Photonic Quantum Information Processing
OPTI 647: Lecture 13

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Recap of lectures 11&12

We've seen how to transform the modes of a field, i.e, the annihilation operator $\hat{\alpha}_i$ of the i-th mode of a field for:

– Phase shifting, beam splitter, SMS, TMS, and general Bogoliubov transformation.

– We obtained the thermal state from tracing out one part of the TMSV.

– We saw the thermal state in Fock basis and coherent basis.

– We saw that the TMS can work as a quantum limited amplifier.

Outline of Lecture 13

1. From unitary Gaussian transformations to symplectic transformations.
2. The covariance matrix, the first moments, and the Wigner function.
3. The (symplectic) transformation of the covariance matrix and the first moments.
4. Towards CV teleportation.

$$\hbar = 1: \ \hat{a} = \frac{\hat{q}_a + i\hat{p}_a}{\sqrt{2}}, \ \hat{a}^\dagger = \frac{\hat{q}_a - i\hat{p}_a}{\sqrt{2}}$$

From unitary Gaussian transformations to symplectic transformations

$$\hat{b}_i = \sum_j A_{ij}\hat{a}_j + B_{ij}\hat{a}_j^\dagger$$

$$\hat{b}_i^\dagger = \sum_j B_{ij}^*\hat{a}_j + A_{ij}^*\hat{a}_j^\dagger$$

$$\boxed{\begin{aligned}[\hat{b}_i, \hat{b}_j^\dagger] &= 1 \\ [\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] &= 0\end{aligned}} \rightarrow \boxed{\begin{aligned}AB^T &= (AB^T)^T \\ AA^\dagger &= BB^\dagger + I\end{aligned}}$$

$$\begin{pmatrix} \vec{b} \\ \vec{b}^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{a}^\dagger \end{pmatrix}$$

$$\hat{a} = \frac{\hat{q}_a + i\hat{p}_a}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{\hat{q}_a - i\hat{p}_a}{\sqrt{2}}$$

$$\hat{b} = \frac{\hat{q}_b + i\hat{p}_b}{\sqrt{2}}, \quad \hat{b}^\dagger = \frac{\hat{q}_b - i\hat{p}_b}{\sqrt{2}}$$

$$\Rightarrow$$

$$\hat{\vec{q}}_a = \frac{1}{\sqrt{2}}\left(\vec{a} + \vec{a}^\dagger\right), \quad \hat{\vec{p}}_a = \frac{-i}{\sqrt{2}}\left(\vec{a} - \vec{a}^\dagger\right)$$

$$\hat{\vec{q}}_b = \frac{1}{\sqrt{2}}\left(\vec{b} + \vec{b}^\dagger\right), \quad \hat{\vec{p}}_b = \frac{-i}{\sqrt{2}}\left(\vec{b} - \vec{b}^\dagger\right)$$

From unitary Gaussian transformations to symplectic transformations

$$\begin{pmatrix} \vec{b} \\ \vec{b}^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{a}^\dagger \end{pmatrix}$$

$$\hat{\vec{q}}_b = \frac{1}{\sqrt{2}}\left(\vec{b} + \vec{b}^\dagger\right) = \frac{1}{\sqrt{2}}(A\vec{a} + B\vec{a}^\dagger + A^*\vec{a}^\dagger + B^*\vec{a})$$

$$= \frac{1}{2}\left[(A + A^* + B + B^*)\hat{\vec{q}}_a + i(A - A^* - B + B^*)\hat{\vec{p}}_a\right]$$

Continuing in the same way we find:

$$\hat{\vec{p}}_b = -\frac{i}{2}\left[(A - A^* + B - B^*)\hat{\vec{q}}_a + i(A + A^* - B - B^*)\hat{\vec{p}}_a\right]$$

$$\begin{pmatrix} \hat{\vec{q}}_b \\ \hat{\vec{p}}_b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + A^* + B + B^* & i(A - A^* - B + B^*) \\ -i(A - A^* + B - B^*) & A + A^* - B - B^* \end{pmatrix} \begin{pmatrix} \hat{\vec{q}}_a \\ \hat{\vec{p}}_a \end{pmatrix}$$

$$\begin{pmatrix} \hat{\vec{q}}_b \\ \hat{\vec{p}}_b \end{pmatrix} = S \begin{pmatrix} \hat{\vec{q}}_a \\ \hat{\vec{p}}_a \end{pmatrix}$$

