Photonic Quantum Information Processing
OPTI 647: Lecture 20

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Announcements:

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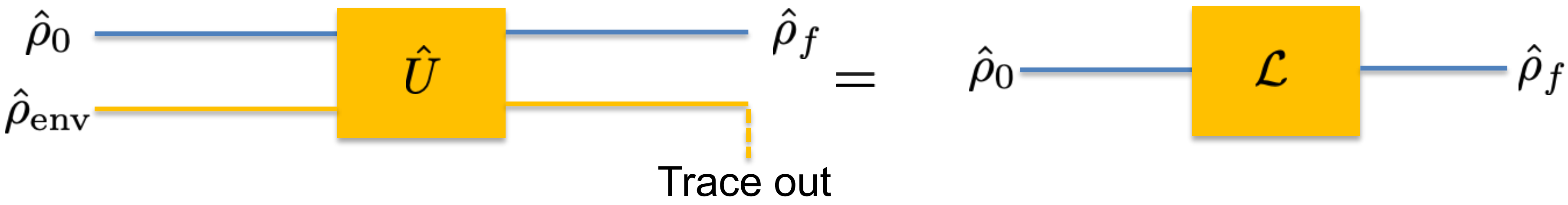
1. The grace period is over: Problem Sets must be delivered on time to be graded.
2. Hand in the last delayed Problem sets by Monday 4th of November at noon.
3. Hand in today's Problem Set by Monday 4th of November at noon.
4. Extra credit for spotting typos!
5. Start considering which advanced problem you want to solve. However, there will be a few more.

Correction: Where, $\hat{A}_m = \langle m|\hat{U}|0\rangle$ are the Kraus operators.

Recap: The concept of single mode a quantum channel



An input state interacts with an environment state via a unitary operation. Then the degrees of freedom of the output are traced out and we get the final state.

The whole procedure can be written as: $\hat{\rho}_f = \mathcal{L}(\hat{\rho}_0)$

Map $\mathcal{L}(.)$ must be:

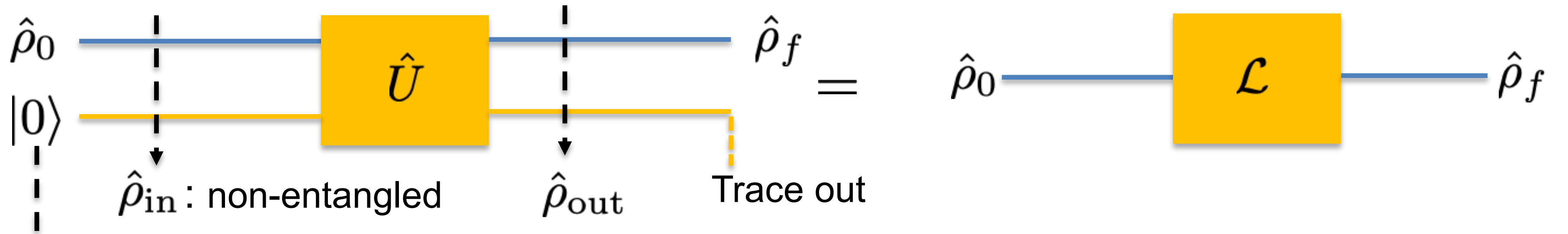
1. Trace preserving.
2. Completely positive.

In short CPTP map.

Trace preserving: So that the output is a valid density matrix (necessary).
Completely positive: So that it maps positive semidefinite operators of any number of modes to semidefinite positive operators (necessary).

CPTP maps, transform any valid density operator (state) into some other valid density operator (state).

Recap: Kraus Operators



We can always assume that the environment is set to vacuum (to be proven later).

Eigensystem of $\hat{\rho}_0$: $\{|k\rangle, \lambda_k\}$ → Orthonormal and complete basis, consisting of vectors $|k\rangle$ (not necessarily Fock states). Positive eigenvalues λ_k.

$$\hat{\rho}_{\rm in} = \sum_k \lambda_k |k0\rangle\langle k0|$$

$$\hat{\rho}_{\rm out} = \sum_k \lambda_k \hat{U}|k0\rangle\langle k0|\hat{U}^\dagger = \sum_{k,n,m,n',m'} \lambda_k \langle nm|\hat{U}|k0\rangle\langle k0|\hat{U}^\dagger|n'm'\rangle |nm\rangle\langle n'm'|$$

We plug in the **identity operator** twice (left and right of the unitary operator). The identity operator can be written in any basis (we used the eigenvectors of $\hat{\rho}_0$).

