Photonic Quantum Information Processing
OPTI 647: Lecture 15

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Recap (partial) of previous lectures

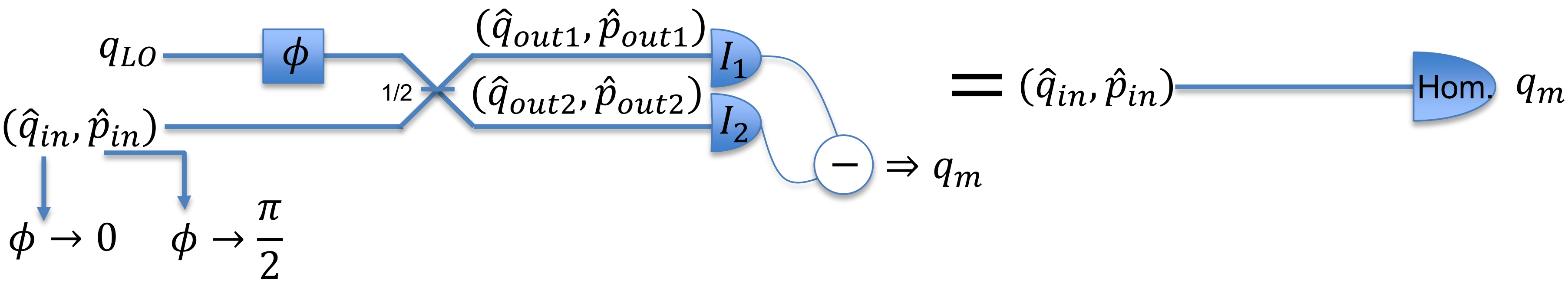
1. Description of a Gaussian state on phase space (covariance matrix, first moments).
2. Covariance matrix and first moments transformations.
3. First impact with the CV teleportation.

Outline of Lecture 15

1. Gaussian measurements (Homodyne/Heterodyne).
2. Revisiting CV teleportation.
3. Von Neumann entropy of Gaussian states.
4. Introduction to fidelity.

$$\hbar = 1: \;\; \hat{a} = \frac{\hat{q}_a + i\hat{p}_a}{\sqrt{2}}, \;\; \hat{a}^\dagger = \frac{\hat{q}_a - i\hat{p}_a}{\sqrt{2}}$$

Homodyne detection



$$\hat{q}_{\mathrm{out}_1} = (\hat{q}_{\mathrm{in}} + q_{\mathrm{LO}})/\sqrt{2},$$
$$\hat{p}_{\mathrm{out}_1} = \hat{p}_{\mathrm{in}}/\sqrt{2},$$
$$\hat{q}_{\mathrm{out}_2} = (\hat{q}_{\mathrm{in}} - q_{\mathrm{LO}})/\sqrt{2},$$
$$\hat{p}_{\mathrm{out}_2} = \hat{p}_{\mathrm{in}}/\sqrt{2}.$$

Operator transformations in the Heisenberg picture. q_{LO} is a classical field ($\sim 10^9$ photons), therefore $p_{LO} \approx 0$.

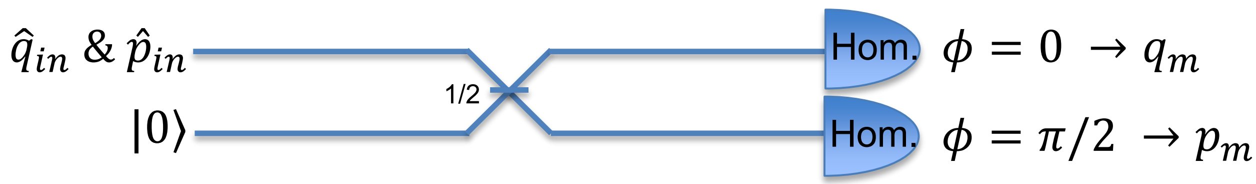
$$I_1 - I_2 = \frac{1}{2}\left(\langle \hat{q}_{out1}^2 + \hat{p}_{out1}^2 \rangle - \langle \hat{q}_{out2}^2 + \hat{p}_{out2}^2 \rangle\right) = q_{LO}\langle \hat{q}_{in} \rangle$$
$$= q_{LO}\, \langle \hat{q}_{in} \rangle = c\; q_{LO} q_m$$

Problem 62: Verify this equation.

c: constant. It has to do with post-processing. We'll take it to be $c = 1$.
q_{LO}: just a known number since it's locally defined, classical field (not operator).

