and idler plane-wave modes

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- Assume monochromatic pump, signal and idler

$$\vec{E}(z,t) = \underbrace{\mathrm{Re}[A_S(z)e^{-j(\omega_S t - k_S z)}]\vec{i}_S}_{\text{signal}} + \underbrace{\mathrm{Re}[A_I(z)e^{-j(\omega_I t - k_I z)}]\vec{i}_I}_{\text{idler}}$$

$$+ \underbrace{\mathrm{Re}[A_P e^{-j(\omega_P t - k_P z)}]\vec{i}_P}_{\text{pump}}, \quad \text{for } 0 \leq z \leq l.$$

– $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$
wave numbers of the signal, idler, and pump fields in terms of the refractive indices of their respective linear polarizations, $\vec{i}_m$ which are all in the x-y plane

– Non-depleting pump, A_P is a constant

– Slowly-varying (in z) signal and idler complex amplitudes

Linear and non-linear polarization terms

- Constitutive Law for 2nd-Order Nonlinear Crystal:

$$\vec{D}(\vec{r},t) = \epsilon_0 \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$$

$$\approx \frac{\epsilon_0 n_S^2(\omega_S) A_S(z) e^{-j(\omega_S t - k_S z)} + \text{cc}}{2} \vec{i}_S$$

$$+ \frac{\epsilon_0 n_I^2(\omega_I) A_I(z) e^{-j(\omega_I t - k_I z)} + \text{cc}}{2} \vec{i}_I$$

$$+ \frac{\epsilon_0 n_P^2(\omega_P) A_P e^{-j(\omega_P t - k_P z)} + \text{cc}}{2} \vec{i}_P$$

$$+ \frac{\epsilon_0 \chi^{(2)} A_I^*(z) A_P e^{-j[(\omega_P - \omega_I)t - (k_P - k_I)z]} + \text{cc}}{2} \vec{i}_S$$

$$+ \frac{\epsilon_0 \chi^{(2)} A_S^*(z) A_P e^{-j[(\omega_P - \omega_S)t - (k_P - k_S)z]} + \text{cc}}{2} \vec{i}_I$$

The first 3 terms are due to linear susceptibility. Except for the possibly different signal, idler, and pump polarizations, it is the three-wave version of what we showed before for a linear dielectric. The last two terms represent the effect of the material's second-order nonlinear susceptibility. we have suppressed the frequency dependence and tensor character

Solving the Helmholtz equation…

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$$
\begin{aligned}
&\frac{\partial^2}{\partial z^2}\left(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\right) \\
&- \quad \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S\right. \\
&+ \quad \left. n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\right) \\
&- \quad \frac{\chi^{(2)}}{c^2}\frac{\partial^2}{\partial t^2}\left(A_I^*(z)A_P e^{-j[(\omega_P-\omega_I)t-(k_P-k_I)z]}\vec{i}_S\right. \\
&+ \quad \left. A_S^*(z)A_P e^{-j[(\omega_P-\omega_S)t-(k_P-k_S)z]}\vec{i}_I\right) = \vec{0}.
\end{aligned}
$$

Solving the Helmholtz equation…(contd.)

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- Performing z differentiation to the first line,

$$\frac{\partial^2}{\partial z^2}\left(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\right)$$

$$= \left[-k_S^2 A_S(z) + 2jk_S\frac{\mathrm{d}A_S(z)}{\mathrm{d}z}\right]e^{-j(\omega_S t-k_S z)}\vec{i}_S$$

$$+ \left[-k_I^2 A_I(z) + 2jk_I\frac{\mathrm{d}A_I(z)}{\mathrm{d}z}\right]e^{-j(\omega_I t-k_I z)}\vec{i}_I - k_P^2 A_P e^{-j(\omega_P - k_P z)}\vec{i}_P,$$

where we have employed the slowly-varying envelope approximation to suppress terms involv $\frac{\partial^2}{\partial z^2}A_m(z)$ for m = S, I.

Solving the Helmholtz equation…(contd.)

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- Performing the t differentiations on the second and third lines,

$$
\begin{aligned}
&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I \\
&+ \quad n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big) \\
&= \quad k_S^2 A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + k_I^2 A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + k_P^2 A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P
\end{aligned}
$$

where we have used $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$.

Solving the Helmholtz equation…(contd.)

