Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

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Reading: For classical coupled-mode equations for parametric interactions:

- B.E.A. Saleh and M.C. Teich, *Fundamentals of Photonics,* (Wiley, New York, 1991) section 19.4.

Introduction

The final major question we shall address this semester is the following. How can we create non-classical light beams that exhibit the signatures we've discussed in our simple one-mode and two-mode analyses? In particular, we will study spontaneous parametric downconversion and optical parametric amplification in second-order nonlinear crystals. These closely-related processes have been and continue to be the primary vehicles for generating non-classical light beams. Given our interest in the system-theoretic aspects of quantum optical communication—and our lack of a serious electromagnetic fields prerequesite—we shall tread lightly, focusing on the coupled-mode equations characterization of collinear configurations, i.e., we shall suppress transverse spatial effects. Nevertheless, we will be able to get to the basic physics of these interactions and provide continuous-time versions of the non-classical signatures that we discussed in single-mode and two-mode forms earlier this term. Today, however, we will begin with a treatment within the classical domain. In the two lectures to follow we will convert today's material into the quantum domain, and then explore the implications of that quantum characterization.

Spontaneous Parametric Downconversion

Slide 3 shows a conceptual picture of spontaneous parametric downconversion (SPDC). A strong laser-beam pump is applied to the entrance facet (at $z = 0$) of a crystalline material that possesses a second-order ($\chi^{(2)}$) nonlinearity. We will only concern ourselves with continuous-wave (cw) pump fields, so this pump beam will be taken to be monochromatic at frequency ω_P. Even though the only light applied to the crystal is at frequency ω_P, three-wave mixing in this nonlinear material can result in the production of lower-frequency signal and idler waves, with center frequencies ω_S

and ω_I, respectively, that emerge—along with the transmitted pump beam—from the crystal's output facet (at $z = l$). This process is *downconversion*, because the signal and idler light arises from a higher-frequency pump beam. The process is deemed *parametric*, because the downconversion is due to the presence of the pump modifying the effective material parameters encountered by the fields propagating at the signal and idler frequencies. It is called *spontaneous*, because there is no illumination of the crystal's input facet at the signal and idler frequencies. Of course, this zero-field input statement is correct in a classical physics description of slide 3. We know, from our quantum description of the electromagnetic field, that the positive-frequency field operator at the crystal's input facet must include components at both the signal and idler frequencies. In SPDC, the $z = 0$ signal and idler frequencies are unexcited, i.e., in their vacuum states. The action of the pump beam in conjunction with the crystal's nonlinearity is responsible for the excitation at these frequencies that is seen at $z = l$. Thus, although a quantum analysis will be required to understand the SPDC process, we will devote the rest of today's effort to a classical treatment of the slide 3 configuration. Nevertheless, we shall get a hint of the quantum future because the signal and idler frequencies, in the classical theory, will obey $\omega_S + \omega_I = \omega_P$. Zero-valued input fields at the signal and idler frequencies cannot account for the energy in non-zero signal and idler output fields. Instead, the energy present in these output fields must come from the pump beam. Rewriting the preceding frequency condition as $\hbar\omega_S + \hbar\omega_I = \hbar\omega_P$ at least *suggests* that a photon fission process—in which a single pump photon spontaneously downconverts into a signal photon plus an idler photon such that energy is conserved—is what is happening in SPDC. In fact, such is the case.

Maxwell's Equations in a Nonlinear Dielectric Medium

We will start our classical analysis of electromagnetic wave propagation in a $\chi^{(2)}$ medium from bedrock: Maxwell's equations for propagation in a source-free region of a nonlinear dielectric. In differential form, and *without* assuming any constitutive laws, we have that

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t), \quad \text{Faraday's law} \tag{1}$$

$$\nabla \cdot \vec{D}(\vec{r},t) = 0, \quad \text{Gauss' law} \tag{2}$$

$$\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(\vec{r},t), \quad \text{Ampère's law} \tag{3}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0, \quad \text{Gauss' law for the magnetic flux density,} \tag{4}$$

where $\vec{E}(\vec{r},t)$ is the electric field, $\vec{D}(\vec{r},t)$ is the displacement flux density, $\vec{H}(\vec{r},t)$ is the magnetic field, and $\vec{B}(\vec{r},t)$ is the magnetic flux density. All of these fields are real valued and in SI units. For dielectrics, we can take

$$\vec{B}(\vec{r},t) = \mu_0 \vec{H}(\vec{r},t), \tag{5}$$

where μ_0 is the permeability of free space, as one of the material's constitutive laws. The other free-space constitutive law is

$$\vec{D}(\vec{r},t) = \epsilon_0 \vec{E}(\vec{r},t), \tag{6}$$

where ϵ_0 is the permittivity of free space.[1] However, for the nonlinear dielectric of interest here we will use

$$\vec{D}(\vec{r},t) = \epsilon_0 \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t), \tag{7}$$

where $\vec{P}(\vec{r},t)$ is the material's polarization, which is a nonlinear function of the electric field.

Our initial objective is to reduce Maxwell's equations to a wave equation for a $+z$-propagating plane wave. Taking the curl of Faraday's law, employing the vector identity

$$\nabla \times [\nabla \times \vec{F}(\vec{r},t)] = \nabla[\nabla \cdot \vec{F}(\vec{r},t)] - \nabla^2 \vec{F}(\vec{r},t), \tag{8}$$

and Ampère's law, we get

$$\nabla[\nabla \cdot \vec{E}(\vec{r},t)] - \nabla^2 \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}[\nabla \times \vec{H}(\vec{r},t)] = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{D}(\vec{r},t). \tag{9}$$

For a $+z$-propagating plane wave whose electric field is orthogonal to the z axis, the preceding result simplifies to

$$\frac{\partial^2}{\partial z^2} \vec{E}(z,t) - \mu_0 \frac{\partial^2}{\partial t^2} \vec{D}(z,t) = \vec{0}. \tag{10}$$

Before moving on to propagation in the nonlinear medium, let's examine the wave solutions to Eq. (10) in free space and in a linear dielectric. Using $\vec{D}(z,t) = \epsilon_0 \vec{E}(\zeta,t)$, for free space, Eq. (10) becomes

$$\frac{\partial^2}{\partial z^2} \vec{E}(z,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}(z,t) = \vec{0}, \tag{11}$$

where we have used $c = 1/\sqrt{\epsilon_0 \mu_0}$. It easily verified—recall Lecture 17—that

$$\vec{E}(z,t) = f(t - z/c)\vec{i}_f, \tag{12}$$

[1] In terms of ϵ_0 and μ_0 we have that $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum, as shown in Lecture 17.

is a solution to Eq. (11) for an arbitrary time function $f(t)$ and unit vector $\vec{i}_f$ in the x-y plane.[2] Moreover, this field is a $+z$-going plane wave, as was noted in Lecture 17.

Now suppose that we are interested in propagation through a linear dielectric. In this case, and for the nonlinear case to follow, it is best to go to the temporal-frequency domain, i.e., we define the Fourier transform of a field $\vec{F}(\vec{r},t)$ by

$$\vec{\mathcal{F}}(\vec{r},\omega) = \int \mathrm{d}t\, \vec{F}(\vec{r},t)e^{j\omega t}. \tag{13}$$

The sign convention here is in keeping with our quantum-optics notion of what constitutes a positive-frequency field, viz., the inverse transform integral is

$$\vec{F}(\vec{r},t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \vec{\mathcal{F}}(\vec{r},\omega)e^{-j\omega t}. \tag{14}$$

The constitutive law for a *linear* dielectric is

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0[1+\boldsymbol{\chi}^{(1)}(\omega)]\vec{\mathcal{E}}(\vec{r},\omega), \tag{15}$$

where the linear susceptibility, $\boldsymbol{\chi}^{(1)}(\omega)$, is a frequency-dependent tensor, so that the polarization,

$$\vec{\mathcal{P}}(\vec{r},\omega) = \epsilon_0\boldsymbol{\chi}^{(1)}(\omega)\vec{\mathcal{E}}(\vec{r},\omega), \tag{16}$$

need *not* be parallel to the electric field. The tensor nature of the linear susceptibility is the anisotropy that we exploited in our discussion, earlier this semester, of wave plates. Thus, if $\vec{\mathcal{E}}(\vec{r},\omega)$ is polarized along a principal axis of the crystal—as we shall assume in what follows—we have that

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0 n^2(\omega)\vec{\mathcal{E}}(\vec{r},\omega), \tag{17}$$

is the appropriate constitutive relation, where $n(\omega)$ is the refractive index at frequency ω for the chosen polarization. Now, if we take the Fourier transform of Eq. (10) and presume fields with no (x,y) dependence with an electric field polarized along a principal axis, we obtain the Helmholtz equation

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z,\omega) = \vec{0}. \tag{18}$$

The $+z$-going plane-wave solution to this equation is

$$\vec{\mathcal{E}}(z,\omega) = \mathrm{Re}[\vec{E}e^{-j(\omega t - kz)}]. \tag{19}$$

where $k \equiv \omega n(\omega)/c$ and $\vec{E}$ is a constant vector in the x-y plane.