Problem 63: Find $I_1 - I_2$ for $\phi = \pi/2$.

Heterodyne detection (Dual homodyne)



Heterodyne and dual homodyne give the same statistics.

Heterodyne measurement: Using a balanced beam splitter, the field is split into two beams. Then a homodyne measurement is performed on each output mode. Vacuum in the lower input port is inevitable, therefore the cost of measuring simultaneously position and momentum is added noise.

Recall:

- The outcome of (many) homodyne measurements is the Wigner function.
- The outcome of (many) heterodyne measurements is the Q function.

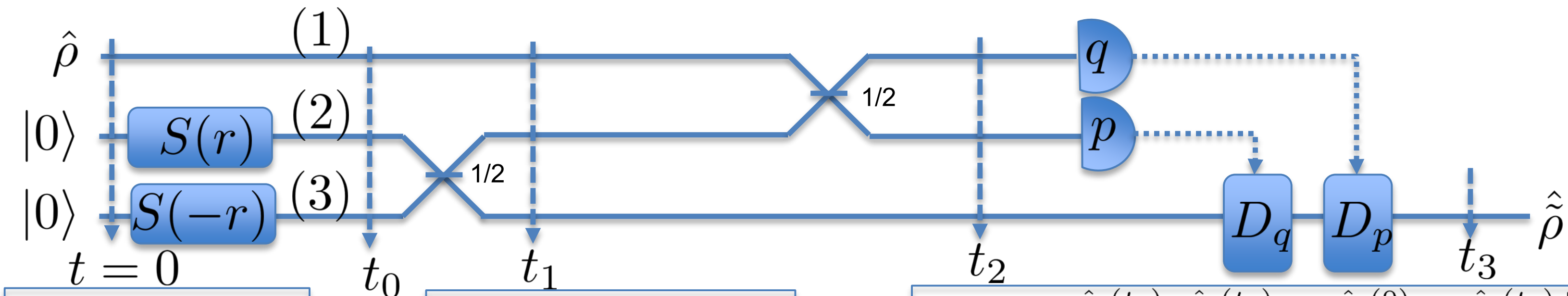
Therefore, heuristically, if the CM of the detected field is V, the correlation matrix in the Q function will be $\Gamma = V + \frac{I}{2}$ (for Gaussian states only), the identity matrix accounts for added noise.

$$W(\vec{x}) = \frac{1}{(2\pi)^N \sqrt{\det V}} \exp\left[-\frac{1}{2} \vec{x}^T V^{-1} \vec{x} \right]$$

$$\vec{x} = (\vec{q} - \vec{d}_q \;\; \vec{p} - \vec{d}_p)$$

$$Q(\vec{x}) = \frac{1}{(2\pi)^N \sqrt{\det \Gamma}} \exp\left[-\tfrac{1}{2} \vec{x}^T \left(V + \tfrac{I}{2}\right)^{-1} \vec{x} \right]$$

CV teleportation



$$\hat{q}_1(t_0) = \hat{q}_1(0)$$
$$\hat{q}_2(t_0) = \hat{q}_2(0)e^{r}$$
$$\hat{q}_3(t_0) = \hat{q}_3(0)e^{-r}$$