Problem 53: derive the matrix S, starting from the Bogoliubov transformation

The symplectic matrix

If a matrix S satisfies the condition: $S\Omega S^T = \Omega$ $\quad \Omega = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$

then we call it **symplectic.**

What does it mean? $S\Omega S^T = \Omega \Rightarrow \begin{cases} AB^T = (AB^T)^T \\ AA^\dagger = BB^\dagger + I \end{cases}$

Problem 54: Derive these relations starting from $S\Omega S^T = \Omega$

Therefore the condition that all symplectic matrices satisfy, has the physical meaning that the transformed field should satisfy commutation relations.

Notation and commutation relations

We will work with the so called **qqpp** representation, which means that we will organize our vectors in the following manner:

Operators

$$\vec{\hat{r}}^{qqpp} \equiv \vec{\hat{r}} = \begin{pmatrix} \vec{\hat{q}} \\ \vec{\hat{p}} \end{pmatrix} = \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \\ \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix}$$

The commutation relations in the **qqpp** representation are:

$$[\hat{r}_k, \hat{r}_{k+n}] = i\Omega_{k,k+n} \qquad \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$S \qquad S^T$

$$[\hat{r}_k, \hat{r}_{k+n}] = i\Omega_{k,k+n} \longrightarrow [\hat{r}'_k, \hat{r}'_{k+n}] = i\Omega_{k,k+n}$$

$S \qquad S^T$

$$\vec{r}' = S\vec{r}$$

Problem 55: Prove that in the qpqp representation, the commutation relation is:

$$[\hat{r}_j, \hat{r}_k] = i\tilde{\Omega}_{j,k}$$

Where: $\vec{\hat{r}}^{qpqp} = \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{q}_n \\ \hat{p}_n \end{pmatrix}$ $\qquad \tilde{\Omega}_{j,k} = \oplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Going from density operators to phase space

$$\hat{\rho}_G = \frac{1}{Z}\exp\left(-\frac{1}{2}(\hat{\vec{r}}-\vec{d})^T G(\hat{\vec{r}}-\vec{d})\right), Z = \mathrm{tr}\hat{\rho}_G$$

Most general Gaussian density operator

Gibbs matrix

Just vectors (numbers)

$$\chi_W(\vec{d}_\alpha) = \mathrm{tr}\left(\rho D(\vec{\alpha},\vec{\alpha}^*)\right) \longrightarrow \chi_W(\vec{d}_\alpha) = \mathrm{tr}\left(\rho e^{i\vec{d}_\alpha^T\Omega\hat{\vec{r}}}\right)$$

$\vec{d}_a^T$ are the displacements in the qqpp representation

$$\chi_W(\vec{d}_\alpha) = \mathrm{tr}\left(\hat{\rho}_G e^{i\vec{d}_\alpha^T\Omega\hat{\vec{r}}}\right)$$

$$W(\vec{r}) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}} d^{2n}\vec{d}_\alpha \; e^{i\vec{d}_\alpha^T\Omega\vec{r}}\chi_W(\vec{d}_\alpha)$$

Wigner function $W(\vec{r})$

All exponentials are at most quadratic (i.e. Gaussian) and the FT is a linear transformation, therefore the outcome will still be quadratic.