Recap: Kraus Operators (continued)

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$$\hat{\rho}_{\text{out}} = \sum_k \lambda_k \hat{U}|k0\rangle\langle k0|\hat{U}^\dagger = \sum_{k,n,m,n',m'} \lambda_k \langle nm|\hat{U}|k0\rangle\langle k0|\hat{U}^\dagger|n'm'\rangle|nm\rangle\langle n'm'|$$

Trace out lower output mode: $\hat{\rho}_f = \sum_{k,n,m,n'} \lambda_k \langle nm|\hat{U}|k0\rangle\langle k0|\hat{U}^\dagger|n'm\rangle|n\rangle\langle n'|$

Last expression is rewritten: $\hat{\rho}_f = \sum_m \hat{A}_m \hat{\rho}_0 \hat{A}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$

Where, $\hat{A}_m = \langle m|\hat{U}|0\rangle$ are the Kraus operators.
Kraus operators are "partial projections" of the unitary operators on an orthonormal and complete basis. They are **not** necessarily unitary themselves.

To have a valid CPTP, the Kraus operators must satisfy: $\sum_m \hat{A}_m^\dagger \hat{A}_m = \hat{I}$

where $\hat{I}$ is the identity operator.

Recap: The pure loss channel

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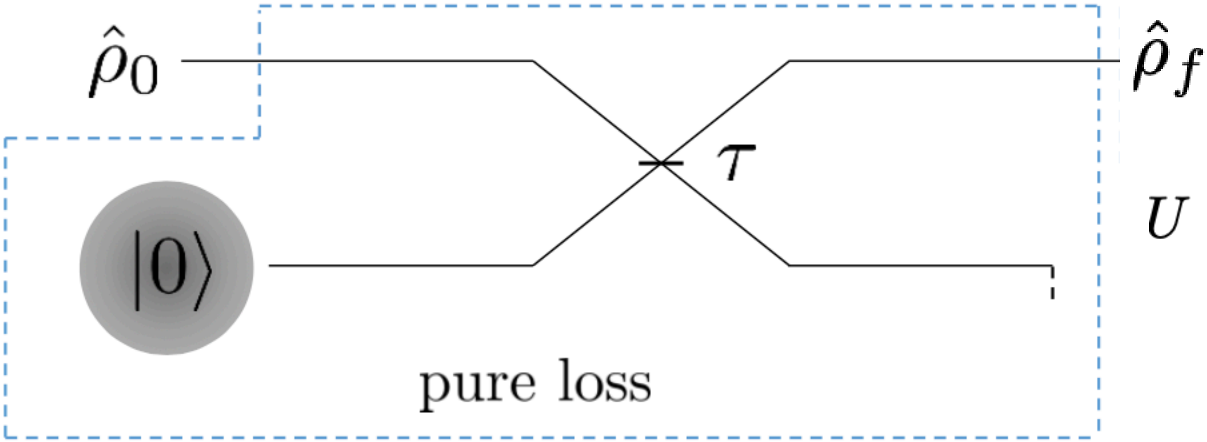

$$U = \begin{pmatrix} \sqrt{\tau} & \sqrt{1-\tau} \\ -\sqrt{1-\tau} & \sqrt{\tau} \end{pmatrix}$$

$$\hat{A}_l = \sqrt{\frac{(1-\tau)^l}{l!}}\tau^{\frac{\hat{n}}{2}}\hat{a}^l \qquad \hat{A}_l^\dagger = \sqrt{\frac{(1-\tau)^l}{l!}}\hat{a}^{\dagger l}\tau^{\frac{\hat{n}}{2}}$$

Choose a basis to represent your unitary, do partial projection. In the same basis express the Krauss operators. See if you get the same result.

Today's plan:

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1. Symmetric logarithmic derivatives.
2. More examples on single mode QFI.
3. Multiple parameters QFI.
4. Attainability of the QFI.
5. Upper bounding the QFI.