- Plugging back both terms in, we get

$$\left(2jk_S\frac{\mathrm{d}A_S(z)}{\mathrm{d}z}e^{-j(\omega_S t-k_S z)}+\frac{\chi^{(2)}\omega_S^2}{c^2}A_I^*(z)A_Pe^{-j[\omega_S t-(k_P-k_I)z]}\right)\vec{i}_S$$

$$+\quad\left(2jk_I\frac{\mathrm{d}A_I(z)}{\mathrm{d}z}e^{-j(\omega_I t-k_I z)}+\frac{\chi^{(2)}\omega_I^2}{c^2}A_S^*(z)A_Pe^{-j[\omega_I t-(k_P-k_S)z]}\right)\vec{i}_I=\vec{0},$$

- SPDC systems in which the signal and idler are in orthogonal linear polarizations. The above equation reduces to two "coupled-mode equations":

$$\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} = j\frac{\omega_S\chi^{(2)}A_P}{2cn_S(\omega_S)}A_I^*(z)e^{j\Delta kz}$$

$$\frac{\mathrm{d}A_I(z)}{\mathrm{d}z} = j\frac{\omega_I\chi^{(2)}A_P}{2cn_I(\omega_I)}A_S^*(z)e^{j\Delta kz},$$

for $0\leq z\leq l$, where $\Delta k\equiv k_P-k_S-k_I$.

Coupled mode equations

- should be solved subject to given initial conditions,
  $\{A_S(0), A_I(0)\} \rightarrow \{A_S(l), A_I(l)\}$
  - yielding,

$$\begin{aligned}\vec{E}(z,t) &= \mathrm{Re}[A_S(l)e^{-j(\omega_S t - k_S l - \omega_S(z-l)/c)}]\vec{i}_S + \mathrm{Re}[A_I(l)e^{-j(\omega_I t - k_I l - \omega_I(z-l)/c)}]\vec{i}_I \\ &+ \mathrm{Re}[A_P e^{-j(\omega_P t - k_P l - \omega_P(z-l)/c)}]\vec{i}_P,\end{aligned}$$

  - where for $z > l$, regular free-space propagation prevails
- Why quantum analysis of SPDC will be needed?
  - If we set, $A_S(0) = A_I(0) = 0,$ we get $A_S(l) = A_I(l) = 0$
  - and hence, $\vec{E}(z,t) = \mathrm{Re}[A_P e^{-j(\omega_P t - k_P l - \omega_P(z-l)/c)}]\vec{i}_P$ for $z > l$.

Conversion to photon-unit fields

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- Time-averaged powers on photodetector active area, $\mathcal{A}$

$$S_m(z) = \frac{c\epsilon_0 n_m(\omega_m)\mathcal{A}}{2}\,|A_m(z)|^2, \quad \text{for } m = S, I, P$$

- Photon-unit ($\sqrt{\text{photons/sec}}$) fields,

$$S_m(z) = \hbar\omega_m|A_m(z)|^2, \quad \text{for } m = S, I, P$$

- Photon-units coupled mode equations:

$$\begin{aligned} \frac{\mathrm{d}A_S(z)}{\mathrm{d}z} &= j\kappa A_I^*(z)e^{j\Delta k z} \\ \frac{\mathrm{d}A_I(z)}{\mathrm{d}z} &= j\kappa A_S^*(z)e^{j\Delta k z} \end{aligned} \qquad \text{for } 0 \le z \le l, \text{ where}$$

$$\kappa \equiv \sqrt{\frac{\hbar\omega_S\omega_I\omega_P}{2c^3\epsilon_0 n_S(\omega_S)n_I(\omega_I)n_P(\omega_P)\mathcal{A}}}\,\chi^{(2)}A_P$$

is a complex-valued coupling constant that is proportional to the pump's complex envelope and the crystal's second-order nonlinear

Solution to photon-unit coupled mode equations

- Solution is given by,

$$
\begin{aligned}
A_S(l) &= \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_S(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_I^*(0)\right]e^{j\Delta kl/2} \\
A_I(l) &= \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_I(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_S^*(0)\right]e^{j\Delta kl/2},
\end{aligned}
$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\Delta k/2)^2},$$

  – can be verified by substitution back into the coupled mode equations

Phase matching

- Inside the crystal, the monochromatic signal, idler, and pump beams—at frequencies ω_S, ω_I, and ω_P, respectively, propagate at their phase velocities given by, $v_m(\omega_m) = \omega_m / k_m$ for $m = S, I, P$.
- The nonlinear interaction governed by the coupled-mode equations is said to be **phase matched** when we have $\Delta k = k_P - k_S - k_I = 0$, i.e., when

$$\omega_P / v_P = \omega_S / v_S + \omega_I / v_I$$

- For a phase-matched system,

$$\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} = j\kappa A_I^*(z) \quad \text{and} \quad \frac{\mathrm{d}A_I(z)}{\mathrm{d}z} = j\kappa A_S^*(z), \quad \text{for } 0 \leq z \leq l$$

phase angle of the coupling between the signal and idler remains the same throughout the interaction. On the other hand, when phase-matching is violated, the phase of the coupling between the signal and idler rotates as these fields propagate

Type-II phase-matched operation at degeneracy

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- Phase Matching for Efficient Coupling: $\Delta k = 0$
  - Conservation of photon momentum: $k_P = k_S + k_I$
  - Type-II system: $\vec{i}_S = \vec{i}_x,\ \vec{i}_I = \vec{i}_y$
  - Operation at Frequency Degeneracy: $\omega_S = \omega_I = \omega_P/2$