Wigner function of Gaussian states

$$W(\vec{r}) = \frac{1}{(2\pi)^n\sqrt{\det V}} \exp\left(-\frac{1}{2}(\vec{r}-\vec{d})^T V^{-1}(\vec{r}-\vec{d})\right)$$

$$\begin{aligned} \hat{\rho}_G &= \frac{1}{Z}\exp^{-\frac{1}{2}(\hat{\vec{r}}-\vec{d})^T G(\hat{\vec{r}}-\vec{d})} \\ Z &= \mathrm{tr}\hat{\rho}_G = \det\left(V + \frac{i\Omega}{2}\right)^{1/2} \\ G &= 2i\Omega\ \mathrm{arccoth}(2iV\Omega) \end{aligned}$$

Displacements or first moments

$$\vec{d} = \langle\hat{\vec{r}}\rangle = \mathrm{tr}(\hat{\rho}\hat{\vec{r}})$$

Covariance matrix (CM) elements or second moments

$$V_{kj} = \frac{1}{2}\langle\{\hat{r}_j - d_j, \hat{r}_k - d_k\}\rangle = \frac{1}{2}\mathrm{tr}(\hat{\rho}\{\hat{r}_j - d_j, \hat{r}_k - d_k\})$$

$$V = \begin{pmatrix} V_{\mathrm{pos}} & V_c \\ V_c^T & V_{\mathrm{mom}} \end{pmatrix}$$

For an N mode field, the covariance matrix is an $N \times N$ positive definite, symmetric, real matrix. It must additionally satisfy:

$$V + \frac{i\Omega}{2} \geq 0$$

The extra condition stems from the commutation relation (i.e. uncertainty principle) and it gives the minimum permitted volume in phase space

Problem 56: Prove that this equation remains the same under symplectic transformations.

The CM captures the “shape” of the state in phase space while the displacements corresponds to how far from the origin of the axes the state is. **The Wigner function is a complete description of the state, is normalized to 1, and gives the correct mean values of observables.**

Vacuum state & coherent state

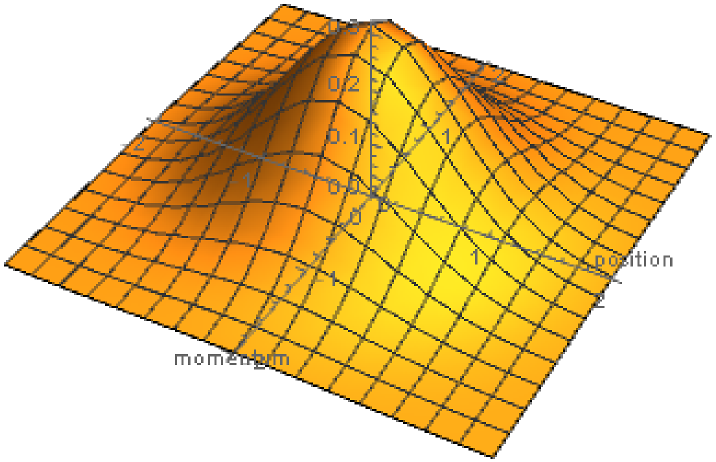
Single mode vacuum: $|0\rangle\langle 0|$ (why is this state Gaussian?)

Remember: $\hbar = 1: \ \hat{a} = \frac{\hat{q}_a + i\hat{p}_a}{\sqrt{2}}, \ \hat{a}^\dagger = \frac{\hat{q}_a - i\hat{p}_a}{\sqrt{2}}$

$$d_i = \langle \hat{r}_i \rangle = \langle 0|\hat{r}_i|0\rangle = 0$$

$$V_{kj} = \frac{1}{2}\langle\{\hat{r}_j - d_j, \hat{r}_k - d_k\}\rangle = \frac{1}{2}\langle 0|\{\hat{r}_j, \hat{r}_k\}|0\rangle = \frac{1}{2}\delta_{jk}$$

$$W(\vec{r}) = \frac{1}{\pi}\exp\left(-\vec{r}^T I \vec{r}\right)$$



Coherent state: $|\alpha\rangle\langle\alpha|$

$$d_1 = \langle \hat{r}_1 \rangle = \langle \alpha|\hat{q}_1|\alpha\rangle = \frac{1}{\sqrt{2}}(\alpha + \alpha^*) = q_1$$

$$d_2 = \langle \hat{r}_2 \rangle = \langle \alpha|\hat{p}_1|\alpha\rangle = \frac{-i}{\sqrt{2}}(\alpha - \alpha^*) = p_1$$

$$V = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Just displaced vacuum.