[2]To show that Eq. (11) provides a solution to Maxwell's equations in free space, however, more work is needed. Faraday's law should be used to derive the associated magnetic field, $\vec{H}(z,t)$, and then it should be verified that $\vec{E}(z,t)$ *and* $\vec{H}(z,t)$ are solutions to the full set of Maxwell's equations. See Lecture 17 for more details.

For a *nonlinear* dielectric we shall employ the following frequency-domain constitutive relation:

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0[1+\boldsymbol{\chi}^{(1)}(\omega)]\vec{\mathcal{E}}(\vec{r},\omega) + \vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega), \tag{20}$$

where $\boldsymbol{\chi}^{(1)}(\omega)$ is the medium's *linear* susceptibility tensor at frequency ω and $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is the *nonlinear* polarization, i.e., $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is a nonlinear function of the electric field. Assuming, as before, a $+z$-going plane wave whose electric field is polarized along a principal axis of the $\boldsymbol{\chi}^{(1)}(\omega)$ tensor, Eq. (18) becomes

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z,\omega) = -\mu_0\omega^2\vec{\mathcal{P}}_{\rm NL}(z,\omega), \tag{21}$$

for the nonlinear dielectric. The left-hand side of this equation includes the medium's linear behavior, with its nonlinear character appearing as a source term on the right-hand side. General solutions to this equation—for arbitrary nonlinearities—are beyond our reach. In the next section, however, we show how to do a coupled-mode analysis that, when converted to quantum form in Lecture 21, will allow us to understand how SPDC produces non-classical light.

Coupled-Mode Equations

Here we shall delve deeper into propagation through a nonlinear dielectric when that material's nonlinear polarization arises from a second-order nonlinearity. Unlike the preceding section, which tried to work in generality, we will now assume that the electric field propagating from $z = 0$ to $z = l$ in the nonlinear crystal consists of three $+z$-going monochromatic plane waves: the frequency-ω_P pump beam; the frequency-ω_S signal beam; and the frequency-ω_I idler beam. Furthermore, we will assume that $\omega_P = \omega_S + \omega_I$ and that the pump is very strong while the signal and idler are very weak. Allowing—as will be necessary to account for the tensor properties of the second-order susceptibility—the pump, signal, and idler to have different linear polarizations along the crystal's principal axes, we will take the electric field to be

$$\begin{aligned}\vec{E}(z,t) &= \underbrace{{\rm Re}[A_S(z)e^{-j(\omega_S t - k_S z)}]\vec{i}_S}_{\text{signal}} + \underbrace{{\rm Re}[A_I(z)e^{-j(\omega_I t - k_I z)}]\vec{i}_I}_{\text{idler}} \\ &+ \underbrace{{\rm Re}[A_P e^{-j(\omega_P t - k_P z)}]\vec{i}_P}_{\text{pump}}, \quad \text{for } 0 \le z \le l. \end{aligned} \tag{22}$$

In this expression: $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$ gives the wave numbers of the signal, idler, and pump fields in terms of the refractive indices, $n_m(\omega_m)$, of their respective linear polarizations, $\vec{i}_m$, which are all in the x-y plane. More importantly, for what will follow, the signal and idler complex envelopes, $A_S(z)$ and $A_I(z)$, are *slowly-varying* functions of z, i.e., they change very little on the scale of their field component's wavelength.[3] Also, the strong pump field has been taken to be non-depleting, i.e., its complex envelope, A_P, is a constant.[4] These assumptions are consistent with SPDC operation.

For the constitutive relation associated with the preceding electric field we will assume that

$$
\begin{aligned}
\vec{D}(z,t) &\approx \frac{\epsilon_0 n_S^2(\omega_S) A_S(z) e^{-j(\omega_S t - k_S z)} + \text{cc}}{2}\,\vec{i}_S \\
&+ \frac{\epsilon_0 n_I^2(\omega_I) A_I(z) e^{-j(\omega_I t - k_I z)} + \text{cc}}{2}\,\vec{i}_I \\
&+ \frac{\epsilon_0 n_P^2(\omega_P) A_P e^{-j(\omega_P t - k_P z)} + \text{cc}}{2}\,\vec{i}_P \\
&+ \frac{\epsilon_0 \chi^{(2)} A_I^*(z) A_P e^{-j[(\omega_P - \omega_I)t - (k_P - k_I)z]} + \text{cc}}{2}\,\vec{i}_S \\
&+ \frac{\epsilon_0 \chi^{(2)} A_S^*(z) A_P e^{-j[(\omega_P - \omega_S)t - (k_P - k_S)z]} + \text{cc}}{2}\,\vec{i}_I,
\end{aligned} \tag{23}
$$

where cc denotes complex conjugate. The first three terms on the right in Eq. (23) are due to the material's linear susceptibility. Except for the possibly different signal, idler, and pump polarizations, it is the three-wave version of what we exhibited in the previous section for a linear dielectric. The last two terms represent the effect of the material's second-order nonlinear susceptibility, $\chi^{(2)}$. Strictly speaking, this susceptibility is a frequency-dependent tensor that produces a nonlinear polarization $\vec{P}_{\text{NL}}(z,t)$ when it is multiplied by the product of two electric-field frequency components. In writing Eq. (23) we have suppressed the frequency dependence and tensor character by our choice of fixed frequencies and polarizations in Eq. (22), and we have only included second-order terms that appear at the signal or idler frequencies, as these are the frequencies that will be of interest in what follows, viz., they represent coupling between the signal and idler which is mediated by the presence of the strong pump beam in the nonlinear crystal.

Let us substitute Eq. (23) into Eq. (10) and exploit the linear independence of $e^{j\omega t}$ and $e^{-j\omega t}$ for $\omega \neq 0$ to restrict our attention to the positive-frequency terms. We

[3]This assumption goes by the acronym SVEA, i.e., the slowly-varying envelope approximation.