BS1

$$\hat{q}_1(t_1) = \hat{q}_1(t_0)$$
$$\hat{q}_2(t_1) = \frac{\hat{q}_2(t_0)+\hat{q}_3(t_0)}{\sqrt{2}}$$
$$\hat{q}_3(t_1) = \frac{\hat{q}_2(t_0)-\hat{q}_3(t_0)}{\sqrt{2}}$$

BS2

$$\hat{q}_1(t_2) = \frac{\hat{q}_1(t_1)-\hat{q}_2(t_1)}{\sqrt{2}} = \frac{\hat{q}_1(0)}{\sqrt{2}} - \frac{\hat{q}_2(t_0)+\hat{q}_3(t_0)}{2}$$
$$\hat{q}_2(t_2) = \frac{\hat{q}_1(t_1)+\hat{q}_2(t_1)}{\sqrt{2}} = \frac{\hat{q}_1(0)}{\sqrt{2}} + \frac{\hat{q}_2(t_0)+\hat{q}_3(t_0)}{2}$$
$$\hat{q}_3(t_2) = \hat{q}_1(0) - \sqrt{2}\hat{q}_1(t_2) - \sqrt{2}\hat{q}_3(0)e^{-r}$$

$$\hat{q}_3(t_2) = \hat{q}_1(0) - \sqrt{2}\hat{q}_1(t_2) - \sqrt{2}\hat{q}_3(0)e^{-r}$$
$$\hat{p}_3(t_2) = \hat{p}_1(0) - \sqrt{2}\hat{p}_2(t_2) + \sqrt{2}\hat{p}_3(0)e^{-r}$$

Hom. Mes.

$$\hat{q}_3(t_2) = \hat{q}_1(0) - q - \sqrt{2}\hat{q}_3(0)e^{-r}$$
$$\hat{p}_3(t_2) = \hat{p}_1(0) - p + \sqrt{2}\hat{p}_3(0)e^{-r}$$

Displacing

$$\hat{q}_3(t_3) = \hat{q}_1(0) - \sqrt{2}\hat{q}_3(0)e^{-r}$$
$$\hat{p}_3(t_3) = \hat{p}_1(0) + \sqrt{2}\hat{p}_3(0)e^{-r}$$

Fidelity

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Uhlmann's fidelity between two states $\hat{\rho}_1$ and $\hat{\rho}_2$: $F(\hat{\rho}_1, \hat{\rho}_2) = tr\sqrt{\sqrt{\hat{\rho}_1}\hat{\rho}_2\sqrt{\hat{\rho}_1}}$

It is a measure of how "close" two density operators are. However it is not a metric since it doesn't satisfy the triangle inequality.

(Some) properties:

1. It is symmetric: $F(\hat{\rho}_1, \hat{\rho}_2)$=$F(\hat{\rho}_2, \hat{\rho}_1)$. *(also a metric property)*
2. Invariant under unitaries: $F(\hat{\rho}_1, \hat{\rho}_2)$=$F\left(U\hat{\rho}_1U^\dagger, U\hat{\rho}_2U^\dagger\right)$.

Problem 64: Prove property 2. Use the fact that for any unitary operator U and positive operator $A, \sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$.

3. It holds: $0 \leq F(\hat{\rho}_1, \hat{\rho}_2) \leq 1$. *(also a metric property)*
4. $F(\hat{\rho}_1, \hat{\rho}_2) = 1$ if and only if $\hat{\rho}_1 = \hat{\rho}_2$. *(also a metric property)*

Special cases:

1. If $[\hat{\rho}_1, \hat{\rho}_2] = 0$, $\hat{\rho}_1 = \sum_i \lambda_i |i\rangle\langle i|$, $\hat{\rho}_2 = \sum_i \mu_i |i\rangle\langle i|$ then $F(\hat{\rho}_1, \hat{\rho}_2) = \sum_i \sqrt{\lambda_i \mu_i}$.
2. If both states are pure: $F(|\Psi\rangle, |\Phi\rangle) = |\langle\Psi|\Phi\rangle|$.
3. If one of the states is pure: $F(|\Psi\rangle, \hat{\rho}) = \sqrt{\langle\Psi|\hat{\rho}|\Phi\rangle}$.