Thermal state (non displaced)

$$\hat{\rho} = (1 - |\lambda|^2) \sum_{n=0}^{\infty} |\lambda|^{2n} |n\rangle\langle n|, \; 0 \le |\lambda| < 1$$

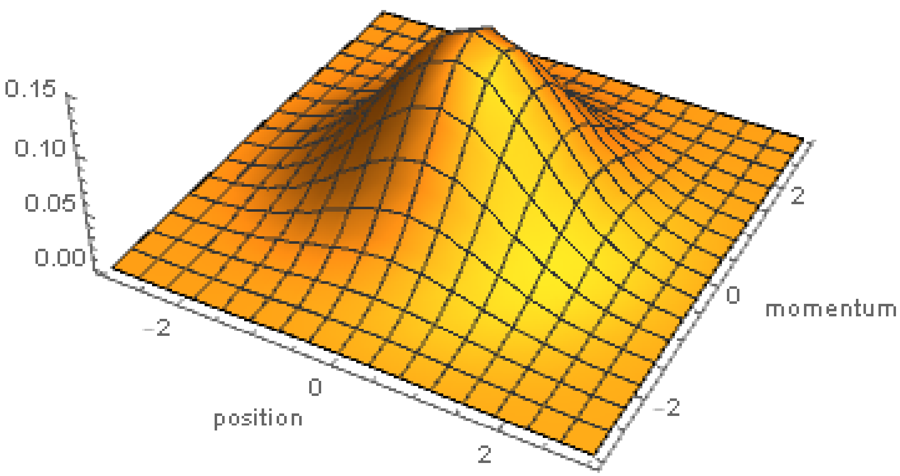
$$d_1 = \langle \hat{q}_1 \rangle = \left(1 - |\lambda|^2\right) \sum_n \frac{|\lambda|^2}{\sqrt{2}} \left\langle n \middle| \left(\hat{a}_1 + \hat{a}_1^\dagger\right) \middle| n \right\rangle = 0$$

$$d_2 = \langle \hat{p}_1 \rangle = \left(1 - |\lambda|^2\right) \sum_n \frac{(-i)|\lambda|^2}{\sqrt{2}} \left\langle n \middle| \left(\hat{a}_1 - \hat{a}_1^\dagger\right) \middle| n \right\rangle = 0$$

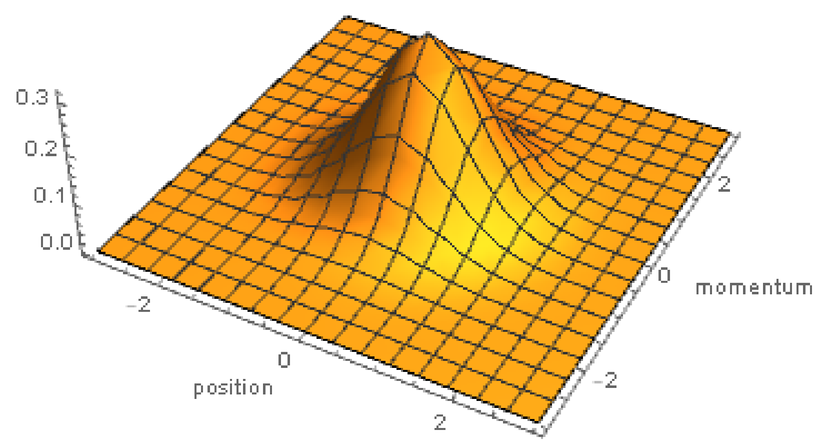
Problem 57: Calculate the covariance matrix (CM) for the thermal state and write down the Wigner function. Will the CM change if we perform a displacement? Why?