[4]Strictly speaking, this no-depletion assumption cannot be exactly correct, as the pump beam supplies the energy for the signal and idler outputs in SPDC. It is a good approximation for SPDC, however, because the signal and idler outputs in typical operation are much weaker than the pump beam.

then find that the electric-field complex envelopes must obey

$$
\begin{aligned}
&\frac{\partial^2}{\partial z^2}\Big(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&- \quad \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S\\
&+ \quad n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&- \quad \frac{\chi^{(2)}}{c^2}\frac{\partial^2}{\partial t^2}\Big(A_I^*(z)A_P e^{-j[(\omega_P-\omega_I)t-(k_P-k_I)z]}\vec{i}_S\\
&+ \quad A_S^*(z)A_P e^{-j[(\omega_P-\omega_S)t-(k_P-k_S)z]}\vec{i}_I\Big) = \vec{0}.
\end{aligned}
\tag{24}
$$

Performing the z differentiations on the first line of Eq. (24) gives

$$
\begin{aligned}
&\frac{\partial^2}{\partial z^2}\Big(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&= \quad \left[-k_S^2 A_S(z) + 2jk_S\frac{\mathrm{d}A_S(z)}{\mathrm{d}z}\right]e^{-j(\omega_S t-k_S z)}\vec{i}_S\\
&+ \quad \left[-k_I^2 A_I(z) + 2jk_I\frac{\mathrm{d}A_I(z)}{\mathrm{d}z}\right]e^{-j(\omega_I t-k_I z)}\vec{i}_I - k_P^2 A_P e^{-j(\omega_P-k_P z)}\vec{i}_P,
\end{aligned}
\tag{25}
$$

where we have employed the slowly-varying envelope approximation to suppress terms involving $\frac{\partial^2}{\partial z^2}A_m(z)$ for $m = S, I$. Performing the t differentiations on the second and third lines of Eq. (24) yields

$$
\begin{aligned}
&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I\\
&+ \quad n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&= \quad k_S^2 A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + k_I^2 A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + k_P^2 A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P,
\end{aligned}
\tag{26}
$$

where we have used $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$. Using Eqs. (25) and (26) in Eq. (24) leads to term cancellations[5] that reduce the latter equation to

$$
\begin{aligned}
&\left(2jk_S\frac{\mathrm{d}A_S(z)}{\mathrm{d}z}e^{-j(\omega_S t-k_S z)} + \frac{\chi^{(2)}\omega_S^2}{c^2}A_I^*(z)A_P e^{-j[\omega_S t-(k_P-k_I)z]}\right)\vec{i}_S\\
&+ \quad \left(2jk_I\frac{\mathrm{d}A_I(z)}{\mathrm{d}z}e^{-j(\omega_I t-k_I z)} + \frac{\chi^{(2)}\omega_I^2}{c^2}A_S^*(z)A_P e^{-j[\omega_I t-(k_P-k_S)z]}\right)\vec{i}_I = \vec{0},
\end{aligned}
\tag{27}
$$

[5]These cancellations are to be expected, as the terms in question are those for a linear dielectric and $k_m = \omega_m n_m(\omega_m)/c$ gives the plane-wave solutions for such media.

where we have used $\omega_P = \omega_S + \omega_I$.

We will be interested in SPDC systems in which the signal and idler are in orthogonal linear polarizations. In this case, the preceding equation can be decomposed into two *coupled-mode* equations:[6]

$$
\begin{aligned}
\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} &= j\frac{\omega_S\chi^{(2)}A_P}{2cn_S(\omega_S)}\,A_I^*(z)e^{j\Delta kz} && (28)\\
\frac{\mathrm{d}A_I(z)}{\mathrm{d}z} &= j\frac{\omega_I\chi^{(2)}A_P}{2cn_I(\omega_I)}\,A_S^*(z)e^{j\Delta kz}, && (29)
\end{aligned}
$$

for $0 \le z \le l$, where $\Delta k \equiv k_P - k_S - k_I$. Equations (28) and (29) should be solved subject to given initial conditions at $z = 0$, i.e., given values for $A_S(0)$ and $A_I(0)$. Once $A_S(l)$ and $A_I(l)$ are found, the resulting electric field for $z > l$ is then

$$
\begin{aligned}
\vec{E}(z,t) &= \mathrm{Re}[A_S(l)e^{-j(\omega_S t - k_S l - \omega_S(z-l)/c)}]\vec{i}_S + \mathrm{Re}[A_I(l)e^{-j(\omega_I t - k_I l - \omega_I(z-l)/c)}]\vec{i}_I \\
&+ \mathrm{Re}[A_P e^{-j(\omega_P t - k_P l - \omega_P(z-l)/c)}]\vec{i}_P, && (30)
\end{aligned}
$$

i.e., free-space plane-wave propagation prevails.[7] Here we can see why quantum mechanics is needed to properly understand the SPDC process shown on slide 3. If $A_S(0) = A_I(0) = 0$, in our classical analysis, then we get $A_S(l) = A_I(l) = 0$ from our coupled-mode equations,[8] and hence $\vec{E}(z,t) = \mathrm{Re}[A_P e^{j(\omega_P t - k_P l - \omega_P(z-l)/c)}]\vec{i}_P$ for $z > l$.

Solution to the Coupled-Mode Equations

So far we have been working with Maxwell's equations—and hence have developed coupled-mode equations—in SI units, i.e., the complex envelopes $A_S(z), A_I(z)$, and A_P have V/m units. Before solving the coupled-mode equations, it will be convenient for us to convert them to photon units, so as to ease the transition we will make—in Lecture 21—from the classical solution to the quantum version. The key to making this conversion is power flow.

Consider a monochromatic, $+z$-going plane wave in an isotropic linear dielectric whose electric and magnetic fields are

$$\vec{E}(z,t) = \mathrm{Re}[Ae^{-j(\omega t - kz)}]\vec{i}_x \quad \text{and} \quad \vec{H}(z,t) = \mathrm{Re}[c\epsilon_0 n(\omega)Ae^{-j(\omega t - kz)}]\vec{i}_y. \qquad (31)$$

[6]If we regard the signal-frequency and idler-frequency components of the total field as *modes*, then these equations clearly couple them through the action of the strong pump beam and the crystal's $\chi^{(2)}$ nonlinearity.

[7]Our analysis assumes that anti-reflection coatings have been applied to the crystal's entrance and exit facets.

[8]If this statement is not immediately obvious, see the next section, in which we present solutions to the photon-units form of the coupled-mode equations

The time-average power (in W) crossing an area $\mathcal{A}$ in a constant-z plane is

$$S(z) = \frac{c\epsilon_0 n(\omega)\mathcal{A}}{2}\,|A|^2. \tag{32}$$

Were A to be written in $\sqrt{\text{photons/s}}$ units—for the chosen area $\mathcal{A}$—we would get[9]

$$S(z) = \hbar\omega|A|^2 \tag{33}$$

for the time-average power (in W) crossing the same area. It follows that

$$A|_{\sqrt{\text{photons/s}}} = \sqrt{\frac{c\epsilon_0 n(\omega)\mathcal{A}}{2\hbar\omega}}\,A|_{\text{V/m}}. \tag{34}$$

Making this substitution in Eqs. (28) and (29) leads to the photon-units coupled-mode equations,

$$\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} = j\kappa A_I^*(z)e^{j\Delta kz} \tag{35}$$

$$\frac{\mathrm{d}A_I(z)}{\mathrm{d}z} = j\kappa A_S^*(z)e^{j\Delta kz}, \tag{36}$$

for $0 \le z \le l$, where

$$\kappa \equiv \sqrt{\frac{\hbar\omega_S\omega_I\omega_P}{2c^3\epsilon_0 n_S(\omega_S)n_I(\omega_I)n_P(\omega_P)\mathcal{A}}}\,\chi^{(2)}A_P \tag{37}$$

is a complex-valued coupling constant that is proportional to the pump's complex envelope and the crystal's second-order nonlinear susceptibility.

The preceding photon-units coupled-mode equations have the following solution,

$$A_S(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_S(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_I^*(0)\right]e^{j\Delta kl/2} \tag{38}$$

$$A_I(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_I(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_S^*(0)\right]e^{j\Delta kl/2}, \tag{39}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\Delta k/2)^2}, \tag{40}$$

as the reader may want to verify by substituting these results into the coupled-mode equations. Equations (38) and (39) have two interesting features that are worth

[9]We have chosen $\sqrt{\text{photons/s}}$ units, which require us to account for a cross-sectional area, to avoid needing an explicit area factor when we examine the continuous-time photodetection statistics of our SPDC model.

noting at this time. The first concerns phase matching. The second is a prelude to our quantum treatment of SPDC.