Fidelity of Gaussian states

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Advanced Problem 5: Part I

Study and present the proof of the main result (Eqs. 14, 15, and 16) in Phys. Rev. Lett. **115**, 260501 and sup. material.

There, they give the formula for the fidelity between any two Gaussian states (pure or mixed).

$$\begin{aligned}
\hat{\rho}_G &= \frac{1}{Z} \exp^{-\frac{1}{2}(\hat{\vec{r}}-\vec{d})^T G (\hat{\vec{r}}-\vec{d})} \\
Z &= \mathrm{tr}\hat{\rho}_G = \det\left(V + \frac{i\Omega}{2}\right)^{1/2} \\
G &= 2i\Omega \, \mathrm{arccoth}(2iV\Omega)
\end{aligned}$$

$\Rightarrow \cdots$

Fidelity in the CV teleportation scheme

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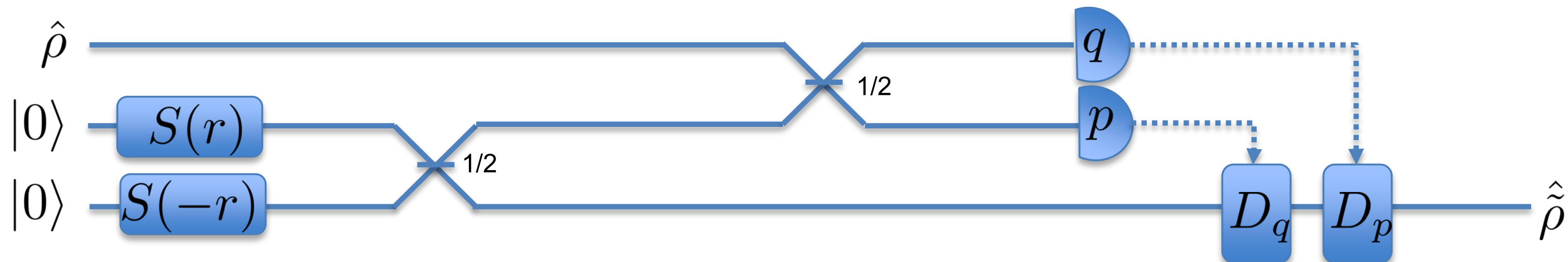

For one mode Gaussian states (V, d) and $(\tilde{V}, \tilde{d})$:

$\hat{\rho}$ $\hat{\tilde{\rho}}$

$$F = F_0 \exp(-\frac{1}{4}\,\delta^T (V + \tilde{V})\delta)$$

$$F_0^2 = \frac{1}{\sqrt{\Delta + \Lambda} - \sqrt{\Lambda}}$$

$$\Delta = \det(V + \tilde{V})$$

$$\Lambda = 4\det\left(V + \frac{i\Omega}{2}\right)\det\left(\tilde{V} + \frac{i\Omega}{2}\right)$$

$$\delta = d - \tilde{d}$$

From slide 6, we can find the $\tilde{V}$ (since there we found $\hat{q}_3(t_3)$ and $\hat{p}_3(t_3)$)

Example: $\hat{\rho} = |\alpha\rangle\langle\alpha|$

$$V = \frac{1}{2}I \qquad \rightarrow \qquad \tilde{V} = \begin{pmatrix} \frac{1}{2} + e^{-4r} + 2e^{-2r} & 0 \\ 0 & \frac{1}{2} + e^{-4r} + 2e^{-2r} \end{pmatrix}$$

$$d = (q_\alpha, p_\alpha) \qquad \tilde{d} = (q_\alpha, p_\alpha)$$

$$F = \frac{1}{2}(1 + \tanh r)$$

$r \rightarrow \infty$: $F = 1$, perfect teleportation.