$$V_{kj} = \tfrac{1}{2}\langle\{\hat{r}_j - d_j, \hat{r}_k - d_k\}\rangle = \tfrac{1}{2}\mathrm{tr}(\hat{\rho}\{\hat{r}_j, \hat{r}_k\}) = \tfrac{1}{2}\tfrac{1+|\lambda|^2}{1-|\lambda|^2}\delta_{jk}, \; j,k = 1,2$$

$\lambda=0.6$



$\lambda=10^{-4}$



Gaussian unitaries in phase space

A given Gaussian state with first moments $\vec{d}$ and CM V, under symplectic transformations is transformed as:

$$\vec{d}' = S\vec{d} \qquad \begin{pmatrix} \hat{\vec{q}}_b \\ \hat{\vec{p}}_b \end{pmatrix} = \begin{pmatrix} A + A^* + B + B^* & i(A - A^* - B + B^*) \\ -i(A - A^* + B - B^*) & A + A^* - B - B^* \end{pmatrix} \begin{pmatrix} \hat{\vec{q}}_a \\ \hat{\vec{p}}_a \end{pmatrix} \Rightarrow$$

$$\Rightarrow \hat{\vec{r}}' = S\hat{\vec{r}} \Rightarrow \langle \hat{\vec{r}}' \rangle = S\langle \hat{\vec{r}} \rangle \Rightarrow \vec{d}' = S\vec{d}$$

Recall that S, is not an operator, is a matrix.

$$V' = SVS^T$$

Exploiting the first moments' transformation, we transform the CM.

$$V'_{kj} = \frac{1}{2}\langle\{\hat{r}'_j - d'_j, \hat{r}'_k - d'_k\}\rangle \equiv \frac{1}{2}\langle\{\hat{R}'_k, \hat{R}'_j\}\rangle = \frac{1}{2}\sum_{l,m} S_{mk}S_{lj}\langle\{\hat{R}_k, \hat{R}_j\}\rangle =$$

$$= \frac{1}{2}\sum_{l,m} S_{mk}S_{lj}V_{kj} = \frac{1}{2}\sum_{l,m} S_{mk}V_{kj}S^T_{jl} \Rightarrow V' = SVS^T$$

Single-mode transformations

Displacement

$$S_\alpha \hat{r}_j = \hat{r}_j + d_{\alpha j}$$

Therefore it leaves the CM the same, the only thing that changes are the first moments.

Phase shift (Rotation)

If we apply the general formula obtained earlier, and plug the Bogoliubov transformation in:

$$S_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$

Orthogonal: passive element

Single-mode squeezing (phases set to π/2)

Again, we have derived everything in the Heisenberg picture. Now we only have to substitute in the general formula.

$$S_r = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$$

Non-orthogonal: active element

Two-modes transformation

Beam splitter $\tau = \cos^2\theta$

$$S_\tau = \begin{pmatrix} \sqrt{\tau} & \sqrt{1-\tau} & 0 & 0 \\ -\sqrt{1-\tau} & \sqrt{\tau} & 0 & 0 \\ 0 & 0 & \sqrt{\tau} & \sqrt{1-\tau} \\ 0 & 0 & -\sqrt{1-\tau} & \sqrt{\tau} \end{pmatrix}$$

Orthogonal: passive element

Two mode squeezer (with phase set to π/2)

Block A Block B

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & \cosh\xi & \sinh\xi & 0 \\ 0 & \sinh\xi & \cosh\xi & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$

Heisenberg

$$S_\xi = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & \cosh\xi & -\sinh\xi \\ 0 & 0 & -\sinh\xi & \cosh\xi \end{pmatrix}$$

Non-orthogonal: active element

Symplectic

Two mode squeezer decomposition

$$S_\xi = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & \cosh\xi & -\sinh\xi \\ 0 & 0 & -\sinh\xi & \cosh\xi \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{\xi} & & & \\ & e^{-\xi} & & \\ & & e^{-\xi} & \\ & & & e^{\xi} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$$

$$S_\xi = S^T_{\tau=1/2}(S_{r=\xi/2} \oplus S^{-1}_{r=\xi/2})S_{\tau=1/2}$$

The TMS is decomposed into a sequence of a beamsplitter, single mode squeezers (with opposite squeezing direction), and another beam splitter which is the transpose of the 1st one.