Inside the crystal, the monochromatic signal, idler, and pump beams—at frequencies ω_S, ω_I, and ω_P, respectively, propagate at their phase velocities, $v_m(\omega_m) = \omega_m/k_m$ for $m = S, I, P$. The nonlinear interaction governed by the coupled-mode equations Eqs. (35) and (36) is said to be *phase matched* when $\Delta k = 0$, i.e., when $\omega_P/v_P = \omega_S/v_S + \omega_I/v_I$. For a phase-matched system the coupled-mode equations simplify to

$$\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} = j\kappa A_I^*(z) \quad \text{and} \quad \frac{\mathrm{d}A_I(z)}{\mathrm{d}z} = j\kappa A_S^*(z), \quad \text{for } 0 \leq z \leq l, \tag{41}$$

which shows that the phase angle of the coupling between the signal and idler remains the same throughout the interaction. On the other hand, when phase-matching is violated, the phase of the coupling between the signal and idler rotates as these fields propagate through the crystal. As a result, the solution to the phase-matched coupled-mode equations,

$$A_S(l) = \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0) \tag{42}$$

$$A_I(l) = \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0), \tag{43}$$

shows increasing amounts of signal-idler coupling with increasing $|\kappa|l$, i.e., with increasing pump power or crystal length. In contrast, far from phase matching—when $|\Delta k/2| \gg |\kappa|$—we get $p \approx j|\Delta k|/2$, whence

$$A_S(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_S(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}A_I^*(0)\right]e^{j\Delta kl/2} \tag{44}$$

$$A_I(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_I(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}A_S^*(0)\right]e^{j\Delta kl/2}, \tag{45}$$

which further reduce to

$$A_S(l) \approx A_S(0) \quad \text{and} \quad A_I(l) \approx A_I(0), \tag{46}$$

when $|\Delta kl/2| \gg 1$, i.e., when the crystal is long enough that the phase mismatch, $\Delta k \neq 0$, rotates the signal-idler coupling phase through many 2π cycles. Phase matching is critical to SPDC; in terms of photon fission, for every 10^6 pump photons, we may get only one signal-idler pair from a phase-matched interaction.

Photon fission is a good place to start our comments about the quantum form of the coupled-mode equations. We have already noted that $\omega_P = \omega_S + \omega_I$ is consistent with the photon-fission energy conservation principle: $\hbar\omega_P = \hbar\omega_S + \hbar\omega_I$. The momentum of a $+z$-going single photon at frequency ω is $+z$-directed with magnitude $\hbar\omega$. Thus our phase-matching condition, $k_P = k_S + k_I$, becomes photon-fission momentum conservation, $\hbar k_P = \hbar k_S + \hbar k_I$, when applied at the single-photon level. Photons being produced in pairs smacks of the two-mode parametric amplifier that we studied earlier in the semester. That system was governed by a two-mode Bogoliubov transformation,

$$\hat{a}_S^{\text{out}} = \mu \hat{a}_S^{\text{in}} + \nu \hat{a}_I^{\text{in}\dagger} \quad \text{and} \quad \hat{a}_I^{\text{out}} = \mu \hat{a}_I^{\text{in}} + \nu \hat{a}_S^{\text{in}\dagger}, \quad \text{where } |\mu|^2 - |\nu|^2 = 1. \tag{47}$$

Comparing Eq. (47) with Eqs. (42) and (43) reveals a great similarity. Indeed, if we change field complex envelopes and their conjugates to annihilation operators and creation operators, respectively, the latter two equations become a two-mode Bogoliubov transformation with[10]

$$\mu \equiv \cosh(|\kappa| l) \quad \text{and} \quad \nu \equiv j \frac{\kappa}{|\kappa|} \sinh(|\kappa| l). \tag{48}$$

The Road Ahead

In the next lecture we shall develop the quantum treatment of SPDC and the optical parametric amplifier (OPA), which is SPDC enhanced by placing the nonlinear crystal inside a resonant optical cavity. We shall also begin studying the non-classical behavior that can be seen in continuous-time photodetection using the outputs from SPDC and the OPA.

[10]Even for the general case of $\Delta k \neq 0$, changing the field complex envelopes and their conjugates into annihilation and creation operators, respectively, converts the classical coupled-mode input-output relation into a two-mode Bogoliubov transformation. When $|\Delta k l/2| \gg 1$, however, that two-mode Bogoliubov transformation will have $\mu \approx 1$ and $\nu \approx 0$.

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6.453 Quantum Optical Communication

Lecture Number 21
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Reading:

- For nonclassical light generation from parametric downconversion:
  - L. Mandel and E. Wolf *Optical Coherence and Quantum Optics,* (Cambridge University Press, Cambridge, 1995) sections 21.7, 22.4.
  - F.N.C. Wong, J.H. Shapiro, and T. Kim, "Efficient generation of polarization-entangled photons in a nonlinear crystal," Laser Phys. **16,** 1516 (2006).
- For Gaussian-state theory of parametric amplifier noise and its quantum signatures:
  - J.H. Shapiro and K.-X. Sun, "Semiclassical versus quantum behavior in fourth-order interference," J. Opt. Soc. Am. B **11**, 1130 (1994).
  - J.H. Shapiro, "Quantum Gaussian noise," Proc. SPIE **5111**, 382 (2003).
  - J.H. Shapiro, "The quantum theory of optical communications," IEEE J. Sel. Top. Quantum Electron. **15**, 1547–1569 (2009); J.H. Shapiro, "Corretions to 'The quantum theory of optical communicataions'," IEEE J. Sel. Top. Quantum Electron. **16**, 698 (2010).

Introduction

In today's lecture we will continue—and complete—our analysis of spontaneous parametric downconversion (SPDC) by converting the classical treatment from Lecture 20 into a continuous-time field operator theory. As was done in Lecture 20, we shall assume continuous-wave (cw) pumping with no pump depletion, and a collinear type-II configuration in which the signal and idler fields are +z-going plane waves that are orthogonally polarized. Moreover, we shall assume that the signal and idler center frequencies are both $\omega_P/2$, i.e., half the pump frequency.[1] This frequency degeneracy

[1]Whereas the analysis in Lecture 20 assumed single-frequency signal and idler beams, the quantum theory requires that we include *all* frequencies, hence our identification of center frequencies for these beams.

of the signal and idler is not required for some nonclassical effects that can be obtained from SPDC, but is necessary for others, e.g., quadrature-noise squeezing. Thus it is worthwhile imposing this condition at the outset. Once we have established the quantum theory for SPDC, we will add cavity enhancement to convert the downconverter into an optical parametric amplifier (OPA). The OPA analysis that we shall perform will employ a simpler, lumped-element theory for the nonlinear interaction in the $\chi^{(2)}$ material that will quickly lead to a Gaussian-state characterization which gives rise to quadrature-noise squeezing. In Lecture 22, we shall finish our survey of the nonclassical signatures produced by $\chi^{(2)}$ interactions. There we shall consider Hong-Ou-Mandel interferometry and the generation of polarization-entangled photon pairs from SPDC, along with the photon-twins behavior of the signal and idler beams from an OPA.

Classical Theory of Spontaneous Parametric Downconversion

Slide 3 reprises our conceptual picture of spontaneous parametric downconversion. A strong, linearly-polarized (along $\vec{i}_P$) cw laser-beam pump at frequency ω_P is applied to the entrance facet (at $z = 0$) of a length-l crystalline material that possesses a $\chi^{(2)}$ nonlinearity. The action of the pump beam in conjunction with the crystal's nonlinearity couples lower-frequency—signal and idler—beams that we shall assume to be linearly polarized along orthogonal directions $\vec{i}_S = \vec{i}_x$ (signal) and $\vec{i}_I = \vec{i}_y$ (idler), respectively, with common center frequency $\omega_P/2$. In Lecture 20 we treated the signal, idler, and (non-depleting) pump inside the crystal as monochromatic plane waves, with positive-frequency, photon-units fields given by

$$E_S^{(+)}(z,t) = A_S(z)e^{-j(\omega_P t/2 - k_S z)} \quad (1)$$

$$E_I^{(+)}(z,t) = A_I(z)e^{-j(\omega_P t/2 - k_I z)} \quad (2)$$

$$E_P^{(+)}(z,t) = A_P e^{-j(\omega_P t - k_P z)}. \quad (3)$$

respectively, for the polarization components of interest. In this representation, $\hbar\omega_P|A_S(z)|^2/2$ and $\hbar\omega_P|A_I(z)|^2/2$ are the signal and idler powers flowing across the z plane, for $0 \leq z \leq l$. For $z > l$, free-space propagation applies, i.e., the positive-frequency, photon-units signal, idler, pump fields in that region are