$r = 0$: $F = 0.5$, teleportation becomes a classical measurement and state preparation scheme with $F = 0.5$. Necessary lower bound for successful teleportation.

Advanced Problem 5: Part II

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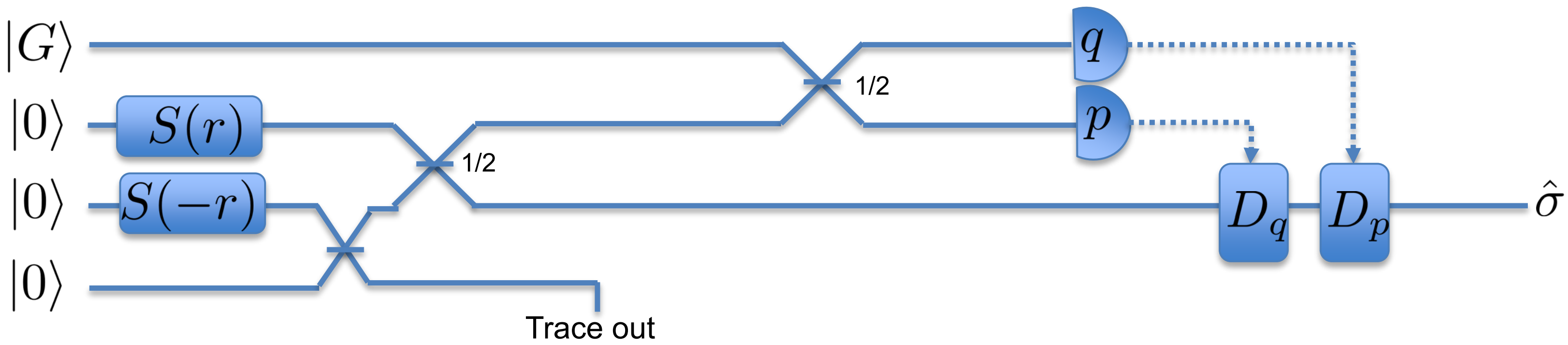

Take $|G\rangle$ to be some pure single mode Gaussian state, with given covariance matrix and zero displacements (1st moments=0). Also consider that both homodyne measurements results give 0 outcome (so that you don't have to perform any displacements). Calculate the fidelity $F(| \ \rangle, \hat{\sigma})$ between the final and initial state.

If you choose Adv. Prob. 5, you must solve both parts I & II

von Neumann entropy

The Shannon entropy measures the uncertainty of a classical probability distribution. In quantum mechanics, the equivalent quantity to a classical distribution is the density operator.

The entropy associated with a (quantum) density operator $\hat{\rho}$:

(2) $S(\hat{\rho}) = -tr(\hat{\rho}\ln\hat{\rho})$ which in general is not trivial to find.

If the eigenvalues of $\hat{\rho}$ are λ_i the von Neumann entropy is:

(3) $S(\hat{\rho}) = -\sum_i \lambda_i \ln\lambda_i$

(we always take $0\ln 0 = 0$)

Problem 65: Go from Eq. (2) to Eq. (3). Can you think of any other measure of uncertainty of mixdness for the density operator?

Properties:

