Photonic Quantum Information Processing
OPTI 647: Lecture 12

Christos Gagatsos
Saikat Guha

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College of Optical Sciences
Meinel 501A

Recap of lecture 11

We've seen how to transform the modes of a field, i.e, the annihilation operator $\hat{\alpha}_i$ of the i-th mode of a field for:

– Phase shifting

– Beam splitter

– Single mode squeezing

We introduced the theory for general Gaussian unitary transformations in the Heisenberg picture.

Correction Lecture 11, slide 7:

$$\langle \hat{N}_{b_1} \rangle - \langle \hat{N}_{b_2} \rangle = \langle \hat{N}_{a_1} + \hat{N}_{a_2} \rangle \cos\phi - i\langle \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1 \rangle \sin\phi$$

Outline of Lecture 12

Two-mode squeezing (TMS):

1. Hamiltonian description of the TMS
2. Bloch-Messiah decomposition of the TMS
3. Derivation of a two-mode squeezed vacuum state (TMSV)
4. Characteristic function (Homework)
5. Tracing out: thermal state. Fock and coherent representations of the thermal state
6. TMS as a non-unitary, irreversible, phase-insensitive, linear amplifier
7. Amplification of a coherent state

Hamiltonian description of the TMS

TMS is described by the macroscopic Hamiltonian:

$$H = \hbar\xi e^{i\phi}\hat{a}_1\hat{a}_2 + \hbar\xi e^{-i\phi}\hat{a}_1^\dagger\hat{a}_2^\dagger$$

$$S(s) = \exp\left(-\tfrac{i}{\hbar}Hs\right)$$

$$S(s) = \exp\left(s^*\hat{a}_1\hat{a}_2 - s\hat{a}_1^\dagger\hat{a}_2^\dagger\right)$$

If there are operators:

$$[K_-, K_+] = 2K_0 \qquad [K_0, K_\pm] = \pm K_\pm \qquad K_0^\dagger = K_0$$

$$K_+ = \hat{a}_1^\dagger\hat{a}_2^\dagger$$

$$K_- = K_+^\dagger$$

$$2K_0 = \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1$$

$$S(s) = e^{-\tau K_+}e^{-2\nu K_0}e^{\tau^* K_-}$$

$$\tau = \frac{s}{|s|}\tanh|s|$$

$$\nu = \ln\left(\cosh|s|\right)$$

Two-mode squeezed vacuum (TMSV)

We will hit the vacuum state $|00\rangle$ with $S(s) = \exp\left(s^* \hat{a}_1 \hat{a}_2 - s \hat{a}_1^\dagger \hat{a}_2^\dagger\right)$

$$K_+ = \hat{a}_1^\dagger \hat{a}_2^\dagger$$

$$K_- = \hat{a}_2 \hat{a}_1$$

$$2K_0 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1$$

$$\tau = \frac{s}{|s|} \tanh |s|$$

$$\nu = \ln\left(\cosh |s|\right)$$

$$S(s)|00\rangle = e^{-\tau K_+} e^{-2\nu K_0} e^{\tau^* K_-} |00\rangle =$$

$$= e^{-\tau K_+} e^{-2\nu K_0} |00\rangle =$$

$$= e^{-\nu} e^{-\tau K_+} |00\rangle =$$

$$= e^{-\nu} \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} K_+^n |00\rangle =$$

$$= \frac{1}{\cosh |s|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{s}{|s|} \tanh |s|\right)^n \hat{a}_1^{\dagger n} \hat{a}_2^{\dagger n} |00\rangle =$$

$$= \frac{1}{\cosh |s|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{s}{|s|} \tanh |s|\right)^n \hat{a}_1^{\dagger n} \hat{a}_2^{\dagger n} |00\rangle =$$

$$= \frac{1}{\cosh |s|} \sum_{n=0}^{\infty} \left(-\frac{s}{|s|} \tanh |s|\right)^n |nn\rangle \equiv |00; s\rangle$$

Bloch-Messiah decomposition of the TMS

$$H = i\hbar\xi\hat{a}_1\hat{a}_2 - i\hbar\xi\hat{a}_1^\dagger\hat{a}_2^\dagger,\ \xi > 0$$