Single parameter, unbiased estimator, Cramér-Rao bound

$$\langle(\theta-\hat{\theta})^2\rangle \geq \frac{1}{\mathbb{E}\left[\left(\frac{\partial \ln L(x;\theta)}{\partial\theta}\right)^2\right]}$$

$$\langle(\theta-\hat{\theta})^2\rangle \geq \frac{1}{\mathbb{E}\left[-\frac{\partial^2 \ln L(x;\theta)}{\partial\theta^2}\right]}$$

$$\langle(\theta-\hat{\theta})^2\rangle \geq I^{-1}(\theta)$$

Fisher information (FI): $I(\theta) = \mathbb{E}\left[-\frac{\partial^2 \ln L(x;\theta)}{\partial\theta^2}\right]$

For n independent measurements: $\langle(\theta-\hat{\theta}_n)^2\rangle \geq \frac{1}{n\mathbb{E}\left[-\frac{\partial^2 \ln L(x;\theta)}{\partial\theta^2}\right]}$

$$\max_{\{\hat{\theta}\}} I(\theta) = H(\theta)$$

The QCRB is always attainable for the single parameter case

$$H(\theta) = -2\frac{\partial^2 F(\hat{\rho}(\theta),\hat{\rho}(\theta+\epsilon))}{\partial\epsilon^2}\bigg|_{\epsilon=0}$$

CRB is a lower bound on the performance of estimators:

$$\left\langle(\theta-\hat{\theta})^2\right\rangle \geq I^{-1}(\theta) \geq H^{-1}(\theta)$$

Classical bound (CCRB) ≥ Quantum bound (QCRB)

Single parameter example

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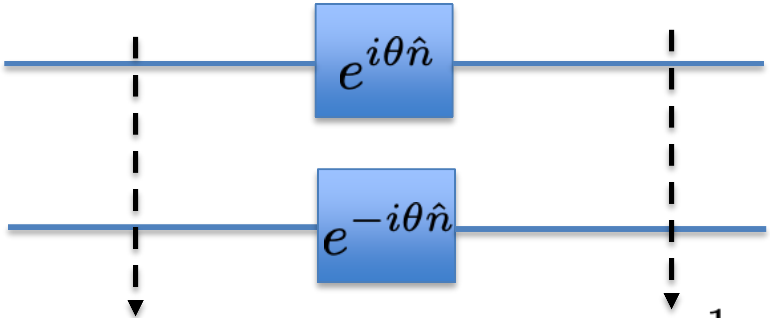

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|n0\rangle + |0n\rangle\right) \qquad |\Psi_\theta\rangle = \frac{1}{\sqrt{2}}\left(e^{i\theta}|n0\rangle + e^{-i\theta}|0n\rangle\right)$$

NOON state (entangled)

$$F(\epsilon) = |\langle\Psi_\theta|\Psi_{\theta+\epsilon}\rangle|^2 = \cdots = 2\cos^2(n\epsilon)$$

$$H = -2\frac{\partial^2 F(\epsilon)}{\partial\epsilon^2}\bigg|_{\epsilon=0} = \cdots = 8n^2 \sim E^2$$

The QFI behaves like energy^2, therefore one expects better estimation performance. Heisenberg limit

Problem 78: Calculate the fidelity and the QFI.

Symmetric logarithmic derivatives (single parameters)

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Let a classical distribution $P = \{p^j\},\ j = 1, 2, \ldots$

What is the distance between the distribution P and P+dP?

$$P + dP = \{p^j + dp^j\},\ j = 1, 2, \ldots$$

We need the notion of metric: $ds^2 = \sum_{jk} g_{jk} dp^i dp^k$

Let two classical random variables A and B. Then their correlation (or mean value):

$$\langle AB \rangle = \sum_{jk} A_j B_k g^{jk}$$

$$\langle AB \rangle = \sum_{j} A_j B_j p^j$$

$$g^{jk} = \frac{\delta^j{}_k}{p^j}$$

$$ds^2 = \sum_{jk} \frac{\delta^j{}_k}{p^j} dp^i dp^k = \sum_j \frac{(dp^j)^2}{p^j} = \sum_j p_j (d \ln p_j)^2$$

We take the derivative wrt to the parameter θ:

$$\left(\frac{ds}{d\theta}\right)^2 = \sum_j p_j \left(\frac{d \ln p_j}{d\theta}\right)^2$$

You've seen this formula before

$$I(\theta) = \mathbb{E}\left[\left(\frac{\partial \ln L(x;\theta)}{\partial\theta}\right)^2\right]$$

$$\left(\frac{ds}{d\theta}\right)^2 = \sum_j p_j \left(\frac{d \ln p_j}{d\theta}\right)^2 = I(\theta)$$