$$E_S^{(+)}(z,t) = A_S(l)e^{-j(\omega_P(t-(z-l)/c)/2 - k_S l)} \quad (4)$$

$$E_I^{(+)}(z,t) = A_I(l)e^{-j(\omega_P(t-(z-l)/c)/2 - k_I l)} \quad (5)$$

$$E_P^{(+)}(z,t) = A_P e^{-j(\omega_P(t-(z-l)/c) - k_P l)}. \quad (6)$$

The coupled-mode equations that the signal and idler satisfy inside the nonlinear crystal were shown last time to be

$$\frac{\mathrm{d}A_S(z)}{\mathrm{d}z} = j\kappa A_I^*(z)e^{j\Delta kz} \tag{7}$$

$$\frac{\mathrm{d}A_I(z)}{\mathrm{d}z} = j\kappa A_S^*(z)e^{j\Delta kz}, \tag{8}$$

for $0 \le z \le l$. Here: $\Delta k \equiv k_P(\omega_P) - k_S(\omega_P/2) - k_I(\omega_P/2)$ quantifies the phase-mismatch between the signal, idler, and pump beams in terms of their respective dispersion relations, $\{\, k_j(\omega) \equiv \omega n_j(\omega)/c : j = S, I, P \,\}$ with $\{\, n_j(\omega) : j = S, I, P \,\}$ denoting the refractive indices for the relevant polarization components; and

$$\kappa \equiv \sqrt{\frac{\hbar\omega_S\omega_I\omega_P}{2c^3\epsilon_0 n_S(\omega_S)n_I(\omega_I)n_P(\omega_P)\mathcal{A}}}\,\chi^{(2)}A_P \tag{9}$$

is a complex-valued coupling constant that is proportional to the pump's complex envelope and the crystal's second-order nonlinear susceptibility. The general solution to these equations is

$$A_S(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_S(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_I^*(0)\right]e^{j\Delta kl/2} \tag{10}$$

$$A_I(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_I(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_S^*(0)\right]e^{j\Delta kl/2}, \tag{11}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\Delta k/2)^2}. \tag{12}$$

However, to get the most efficient interaction, we need phase-matched operation, i.e., $\Delta k = 0$, in which case the solution to Eqs. (7) and (8) reduces to

$$A_S(l) = \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0) \tag{13}$$

$$A_I(l) = \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0), \tag{14}$$

indicating increasing amounts of signal-idler coupling with increasing $|\kappa|l$, i.e., with increasing pump power or crystal length.

Quantum Theory of Spontaneous Parametric Downconversion

At the end of Lecture 20 we noted that the SPDC's frequency-sum condition, $\omega_P = \omega_S + \omega_I$, and its phase-matching condition, $k_P = k_S + k_I$, could be interpreted as energy conservation and momentum conservation, respectively, for a photon fission process in which a single pump photon divides into a signal photon and an idler photon. We also noted, in that lecture, that the solutions to the coupled-mode equations, which we reprised in the previous section, are a two-mode Bogoluibov transformation, similar to what we saw earlier in the semester for our two-mode optical parametric amplifier. It is now time for us to go beyond these precursors and establish the quantum field-operator theory for cw collinear SPDC at frequency degeneracy.[2]

Suppose that $\hat{E}_S^{(+)}(z,t)$ and $\hat{E}_I^{(+)}(z,t)$ for $0 \leq z \leq l$ are the positive-frequency, photon-units $+z$-going plane-wave field operators for the $\vec{i}_x$ and $\vec{i}_y$ polarization components of the signal and idler, respectively.[3] Because we must preserve δ-function commutators for the signal and idler field operators leaving the nonlinear crystal, we must include all frequencies in them. Hence we shall take $\hat{E}_S^{(+)}(z,t)$ and $\hat{E}_I^{(+)}(z,t)$ to have the following Fourier decompositions:

$$\hat{E}_S^{(+)}(z,t) = \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_S(z,\omega)e^{-j[(\omega_P/2+\omega)t - k_S(\omega_P/2+\omega)z]}, \tag{15}$$

$$\hat{E}_I^{(+)}(z,t) = \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_I(z,\omega)e^{-j[(\omega_P/2-\omega)t - k_I(\omega_P/2-\omega)z]}. \tag{16}$$

In these expressions, $\hat{A}_S(z,\omega)$ is the plane-wave field-component annihilation operator for the signal beam at frequency shift ω from frequency degeneracy, and $\hat{A}_I(z,\omega)$ is the plane-wave field-component annihilation operator for the idler beam at frequency shift $-\omega$ from frequency degeneracy.[4] At the crystal's entrance and exit facets, the signal and idler fields operators must have the following non-zero commutators that apply for free-space fields,

$$[\hat{E}_S^{(+)}(z,t), \hat{E}_S^{(+)\dagger}(z,u)] = [\hat{E}_I^{(+)}(z,t), \hat{E}_I^{(+)\dagger}(z,u)] = \delta(t-u), \quad \text{for } z = 0, l, \tag{17}$$

which imply that

$$[\hat{A}_S(z,\omega), \hat{A}_S^\dagger(z,\omega')] = [\hat{A}_I(z,\omega), \hat{A}_I^\dagger(z,\omega')] = 2\pi\delta(\omega-\omega'), \quad \text{for } z = 0, l, \tag{18}$$

are the only non-zero frequency-domain commutators at the crystal's input and output. Any proper quantized form of the coupled-mode equations and their solutions must preserve these commutator brackets.

[2] The basic concepts we shall develop can be extended to non-degenerate, non-collinear operation, but we shall not do so.

[3] A full field-operator treatment should include *all* spatial modes, not just the $+z$-going plane-wave modes, and *both* polarizations for all such modes. However, we shall limit our consideration to these polarizations of the $+z$-going signal and idler plane waves. For coherent (homodyne or heterodyne) detection measurements, spatial and polarization mode selection automatically occurs by choice of the local oscillator, so our assumption is easily enforced in such measurement scenarios. For direct detection, however, other spatial modes and polarizations may have to be included, depending on the SPDC and measurement configuration.

[4] This sign convention is convenient because the coupled-mode equations for classical versions of these Fourier decompositions link $A_S(z,\omega)$ to $A_I^*(z,\omega)$ and vice versa.

We shall assume that the downconverter is phase-matched at frequency degeneracy, viz.,

$$\Delta k(\omega) \equiv k_P(\omega_P) - k_S(\omega_P/2+\omega) - k_I(\omega_P/2-\omega), \tag{19}$$

satisifes $\Delta k(0) = 0$, and that group-velocity dispersion can be neglected, so that

$$\Delta k(\omega) \approx \omega \Delta k' \tag{20}$$

holds, where

$$\Delta k' \equiv \left.\frac{\mathrm{d}\Delta k}{\mathrm{d}\omega}\right|_{\omega=0} = -\left.\frac{\mathrm{d}k_S(\omega_P/2+\omega)}{\mathrm{d}\omega}\right|_{\omega=0} - \left.\frac{\mathrm{d}k_I(\omega_P/2-\omega)}{\mathrm{d}\omega}\right|_{\omega=0}. \tag{21}$$