1. *The vN entropy is non-negative. It is zero only for pure states.*
2. *The vN entropy is invariant under unitaries:* $S\left(U\hat{\rho}U^\dagger\right) = S(\hat{\rho})$.
3. $S(\hat{\rho}_1 \otimes \hat{\rho}_2 \otimes \cdots \otimes \hat{\rho}_N) = S(\hat{\rho}_1) + S(\hat{\rho}_2) + \cdots + S(\hat{\rho}_N)$.
4. *The state that maximizes the vN entropy is the* $\rho = I/d$ *(completely mixed state), where* d *is the dimension of the Hilbert state. For that case* $S(\rho) = \ln d$.
5. *For a pure state* $\hat{\rho}_{AB}$, *then* $S(\hat{\rho}_A) = S(\hat{\rho}_B)$. *Where* $\hat{\rho}_A = tr_B(\hat{\rho}_{AB})$ *and* $\hat{\rho}_B = tr_A(\hat{\rho}_{AB})$.
6. $S\left(\sum_i p_i\hat{\rho}_i\right) = H(p_i) + \sum_i p_i S(\hat{\rho}_i)$. *Where* $H(.)$ *is the Shannon entropy,* p_i *is a probability distribution, and the density operators* $\hat{\rho}_i$ *have support on orthogonal subspaces.*
7. $S\left(\sum_i p_i |i\rangle\langle i| \otimes \hat{\rho}_i\right) = H(p_i) + \sum_i p_i S(\hat{\rho}_i)$. *Where* p_i *is a probability distribution,* $|i\rangle$ *is an orthogonal basis for a system* A, *and* $\hat{\rho}_i$ *is any set of density ops for another system* B

von Neumann Entropy of Gaussian states

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vNE is invariant under unitary operations (as it depends only on the density operator's eigenvalues). In the phase space description we translated unitary operators to symplectic transformations. Therefore, in the Gaussian regime, the vNE is invariant under symplectic transformations and the vNE of any Gaussian state will be given by the eigenvalues of some multimode thermal state.

$$S(\hat{\rho}_{th}) = S\left(\hat{\rho}_{th,1} \otimes \hat{\rho}_{th,2} \otimes \cdots \otimes \hat{\rho}_{th,N}\right) = S\left(\hat{\rho}_{th,1}\right) + S\left(\hat{\rho}_{th,2}\right) + \cdots + S\left(\hat{\rho}_{th,N}\right)$$

Thermal states are diagonal on Fock basis → Easy to find the vNE.

$$\hat{\rho}_{th,i} = \frac{1}{M_i+1} \sum_{n=0}^{\infty} \left(\frac{M_i}{M_i+1}\right)^n |n\rangle\langle n| \rightarrow S(\hat{\rho}_{th,i}) = \sum_{i=1}^{N} (M_i+1)\ln(M_i+1) - M_i \ln M_i$$

Problem 66: Prove it

$$S(\hat{\rho}_{th,i}) = g(M_i) \qquad g(x) = \begin{cases} (x+1)\ln(x+1) - x\ln x, & x > 0 \\ 0, & x = 0 \end{cases}$$

M_i: thermal of photons of the i^{th} mode.

$$S(\hat{\rho}_{th}) = \sum_{i=1}^{N} g(M_i)$$

von Neumann Entropy of Gaussian states

$$S(\hat{\rho}_{th}) = \sum_{i=1}^{N} g(M_i) = \sum_{i=1}^{N} g(\nu_i - \tfrac{1}{2})$$

To calculate the vNE of a Gaussian state, we must simply find the symplectic eigenvalues and feed it to the $g(.)$ function.

Symplectic eigenvalues of any CM V: $V = S\, V_{th,N} S^T$

$$V_{\mathrm{th},N} = \begin{pmatrix} \nu_1 & & & & & \\ & \ddots & & & & \\ & & \nu_N & & & \\ & & & \nu_1 & & \\ & & & & \ddots & \\ & & & & & \nu_N \end{pmatrix} = \begin{pmatrix} M_1 + \frac{1}{2} & & & & & \\ & \ddots & & & & \\ & & M_N + \frac{1}{2} & & & \\ & & & M_1 + \frac{1}{2} & & \\ & & & & \ddots & \\ & & & & & M_N + \frac{1}{2} \end{pmatrix}$$

Upcoming topics

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1. Recap of most of the staff we discussed so far.
2. Non-Gaussian states: why they are important and what are they. Cat states. Photon subtraction.
3. Probabilistic, noiseless amplification.
4. Discrete variables teleportation and application of fidelity.
5. More optical circuits other than teleportation (e.g. entanglement swapping).
6. Introduction to metrology/sensing (using the fidelity as starting point to introduce the quantum Fisher information metric).