Symmetric logarithmic derivatives (single parameters), continued

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$$\left(\frac{ds}{d\theta}\right)^2 = \sum_j p_j \left(\frac{d\ln p_j}{d\theta}\right)^2 = I(\theta) \quad \xrightarrow{?} \quad \text{QFI } H(\theta)$$

In quantum mechanics, probability distributions are upgraded to density operators. But what happens to its derivatives?

$$H(\theta) = \mathrm{tr}\left(\hat{\rho}_\theta \hat{L}^2\right)$$

$\hat{L}$: The operator which represents the derivative

$\hat{\rho}_\theta$: Final state (with the unknown parameter imprinted on it)

If $\frac{\partial \hat{\rho}_\theta}{\partial \theta} = \hat{L}\hat{\rho}_\theta$ (LRD), then we go back to a classical-like case: $\frac{\partial \hat{\rho}_\theta}{\partial \theta} = \hat{L}\rho_\theta \Rightarrow \hat{L} = \frac{\partial \ln \hat{\rho}_\theta}{\partial \theta}$

The QFI is given by the symmetric logarithmic derivatives (SLD):

$$\frac{\partial \hat{\rho}_\theta}{\partial \theta} = \frac{1}{2}\left(\hat{L}\hat{\rho}_\theta + \hat{\rho}_\theta \hat{L}\right)$$ Lyapunov equation, which we solve for $\hat{L}$

Problem 79: Prove that the SLD is Hermitian

$$\hat{L} = \hat{L}^\dagger$$

Multiple parameters

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Let us have a multiple unknown parameters to be estimated simultaneously:

$$\vec{\theta} = (\theta_1, \ldots, \theta_N)$$

For multiple parameters the generalization is straightforward.

via Fidelity: $H_{ij} = -2 \frac{\partial^2 F(\vec{\epsilon})}{\partial \epsilon_i \partial \epsilon_j}\Big|_{\vec{\epsilon}=\vec{0}}$ $\quad \vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)$

via SLD's: $H_{ij} = \frac{1}{2} \mathrm{tr}\left[\hat{\rho}_{\vec{\theta}} \left(\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i\right)\right]$

One SLD for each parameter:

$$\frac{\partial \hat{\rho}_{\vec{\theta}}}{\partial \theta_i} = \frac{1}{2}\left(\hat{L}_i \hat{\rho}_{\vec{\theta}} + \hat{\rho}_{\vec{\theta}} \hat{L}_i\right)$$

From numbers (single parameter), we go to matrices (multiple parameters). The CRB now is a matrix inequality in the positive semidefinite sense:

$$\mathrm{Cov}(\vec{\theta}) \geq I^{-1} \geq H^{-1}$$

Where all Fisher matrices are positive semidefinite (eigenvalues non-negative) and symmetric, just like the covariance matrices. The off-diagonal elements represent correlated error.

The QFI is attainable when: $[\hat{L}_i, \hat{L}_j] = 0$ (one shot), and $\mathrm{tr}\left(\hat{\rho}_{\vec{\theta}} [\hat{L}_i, \hat{L}_j]\right) = 0$ (asymptotic limit of many measurements). In that case **the measurement (POVM)** is given by **the eigenvectors of the SLD's**.

For single parameter problems: $[\hat{L}, \hat{L}] = 0$. The single parameter QFI is always attainable.

Multiple Parameters Attainability (continued):

CFI: $I_{ij} = \mathbb{E}\left[-\frac{\partial^2 \ln P(\vec{x};\vec{\theta})}{\partial\theta_i \partial\theta_j}\right] = \mathbb{E}\left[\frac{\partial \ln P(\vec{x};\vec{\theta})}{\partial\theta_i}\frac{\partial \ln P(\vec{x};\vec{\theta})}{\partial\theta_j}\right]$

QFI: $H_{ij} = \frac{1}{2}\mathrm{tr}\left[\hat{\rho}_{\vec{\theta}}\left(\hat{L}_i\hat{L}_j + \hat{L}_j\hat{L}_i\right)\right]$