Emboldened by last lecture's comment about Bogoliubov transformations, as well as our earlier quantization of the classical harmonic oscillator, we shall *assume* that $\hat{A}_S(z,\omega)$ and $\hat{A}_I(z,\omega)$ obey the following coupled-mode equations:

$$\frac{\partial \hat{A}_S(z,\omega)}{\partial z} = j\kappa \hat{A}_I^\dagger(z,\omega) e^{j\omega\Delta k' z} \tag{22}$$

$$\frac{\partial \hat{A}_I(z,\omega)}{\partial z} = j\kappa \hat{A}_S^\dagger(z,\omega) e^{j\omega\Delta k' z}, \tag{23}$$

for $0 \le z \le l$, where κ is the *same* coupling constant from the classical theory, i.e., Eq. (9).[5] These equations have the following solution, cf. Eqs. (10) and (11):

$$\hat{A}_S(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_S(0,\omega) + j\kappa l \frac{\sinh(pl)}{pl}\,\hat{A}_I^\dagger(0,\omega)\right] e^{j\omega\Delta k' l/2} \tag{24}$$

$$\hat{A}_I(l,\omega) = \left[\left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right)\hat{A}_I(0,\omega) + j\kappa l \frac{\sinh(pl)}{pl}\,\hat{A}_S^\dagger(0,\omega)\right] e^{j\omega\Delta k' l/2}, \tag{25}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\omega\Delta k'/2)^2}. \tag{26}$$

To verify that these solution preserve free-space commutator brackets, let us define

$$\mu(\omega) = \left(\cosh(pl) - \frac{j\omega\Delta k' l}{2}\frac{\sinh(pl)}{pl}\right) e^{j\omega\Delta k' l/2} \tag{27}$$

$$\nu(\omega) = j\kappa l \frac{\sinh(pl)}{pl} e^{j\omega\Delta k' l/2}, \tag{28}$$

[5]We have assumed that the strong, non-depleting pump is in a coherent state such that—as in the case of the local oscillator beam for homodyne and heterodyne detection—it acts classically in SPDC.

so that Eqs. (24) and (25) become

$$\hat{A}_S(l,\omega) = \mu(\omega)\hat{A}_S(0,\omega) + \nu(\omega)\hat{A}_I^\dagger(0,\omega) \quad (29)$$

$$\hat{A}_I(l,\omega) = \mu(\omega)\hat{A}_I(0,\omega) + \nu(\omega)\hat{A}_S^\dagger(0,\omega). \quad (30)$$

Now, because

$$|\mu(\omega)|^2 - |\nu(\omega)|^2 = \left[\cosh^2(pl) + \left(\frac{\omega\Delta k'}{2p}\right)^2 \sinh^2(pl)\right] - \left(\frac{|\kappa|}{p}\right)^2 \sinh^2(pl) \quad (31)$$

$$= \cosh^2(pl) - \sinh^2(pl) = 1, \quad (32)$$

Eqs. (29) and (30) are a two-mode Bogoliubov transformation that ensures proper commutator preservation.[6]

Gaussian-State Characterization of SPDC

Equations (29) and (30) allow us an immediate insight into the joint state of the signal and idler produced by spontaneous parametric downconversion, i.e., the joint state of the signal and idler beams emerging from the crystal at $z = l$ when the signal and idler inputs at $z = 0$ are in their vacuum states. In particular, the linearity of these equations, combined with the fact that the vacuum state is zero-mean and Gaussian, tells us that the signal and idler outputs will be in a zero-mean jointly Gaussian state. Hence they are completely characterized by their phase-insensitive and phase-sensitive correlation functions, of which the only non-zero ones are $\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle$, $\langle \hat{A}_I^\dagger(l,\omega)\hat{A}_I(l,\omega')\rangle$, and $\langle \hat{A}_S(l,\omega)\hat{A}_I(l,\omega')\rangle$. These correlations are easily computed, e.g., for the signal's phase-insensitive correlation function we have that

$$\begin{aligned}
&\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle \\
&= \langle [\mu^*(\omega)\hat{A}_S^\dagger(0,\omega) + \nu^*(\omega)\hat{A}_I(0,\omega)][\mu(\omega')\hat{A}_S(0,\omega') + \nu(\omega')\hat{A}_I^\dagger(0,\omega')]\rangle \quad (33)\\
&= \mu^*(\omega)\mu(\omega')\langle \hat{A}_S^\dagger(0,\omega)\hat{A}_S(0,\omega')\rangle + \mu^*(\omega)\nu(\omega')\langle \hat{A}_S^\dagger(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle \\
&+ \nu^*(\omega)\mu(\omega')\langle \hat{A}_I(0,\omega)\hat{A}_S(0,\omega')\rangle + \nu^*(\omega)\nu(\omega')\langle \hat{A}_I(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle. \quad (34)
\end{aligned}$$

Now, because the input fields are in their vacuum states, all their normally-ordered correlation functions vanish, so, using the commutator (18), we get

$$\langle \hat{A}_I(0,\omega)\hat{A}_I^\dagger(0,\omega')\rangle = 2\pi\delta(\omega - \omega'), \quad (35)$$

[6]Our proof has assumed that p is real valued, i.e., it applies for frequencies low enough to give $|\omega\Delta k'/2| \le |\kappa|$. At higher frequencies, where $|\omega\Delta k'/2| > |\kappa|$ prevails, p becomes imaginary, but a similar calculation—left to the reader—will show that Eqs. (29) and (30) still constitute a two-mode Bogoliubov transformation and hence commutator preserving.

whence

$$\langle \hat{A}_S^\dagger(l,\omega)\hat{A}_S(l,\omega')\rangle = 2\pi|\nu(\omega)|^2\delta(\omega-\omega'). \tag{36}$$

Similar calculations yield

$$\langle \hat{A}_I^\dagger(l,\omega)\hat{A}_I(l,\omega')\rangle = 2\pi|\nu(\omega)|^2\delta(\omega-\omega'), \tag{37}$$

and

$$\langle \hat{A}_S(l,\omega)\hat{A}_I(l,\omega')\rangle = 2\pi\mu(\omega)\nu(\omega)\delta(\omega-\omega'), \tag{38}$$

for the other correlation functions that we need.

For future use it will be valuable to find the phase-insensitive and phase-sensitive correlation functions for the baseband signal and idler field operators defined by

$$\hat{E}_S^{(+)}(l,t) = \hat{E}_S(t)e^{-j(\omega_P t/2 - k_S(\omega_P/2)l)} \quad \text{and} \quad \hat{E}_I^{(+)}(l,t) = \hat{E}_I(t)e^{-j(\omega_P t/2 - k_I(\omega_P/2)l)}. \tag{39}$$

Using the Fourier relations

$$\begin{aligned} \hat{E}_S(t) &= \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_S(l,\omega)e^{-j\omega(t-k'_S l)}, && (40)\\ \hat{E}_I(t) &= \int \frac{\mathrm{d}\omega}{2\pi}\,\hat{A}_I(l,\omega)e^{j\omega(t+k'_I l)}, && (41) \end{aligned}$$

where

$$k'_S \equiv \left.\frac{\mathrm{d}k_S(\omega_P/2+\omega)}{\mathrm{d}\omega}\right|_{\omega=0} \quad \text{and} \quad k'_I \equiv \left.\frac{\mathrm{d}k_I(\omega_P/2-\omega)}{\mathrm{d}\omega}\right|_{\omega=0}, \tag{42}$$

together with the frequency-domain correlation functions derived above, we find that the non-zero correlations of the baseband field operators are stationary—dependent on time-difference only—and given by

$$\begin{aligned} K_{SS}^{(n)}(\tau) &\equiv \langle \hat{E}_S^\dagger(t+\tau)\hat{E}_S(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,|\nu(\omega)|^2 e^{j\omega\tau} && (43)\\ K_{II}^{(n)}(\tau) &\equiv \langle \hat{E}_I^\dagger(t+\tau)\hat{E}_I(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,|\nu(-\omega)|^2 e^{j\omega\tau} && (44)\\ K_{SI}^{(p)}(\tau) &\equiv \langle \hat{E}_S(t+\tau)\hat{E}_I(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\,\mu(-\omega)\nu(-\omega)e^{j\omega(\tau+\Delta k' l)}, && (45) \end{aligned}$$

with $^{(n)}$ denoting the phase-insensitive (normally-ordered) auto-correlation functions and $^{(p)}$ denoting the phase-sensitive cross-correlation function. We have made all of these expressions employ $e^{j\omega\tau}$ inverse Fourier kernels so that—in keeping with our definition of noise spectral densities for real-valued classical random processes—we can say that

$$\mathcal{S}_{SS}^{(n)}(\omega) = |\nu(\omega)|^2,\ \mathcal{S}_{II}^{(n)}(\omega) = |\nu(-\omega)|^2,\ \text{and}\ \mathcal{S}_{SI}^{(p)}(\omega) = \mu(-\omega)\nu(-\omega)e^{j\omega\Delta k' l}, \tag{46}$$

are their corresponding spectral densities.