Symplectic diagonalization and some other general statements

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Williamson's theorem:
Any positive semi-definite matrix can decomposed as:

$$V = S\mathrm{diag}\left(\nu_1, \ldots, \nu_n; \nu_1, \ldots, \nu_n\right) S^T$$

Where S is symplectic.

The symplectic eigenvalues are given by the ordinary eigenvalues of: $|i\Omega V|$

$$|A| = \sqrt{A^\dagger A}$$

Euler's theorem:
Any symplectic matrix S can be decomposed as:

$$S = O(\Delta \oplus \Delta^{-1})P$$

Where O, P are orthogonal and symplectic and Δ is diagonal.

$(\Delta \oplus \Delta^{-1})$: Always symplectic

The above two theorems say that we can derive any Gaussian transformation by a sequence of beam splitters and phases (the P matrix), squeezing (the Δ matrices), and beam splitters and phases again (the O matrix). Then we displace.

Any pure state → We apply said recipe to vacuum (which has diagonal CM).
Any mixed state → We apply said recipe to a non-displaced thermal state (which has diagonal CM).

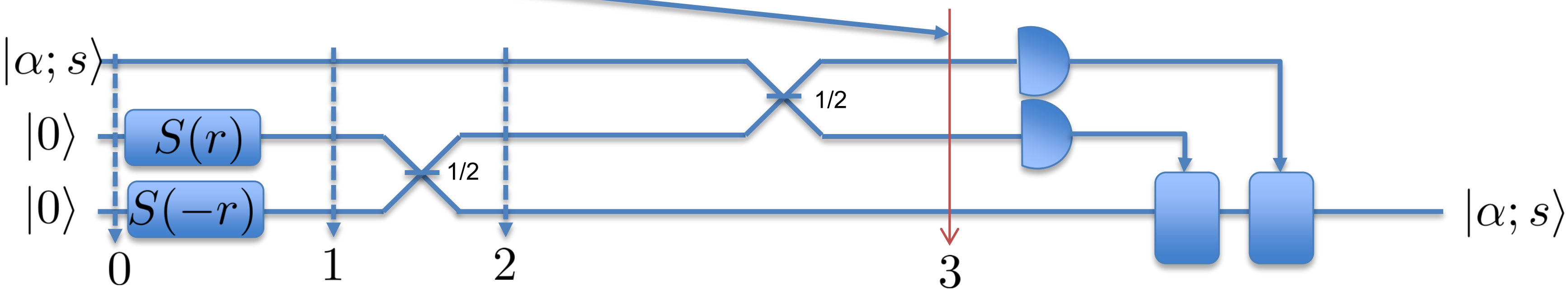
Problem 58

Initially we have the vacuum state $|0\rangle$. Find the covariance matrix and the first moments. If:

i. We squeeze first and then we displace.
ii. We displace first and then we squeeze.
iii. Can you find the same result as the one we found in the Heisenberg picture? Which way you think is easier?

Towards CV teleportation

… Not the full thing yet since we still need to talk about CV measurements…



Problem 58: Calculate the CM and first moments where the red line is.

$s \in \mathbb{C}, \;\; r \in \mathbb{R}$

$|\alpha; s\rangle$ Unknown squeezed displaced state to be teleported

Upcoming topics

Next lecture:

1. Partial trace (tracing out) on the CM level. More on thermal states and symplectic decomposition. Entropy of a Gaussian state.
2. Calculating mean values of observables using the Wigner function.
3. Homodyne measurement and finishing off the CV teleportation
4. (If there's time) Non-Gaussian states:
   - Fock states
   - Cat states
   - Post selection and photon addition/subtraction