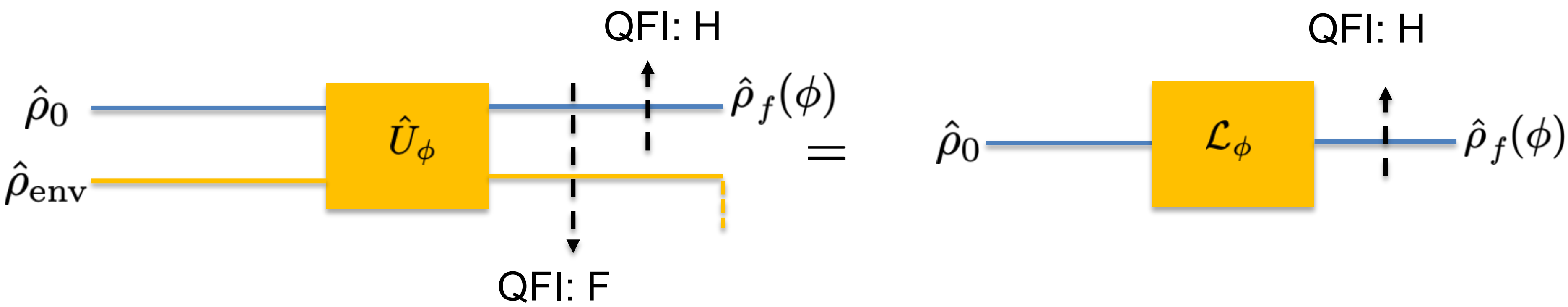
These are metrics on the parametric space of probability distributions (or density matrices) parametrized by $\vec{\theta}$:

$$ds^2_{\mathrm{cl}} \equiv d(P, P + dP)^2 = \sum_{ij} I_{ij} d\theta^i d\theta^j$$

$$ds^2_{\mathrm{q}} \equiv d(\hat{\rho}_{\vec{\theta}}, \hat{\rho}_{\vec{\theta}} + d\hat{\rho}_{\vec{\theta}})^2 = \sum_{ij} H_{ij} d\theta^i d\theta^j$$

If the SLD's commute then the evolution of each parameter becomes indipedent and therefore the problem becomes classical-like.

Upper Bound on the QFI



$F \geq H$ because F accounts for information lost in the environment. Specifically, for single parameter estimation the QFI is the CFI optimized over all measurements. Between F and H, F corresponds to a maximization over a larger set of measurements since it involves a larger Hilbert space. Therefore, a larger maximum might exist (worst case the F=H).

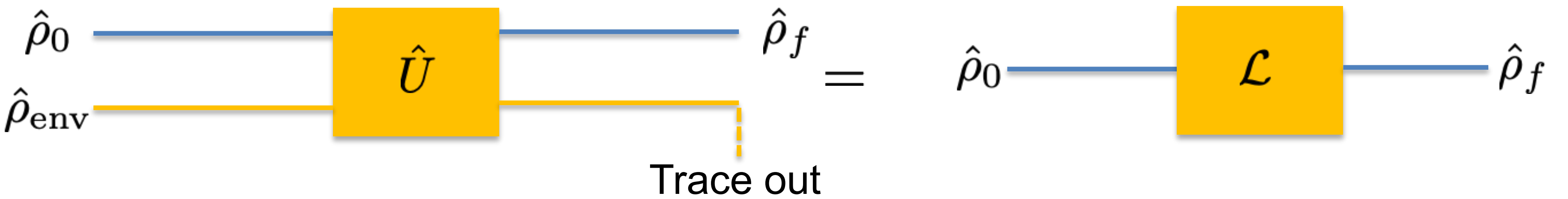
Article | Published: 27 March 2011

General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology

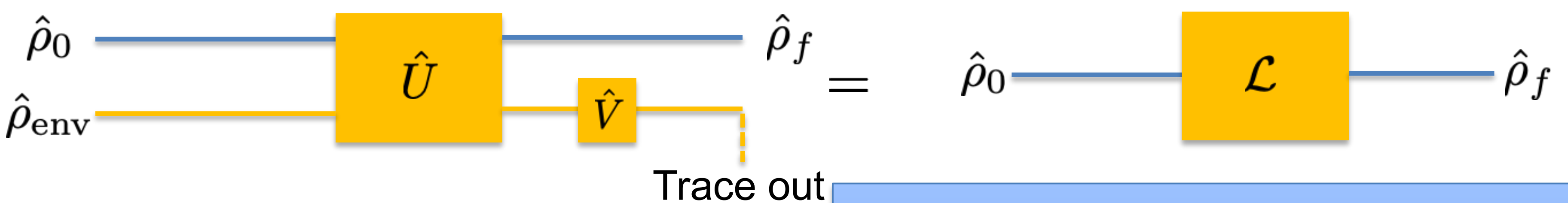
B. M. Escher, R. L. de Matos Filho & L. Davidovich