Physically, $\mathcal{S}^{(n)}_{SS}(\omega)/2\pi$ is the average photon-flux per unit bilateral bandwidth (in rad/s) in the signal beam at frequency $\omega_P/2+\omega$, and $\mathcal{S}^{(n)}_{II}(\omega)/2\pi$ is the average photon-flux per unit bilateral bandwidth (in rad/s) in the idler beam at frequency $\omega_P/2-\omega$. These functions are usually referred to as the fluorescence spectra of the signal and idler, respectively. SPDC is usually performed in the regime wherein $|\kappa|l \ll 1$ so that we can employ $p \approx j\omega|\Delta k'|/2$ at all relevant detunings from degeneracy, i.e., for all ω values of interest. This low-gain condition leads to the following approximations for the Bogoliubov functions in the vicinity of frequency degeneracy[7]

$$\mu(\omega) \approx 1 \quad \text{and} \quad \nu(\omega) \approx j\kappa l \frac{\sin(\omega\Delta k' l/2)}{\omega \Delta k' l/2} e^{j\omega\Delta k' l/2}. \tag{47}$$

It follows that the signal and idler fluorescence spectra are equal, and given by

$$\mathcal{S}^{(n)}_{SS}(\omega) = \mathcal{S}^{(n)}_{II}(\omega) \approx |\kappa|^2 l^2 \left(\frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2}\right)^2. \tag{48}$$

Thus, they peak at $\omega = 0$, i.e., frequency degeneracy, where the phase-matching condition is satisfied. More importantly, we see that these fluorescence spectra are consistent with the photon fission interpretation of SPDC, in that the signal beam's fluorescence spectrum at $\omega_P/2+\omega$ equals the idler beam's fluorescence spectrum at $\omega_P/2-\omega$. The phase-sensitive cross-spectral density, $\mathcal{S}^{(p)}_{SI}(\omega)$, in the low-gain regime, is

$$\mathcal{S}^{(p)}_{SI}(\omega) \approx j\kappa l \frac{\sin(\omega\Delta k' l/2)}{\omega\Delta k' l/2} e^{j\omega\Delta k' l/2}. \tag{49}$$

We shall work further with these low-gain spectra, and their associated correlation functions, in Lecture 22, when we study the Hong-Ou-Mandel dip and SPDC generation of polarization-entangled photon pairs. For the rest of today's lecture, however, we will turn our attention to cavity-enhanced SPDC, i.e., the optical parametric amplifier.

The Doubly-Resonant Optical Parametric Amplifier

To go beyond the low-gain regime in cw SPDC we need the optical parametric amplifier (OPA), shown schematically on slide 10 as a $\chi^{(2)}$ crystal inside an optical cavity formed by two mirrors. These mirrors are anti-reflection coated for the pump frequency ω_P, so the pump makes a single pass, from left to right, through through the crystal. We will assume that the mirror on the left is a perfect reflector at the frequency $\omega_P/2$, while the mirror on the right is lossless and highly reflecting at

[7] These approximations violate strict commutator preservation, i.e., $|\mu(\omega)|^2 - |\nu(\omega)|^2 = 1$ is only satisfied to first order in $|\kappa|$.

this frequency. As a result, the spontaneously generated signal and idler photons—resulting from frequency-degenerate downconversion in the $\chi^{(2)}$ crystal—bounce back and forth between the mirrors many times before exiting through the highly-reflecting mirror. This optical feedback process greatly enhances the nonlinear interaction by making the crystal act as though it was much longer than it is. Of course, this feedback is only effective when it is *positive* feedback, which in this case means that $\omega_P/2$ must be a resonant frequency of the cavity, i.e., the roundtrip phase delay inside the cavity at frequency $\omega_P/2$ must be an integer multiple of 2π. In what follows we shall assume that the cavity is resonant for both the signal and idler polarizations at frequency $\omega_P/2$.

Although it is possible to analyze this OPA arrangement by imposing cavity mirrors around the SPDC analysis we've given earlier in this lecture, a much simpler route to getting to the essential physics employs a lumped-element treatment for intracavity modes that are resonant at frequency $\omega_P/2$ for both the signal and idler ($\vec{i}_x$ and $\vec{i}_y$) polarizations. We shall use $\hat{E}_S^{\text{in}}(t)$ and $\hat{E}_I^{\text{in}}(t)$ to denote the vacuum-state, baseband field operators of the relevant signal and idler polarizations that are incident on the cavity in slide 10 from the right, while $\hat{a}_S(t)$ and $\hat{a}_I(t)$ will be the photon annihilation operators for the associated intracavity modes.[8] The equations of motion for the OPA system then turn out to be

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\Gamma\right)\hat{a}_S(t) = G\Gamma\hat{a}_I^\dagger(t)+\sqrt{2\Gamma}\hat{E}_S^{\text{in}}(t) \tag{50}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\Gamma\right)\hat{a}_I(t) = G\Gamma\hat{a}_S^\dagger(t)+\sqrt{2\Gamma}\hat{E}_I^{\text{in}}(t), \tag{51}$$

where $0 < G < 1$ is the normalized OPA gain[9] and $\Gamma > 0$ is the linewidth of the signal and idler intracavity modes. Once Eqs. (50) and (51) have been solved for the intracavity modes as functions of the input field operators, the baseband field operators for the signal and idler outputs follow from

$$\hat{E}_S^{\text{out}}(t) = \sqrt{2\Gamma}\hat{a}_S(t) - \hat{E}_S^{\text{in}}(t) \quad \text{and} \quad \hat{E}_I^{\text{out}}(t) = \sqrt{2\Gamma}\hat{a}_I(t) - \hat{E}_I^{\text{in}}(t). \tag{52}$$

Frequency-domain techniques—as we used above to obtain our SPDC input-output relations—can be used to derive the following two-mode Bogoliubov relation between the Fourier transforms[10] of the input and output field operators,

$$\hat{\mathcal{E}}_S^{\text{out}}(\Omega) = \mu(\Omega)\hat{\mathcal{E}}_S^{\text{in}}(\Omega)+\nu(\Omega)\hat{\mathcal{E}}_I^{\text{in}\dagger}(\Omega) \tag{53}$$

$$\hat{\mathcal{E}}_I^{\text{out}}(\Omega) = \mu^*(\Omega)\hat{\mathcal{E}}_I^{\text{in}}(\Omega)+\nu^*(\Omega)\hat{\mathcal{E}}_S^{\text{in}\dagger}(\Omega), \tag{54}$$

[8] The field operators $\hat{E}_m^{\text{in}}(t)$ for $m = S, I$ have the usual δ-function commutator with their adjoints, $[\hat{E}_m^{\text{in}}(t), \hat{E}_m^{\text{in}\dagger}(u)] = \delta(t-u)$ for $m = S, I$, while the intracavity annihilation operators $\hat{a}_m(t)$ for $m = S, I$ have the canonical commutation relation, $[\hat{a}_m(t), \hat{a}_m^\dagger(t)] = 1$ for $m = S, I$, with their adjoints.

[9] Here, $G^2 = P_P/P_T$, where P_P is the pump power and P_T is the threshold power, i.e., the pump power value for which the OPA breaks into oscillation and becomes an optical parametric oscillator.