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} \cosh\xi & 0 \\ 0 & \cosh\xi \end{matrix}} & \boxed{\begin{matrix} 0 & \sinh\xi \\ \sinh\xi & 0 \end{matrix}} \\ \boxed{\begin{matrix} 0 & \sinh\xi \\ \sinh\xi & 0 \end{matrix}} & \boxed{\begin{matrix} \cosh\xi & 0 \\ 0 & \cosh\xi \end{matrix}} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$

Matrix A is proportional to the identity matrix and as sequence it (trivially) commutes with matrix B. That means that A is always diagonal in the eigenvectors' basis of matrix B. All we have to do is to diagonalize matrix B.

$$A = A_D,\ B = UB_DU^\dagger,\ B_D = \begin{pmatrix} -\sinh\xi & 0 \\ 0 & \sinh\xi \end{pmatrix}, U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$U^* = U,\ V = U,\ V^\dagger = U^\dagger,\ V^T = U^\dagger$$

Bloch-Messiah decomposition of the TMS

$$\begin{pmatrix} \vec{b} \\ \vec{b}^\dagger \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} A_D & B_D \\ B_D^* & A_D^* \end{pmatrix} \begin{pmatrix} V^\dagger & 0 \\ 0 & V^T \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{a}^\dagger \end{pmatrix}$$

General formula

$$\begin{pmatrix} \vec{b} \\ \vec{b}^\dagger \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A & B_D \\ B_D & A \end{pmatrix} \begin{pmatrix} U^\dagger & 0 \\ 0 & U^\dagger \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{a}^\dagger \end{pmatrix}$$

For our example

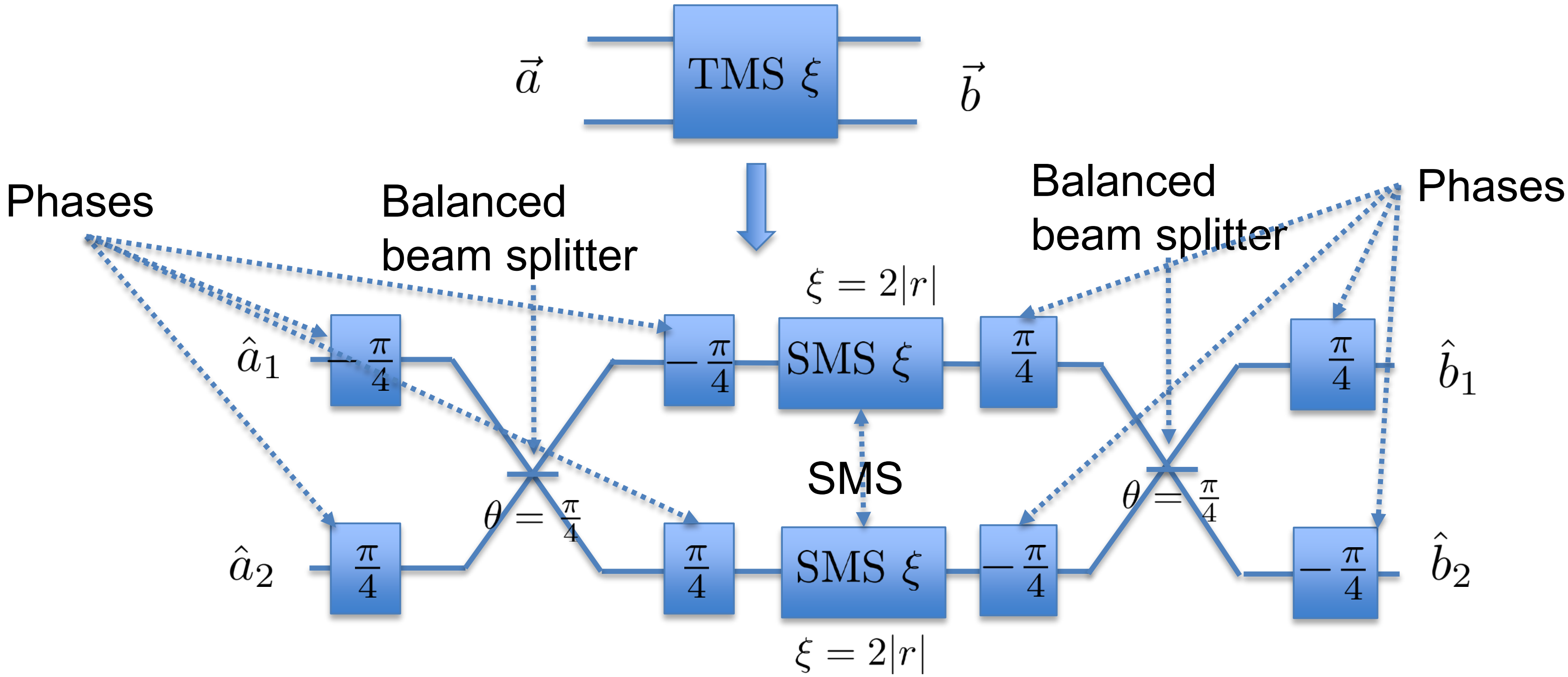
$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cosh\xi & 0 & -\sinh\xi & 0 \\ 0 & \cosh\xi & 0 & \sinh\xi \\ -\sinh\xi & 0 & \cosh\xi & 0 \\ 0 & \sinh\xi & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$