Nature Physics **7**, 406–411 (2011) | Cite this article

Upper Bound on the QFI: Unitary Equivalence of Kraus operators



$$\hat{\rho}_f = \sum_m \hat{A}_m \hat{\rho}_0 \hat{A}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$



$$\hat{\rho}_f = \sum_m \hat{K}_m \hat{\rho}_0 \hat{K}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$

Unitary Equivalence of Kraus operators

$$K_n = \sum_{n,m} V_{mn} A_m$$ Where V_{mn} are the elements of a unitary matrix

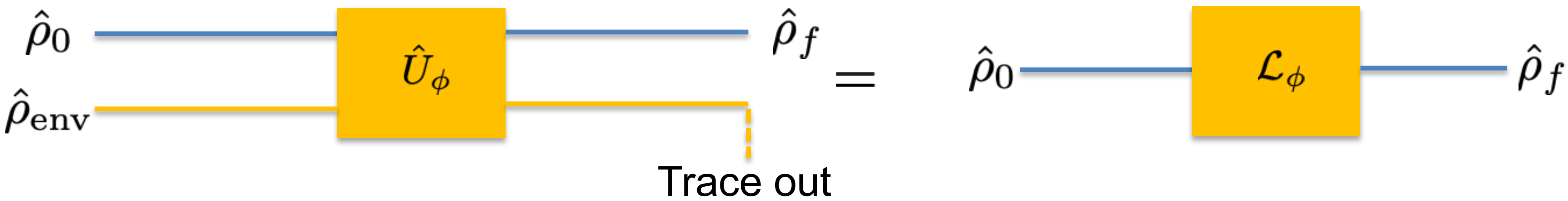
The channel remains the same: We are free to use whichever basis we like for to represent the environment on.

$$\hat{A}_m = \langle m|\hat{U}|l\rangle$$

Arbitrary basis on which the environment is represented.

Problem 80: Prove the unitary equivalence of Kraus operators rigorously.

Upper Bound on the QFI: Unitary Equivalence of Kraus operators (continued)



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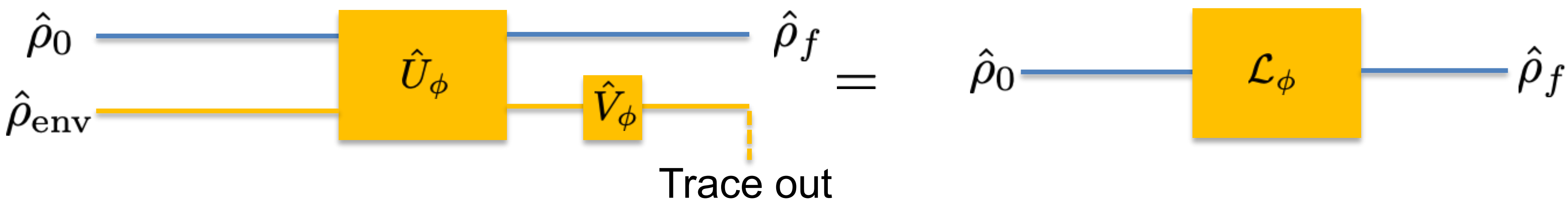

$$\hat{\rho}_f = \sum_m \hat{K}_m \hat{\rho}_0 \hat{K}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$

The unitary V on the environment can have dependence on the same parameters as the unitary U which implements the interaction! Nothing prohibits that.

Upper Bound on the QFI: Unitary Equivalence of Kraus operators (continued)

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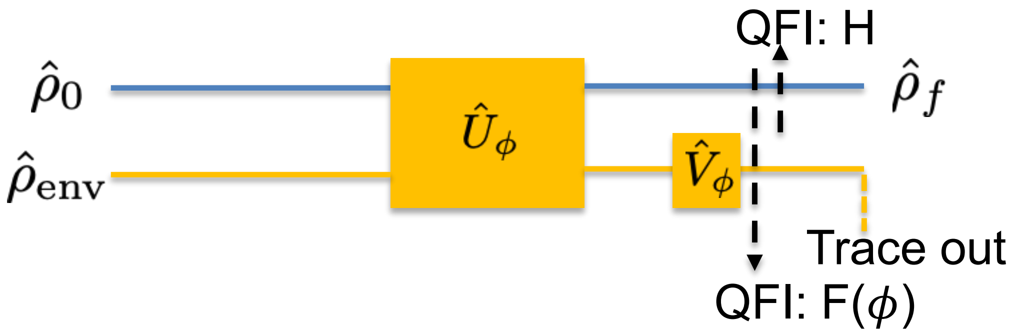

$$\hat{\rho}_f = \sum_m \hat{K}_m \hat{\rho}_0 \hat{K}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$

The Kraus operators will have ϕ dependence.