- Solution for the phase-matched case:

$$A_S(l) = \cosh(|\kappa| l) A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa| l) A_I^*(0)$$

$$A_I(l) = \cosh(|\kappa| l) A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa| l) A_S^*(0)$$

Non-phase matched case

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Solution for the phase-matched case:

$$A_S(l) = \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0)$$

$$A_I(l) = \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0)$$

shows increasing amounts of signal-idler coupling with increasing $|\kappa|l$, i.e., with increasing pump power or crystal length. In contrast, far from phase matching—when $|\Delta k/2| \gg |\kappa|$—we get $p \approx j|\Delta k|/2$, whence

Solution far from phase matching:

$$A_S(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_S(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}A_I^*(0) \right] e^{j\Delta kl/2}$$

$$A_I(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_I(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}A_S^*(0) \right] e^{j\Delta kl/2}$$

$$A_S(l) \approx A_S(0) \quad \text{and} \quad A_I(l) \approx A_I(0), \text{ when } |\Delta kl/2| \gg 1,$$

i.e., when the crystal is long enough that the phase mismatch rotates the signal-idler coupling phase through many 2pi cycles

Prelude to quantum analysis

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- Photon fission: $\hbar\omega_P = \hbar\omega_S + \hbar\omega_I$
- Photons being produced in pairs reminds us of the two-mode parametric amplifier that we studied earlier in the semester. That system was governed by a two-mode Bogoliubov transformation:

$$\hat{a}_S^{\text{out}} = \mu\hat{a}_S^{\text{in}} + \nu\hat{a}_I^{\text{in}\dagger} \quad \text{and} \quad \hat{a}_I^{\text{out}} = \mu\hat{a}_I^{\text{in}} + \nu\hat{a}_S^{\text{in}\dagger}, \quad \text{where } |\mu|^2 - |\nu|^2 = 1.$$

- See the solution to the phase-matched case with,

$$\begin{aligned} A_S(l) &= \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0) \\ A_I(l) &= \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0) \end{aligned}$$

$$\mu \equiv \cosh(|\kappa|l) \quad \text{and} \quad \nu \equiv j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)$$

SPDC with **quantum** analysis

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- Coupled mode equations of **field operators**,

$$\frac{\partial \hat{A}_S(z,\omega)}{\partial z} = j\kappa \hat{A}_I^\dagger(z,\omega)e^{j\omega\Delta k' z}$$

$$\frac{\partial \hat{A}_I(z,\omega)}{\partial z} = j\kappa \hat{A}_S^\dagger(z,\omega)e^{j\omega\Delta k' z}$$

  - Solution:

$$\hat{A}_S(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_S(0,\omega) + j\kappa l\frac{\sinh(pl)}{pl}\,\hat{A}_I^\dagger(0,\omega)\right]e^{j\omega\Delta k' l/2}$$

$$\hat{A}_I(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_I(0,\omega) + j\kappa l\frac{\sinh(pl)}{pl}\,\hat{A}_S^\dagger(0,\omega)\right]e^{j\omega\Delta k' l/2},$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\omega\Delta k'/2)^2}.$$

Solution to quantum analysis (contd)

- To verify these satisfy commutation relations, define

$$\mu(\omega) = \left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)e^{j\omega\Delta k' l/2}$$

$$\nu(\omega) = j\kappa l\frac{\sinh(pl)}{pl}e^{j\omega\Delta k' l/2},$$

  – so the coupled mode equations become:

$$\hat{A}_S(l,\omega) = \mu(\omega)\hat{A}_S(0,\omega) + \nu(\omega)\hat{A}_I^\dagger(0,\omega)$$

$$\hat{A}_I(l,\omega) = \mu(\omega)\hat{A}_I(0,\omega) + \nu(\omega)\hat{A}_S^\dagger(0,\omega)$$

$$|\mu(\omega)|^2 - |\nu(\omega)|^2 = \left[\cosh^2(pl) + \left(\frac{\omega\Delta k'}{2p}\right)^2\sinh^2(pl)\right] - \left(\frac{|\kappa|}{p}\right)^2\sinh^2(pl)$$

$$= \cosh^2(pl) - \sinh^2(pl) = 1,$$ hence, a two-mode Bogoliubov transformation that ensures proper commutator preservation.

Plan for the rest of this course

1. Next lecture: Continue today's discussion.
2. Recap on Gaussian formalism on phase space. Quantum channels. Description of measurements. (3-4 lectures).
3. Quantum sensing (3-4 lectures).
4. Soft introduction to Optical Quantum Computers (2-3 lectures).
5. Quantum discrimination (1-2 lectures).
6. Quantum Communications (2-3 lectures).