[10] Our sign convention for these transforms is $\hat{\mathcal{E}}_S(\Omega) = \int \mathrm{d}t\, \hat{E}_S(t)e^{j\Omega t}$ and $\hat{\mathcal{E}}_I(\Omega) = \int \mathrm{d}t\, \hat{E}_I(t)e^{-j\Omega t}$.

where

$$\mu(\Omega) \equiv \frac{1+G^2+\Omega^2/\Gamma^2}{1-G^2-\Omega^2/\Gamma^2-2j\Omega/\Gamma} \tag{55}$$

$$\nu(\Omega) \equiv \frac{2G}{1-G^2-\Omega^2/\Gamma^2-2j\Omega/\Gamma}. \tag{56}$$

It easily shown that $|\mu(\Omega)|^2 - |\nu(\Omega)|^2 = 1$ and that Eqs. (53) and (54) give rise to the proper commutator brackets. More importantly, Eqs.(53) and (54) are linear and their driving terms are vacuum-state field operators. It follows that $\hat{E}_S^{\text{out}}(t)$ and $\hat{E}_I^{\text{out}}(t)$ will be in a zero-mean jointly Gaussian state. Paralleling the approach used to find the correlation functions for spontaneous parametric downconversion, we can show that this jointly Gaussian state is completely characterized by the following spectral densities and stationary correlation functions:

$$\mathcal{S}_{mm}^{(n)}(\Omega) = \int \mathrm{d}\tau\, K_{mm}^{(n)}(\tau)e^{-j\Omega\tau} = |\nu(\Omega)|^2 \tag{57}$$

$$= \frac{4G^2}{(1-G^2-\Omega^2/\Gamma^2)^2+4\Omega^2/\Gamma^2}, \quad \text{for } m = S, I, \tag{58}$$

$$\mathcal{S}_{SI}^{(p)}(\Omega) = \int \mathrm{d}\tau\, K_{SI}^{(p)}(\tau)e^{-j\Omega\tau} = \mu^*(\Omega)\nu(\Omega) \tag{59}$$

$$= \frac{2G(1+G^2+\Omega^2/\Gamma^2)}{(1-G^2-\Omega^2/\Gamma^2)^2+4\Omega^2/\Gamma^2}, \tag{60}$$

and

$$K_{mm}^{(n)}(\tau) = \langle \hat{E}_m^{\text{out}\dagger}(t+\tau)\hat{E}_m^{\text{out}}(t)\rangle = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} - \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], \text{ for } m = S, I, \tag{61}$$

$$K_{SI}^{(p)}(\tau) = \langle \hat{E}_S^{\text{out}}(t+\tau)\hat{E}_I^{\text{out}}(t)\rangle = \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} + \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right]. \tag{62}$$

In the next section, we will show how the preceding spectra lead to quadrature-noise squeezing.

Quadrature-Noise Squeezing from an OPA

From our previous work on two-mode parametric amplifiers, we expect that the $\pm 45°$ polarizations at the output of our continuous-time OPA should exhibit quadrature-noise squeezing. Let's show that this is so for the $+45°$ case. The baseband field operator for this polarization is

$$\hat{E}_{+45}^{\text{out}}(t) \equiv \frac{\hat{E}_S^{\text{out}}(t)+\hat{E}_I^{\text{out}}(t)}{\sqrt{2}}. \tag{63}$$

This field operator is in a zero-mean Gaussian state whose phase-insensitive and phase-sensitive correlation functions are

$$\begin{aligned} K^{(n)}(\tau) &\equiv \langle \hat{E}^{\text{out}\dagger}_{+45}(t+\tau)\hat{E}^{\text{out}}_{+45}(t)\rangle = \frac{K^{(n)}_{SS}(\tau)+K^{(n)}_{II}(\tau)}{2} && (64)\\ &= \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} - \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], && (65) \end{aligned}$$

and

$$\begin{aligned} K^{(p)}(\tau) &\equiv \langle \hat{E}^{\text{out}}_{+45}(t+\tau)\hat{E}^{\text{out}}_{+45}(t)\rangle = \frac{K^{(p)}_{SI}(\tau)+K^{(p)}_{SI}(-\tau)}{2} && (66)\\ &= \frac{G\Gamma}{2}\left[\frac{e^{-(1-G)\Gamma|\tau|}}{1-G} + \frac{e^{-(1+G)\Gamma|\tau|}}{1+G}\right], && (67) \end{aligned}$$

respectively. The spectral densities associated with these correlation functions are

$$\begin{aligned} \mathcal{S}^{(n)}(\Omega) &\equiv \int \mathrm{d}\tau\, K^{(n)}(\tau)e^{-j\Omega\tau} = |\nu(\Omega)|^2 && (68)\\ \mathcal{S}^{(p)}(\Omega) &\equiv \int \mathrm{d}\tau\, K^{(p)}(\tau)e^{-j\Omega\tau} = \mu^*(\Omega)\nu(\Omega). && (69) \end{aligned}$$

Now, consider the balanced homodyne measurement system—shown on slide 11–for detecting the θ-quadrature of $\hat{E}^{\text{out}}_{+45}(t)$. Here we have assumed unity quantum efficiency photodetectors, and omitted the low-pass filter. From our continuous-time theory of homodyne detection we know that the photocurrent difference $\Delta i(t)$ has statistics that are equivalent to those of the operator

$$\Delta\hat{i}(t) = 2q\sqrt{\frac{P_{\text{LO}}}{\hbar\omega_P/2}}\,\text{Re}[\hat{E}^{\text{out}}_{+45}(t)e^{-j\theta}]. \qquad (70)$$

Because $\hat{E}^{\text{out}}_{+45}(t)$ is in a zero-mean, statistically-stationary Gaussian state, the homodyne measurement will yield a zero-mean, stationary Gaussian random process whose covariance function is

$$\begin{aligned} K_{\Delta i\Delta i}(\tau) &\equiv \langle \Delta i(t+\tau)\Delta i(t)\rangle && (71)\\ &= q^2\frac{P_{\text{LO}}}{\hbar\omega_P/2}\{\delta(\tau)+K^{(n)}(\tau)+K^{(n)}(-\tau)+2\text{Re}[K^{(p)}(\tau)e^{-2j\theta}]\}. && (72) \end{aligned}$$

The photocurrent-noise spectral density that will be observed using a spectrum analyzer at the homodyne system's output is thus

$$
\begin{aligned}
\mathcal{S}_{\Delta i \Delta i}(\Omega) &= \int \mathrm{d}\tau\, K_{\Delta i \Delta i}(\tau) e^{-j\Omega\tau} && (73)\\
&= q^2 \frac{P_{\mathrm{LO}}}{\hbar\omega_P/2}[1 + 2|\nu(\Omega)|^2 + 2\mathrm{Re}(\mu^*(\Omega)\nu(\Omega)e^{-2j\theta})] && (74)\\
&= q^2 \frac{P_{\mathrm{LO}}}{\hbar\omega_P/2}|\mu(\Omega) + \nu(\Omega)e^{-2j\theta}|^2. && (75)
\end{aligned}
$$

Were $\hat{E}^{\mathrm{out}}_{+45}(t)$ in a coherent state, this homodyne receiver's photocurrent-noise spectral density would be

$$
\mathcal{S}_{\Delta i \Delta i}(\Omega)|_{\mathrm{CS}} = q^2 \frac{P_{\mathrm{LO}}}{\hbar\omega_P/2}, \tag{76}
$$

representing the shot-noise limit of semiclassical theory. The normalized photocurrent-noise spectral density,

$$
\frac{\mathcal{S}_{\Delta i \Delta i}(\Omega)}{\mathcal{S}_{\Delta i \Delta i}(\Omega)|_{\mathrm{CS}}} = |\mu(\Omega) + \nu(\Omega)e^{-2j\theta}|^2, \tag{77}
$$

contains contains phase-sensitive noise that, as shown in the left panel on slide 12, goes well below the shot-noise level at $\theta = \pm\pi/2$ for $\Omega = 0$. As shown in the right panel on slide 12, the strongest quadrature-noise squeezing is limited to frequencies below the cavity linewidth.

The Road Ahead

In the next lecture we shall use the results developed today for SPDC and the OPA to study additional signatures of nonclassical light that can be obtained from these nonlinear optical systems. Of particular interest will be Hong-Ou-Mandel interferometry, as it relates to the important notion of distinguishability. We will also connect our treatment of SPDC with the concept of a biphoton.