If we minimize the bound F with respect to V, we find an upper bound to QFI. Actually if we optimize over all possible V's, the bound is attainable, i.e., in that way we can calculate the QFI H. In general this is a difficult task…

The intuition behind it is that V, will "clear up" information on unknown the parameter in the output environment mode, and make clear what is going on on the upper output mode.

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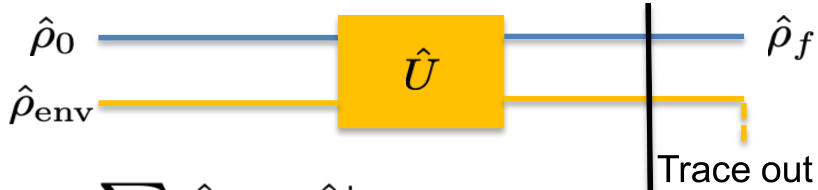
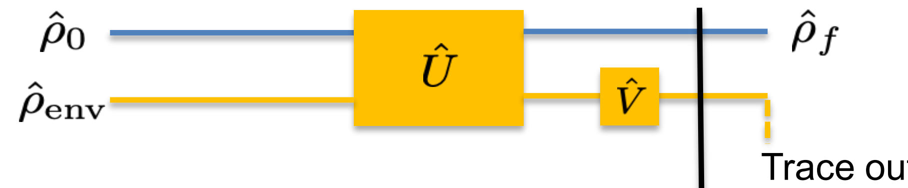


$$\hat{\rho}_f = \sum_m \hat{A}_m \hat{\rho}_0 \hat{A}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$



$$\hat{\rho}_f = \sum_m \hat{K}_m \hat{\rho}_0 \hat{K}_m^\dagger \equiv \mathcal{L}(\hat{\rho}_0)$$

Another way to understand the unitary equivalence of Kraus operators: The purification of a mixed state (in this case $\hat{\rho}_f$), is not unique.

The purification of a state has higher QFI. This is apparent from **Uhlmann's theorem** (one way to define fidelity):

$$F(\hat{\rho}, \hat{\sigma}) = \max_{|\psi_{\hat{\sigma}}\rangle} |\langle \psi_{\hat{\rho}} | \psi_{\hat{\sigma}} \rangle|^2$$

Maximization over all possible purifications.

Problem 81: Argue we the inequality $F(\hat{\rho}, \hat{\sigma}) \geq |\langle \psi_{\hat{\rho}} | \psi_{\hat{\sigma}} \rangle|^2$ is valid. Then prove that the purification of a state has greater QFI than the initial mixed state (consider single parameter estimation). You should use the fidelity based definition of the QFI.

Advanced Problem 12: Tight bound for thermal loss channel

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Study the papers:

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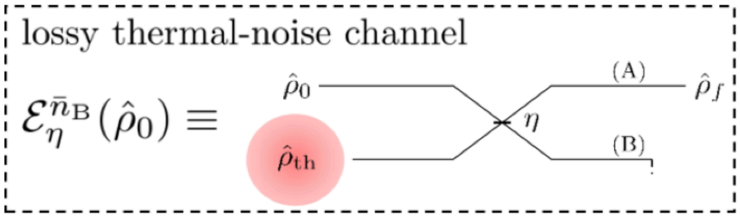
Bounding the quantum limits of precision for phase estimation with loss and thermal noise

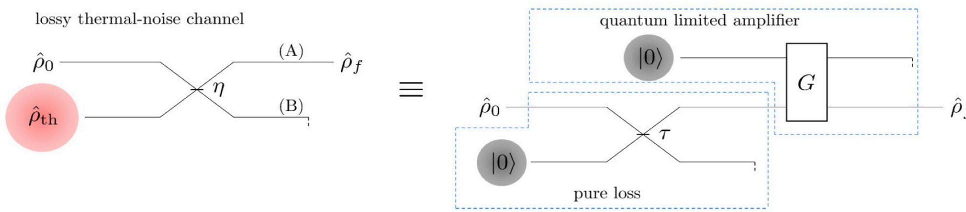
Christos N. Gagatsos, Boulat A. Bash, Saikat Guha, and Animesh Datta

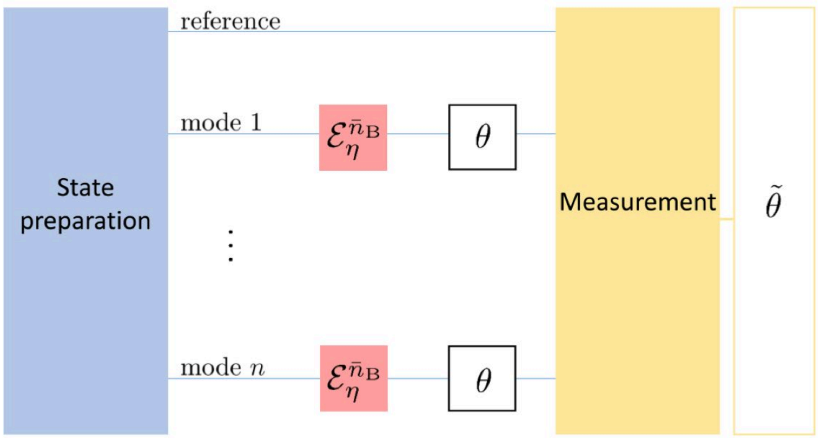
Phys. Rev. A **96**, 062306 – Published 4 December 2017

For **single** phase estimation only under the thermal loss channel, find a better bound than mine! (difficult problem, but probably publishable.)

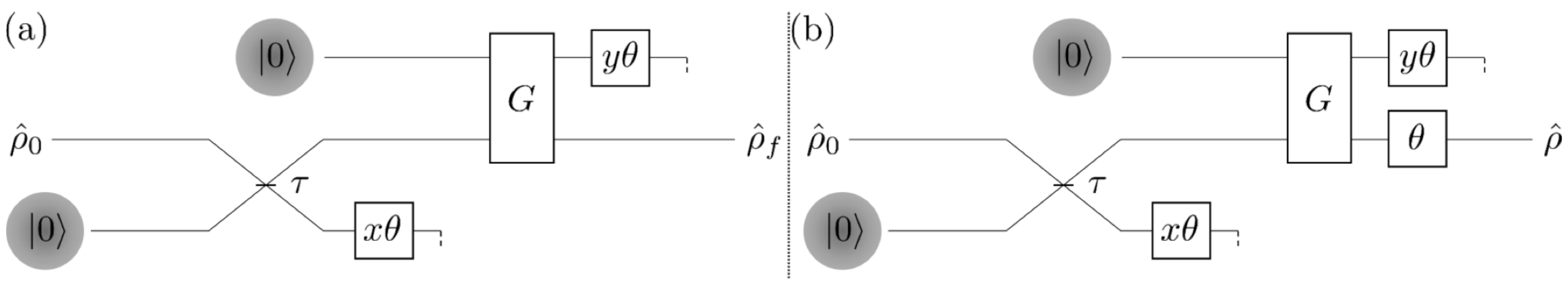
QFI upper bound for phase estimation over a thermal loss channel







From intuition I use phase unitaries to “clear up” information from the both the environments (because the unknown parameter is phase). I optimize over (x, y) and I get a nice bound, which is not the best possible.



General comments on the QFI approach

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In general the strategy is as follows:

i. We're given a sensing task of multiple parameters.
ii. Calculate the QFI.
iii. See if the QFI is attainable.
iv. If a POVM which attains the QFI exists, find it.

The "most reasonable thing" would be to optimize the CFI over all measurements: Extremely difficult, ergo the approach above.
Even worse, nothing guarantees that even if a POVM which attains the QFI exists, that this POVM will be a reasonable (implementable by usual techiques) one. In many, cases we calculate the CFI for measurements *we can* do, and then compare them to the QFI.

Next lectures: Further topics on quantum estimation theory

1. More on SLD's.

2. If there's interest: More on derivations (let me know, or come and find me to discuss more math).

3. More useful QFI formulas for special cases:
   i. Pure states and unitary dynamics (SLD).
   ii. Gaussian states (Fidelity).

4. Examples/applications.