Doesn't mix annihilation and creation ops: Beam splitter and phases.

Does mix annihilation and creation ops of the same mode: Single mode squeezers

Doesn't mix annihilation and creation ops: Beam splitter and phases

Bloch-Messiah decomposition of the TMS



$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cosh\xi & 0 & -\sinh\xi & 0 \\ 0 & \cosh\xi & 0 & \sinh\xi \\ -\sinh\xi & 0 & \cosh\xi & 0 \\ 0 & \sinh\xi & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$

Problems

Problem 50:

Decompose the unitary transformation, generated by the TMS Hamiltonian below, using the Bloch-Messiah reduction.

Be careful to define your vectors in a consistent way.

$$H = \hbar\xi e^{i\phi}\hat{a}_1\hat{a}_2 + e^{-i\phi}\hbar\xi\hat{a}_1^\dagger\hat{a}_2^\dagger, \ \xi > 0$$

Problem 51:

Calculate the characteristic function

$$\chi_W(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*) = \chi_W(\vec{\alpha}, \vec{\alpha}^*) = \mathrm{tr}\left(\rho D(\vec{\alpha}, \vec{\alpha}^*)\right)$$

For $\rho = |00; s\rangle\langle 00; s|, \ |00; s\rangle = \frac{1}{\cosh|s|}\sum_{n=0}^{\infty}\left(-\frac{s}{|s|}\tanh|s|\right)^n |nn\rangle$

$$D(\vec{\alpha}, \vec{\alpha^*}) = \exp\left(\vec{\alpha}\cdot\vec{a}^\dagger - \vec{\alpha}^*\cdot\vec{a}\right)$$

$$\vec{\alpha}\cdot\vec{a}^\dagger = \alpha_1\hat{a}_1^\dagger + \alpha_2\hat{a}_2^\dagger$$

Hint: Do something similar as in slide 16, Lecture 11, when we commuted SMS and displacement.

Hint for the problem 50

The transformation for the given Hamiltonian is:

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_1^\dagger \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\xi & 0 & 0 & -ie^{i\phi}\sinh\xi \\ 0 & \cosh\xi & -ie^{i\phi}\sinh\xi & 0 \\ 0 & ie^{-i\phi}\sinh\xi & \cosh\xi & 0 \\ ie^{-i\phi}\sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_1^\dagger \\ \hat{a}_2^\dagger \end{pmatrix}$$

Tracing out one mode of TMSV

$$|00;s\rangle = \frac{1}{\cosh|s|}\sum_{n=0}^{\infty}\left(-\frac{s}{|s|}\tanh|s|\right)^n |nn\rangle = \sqrt{1-|\lambda|^2}\sum_{n=0}^{\infty}\lambda^n|nn\rangle$$

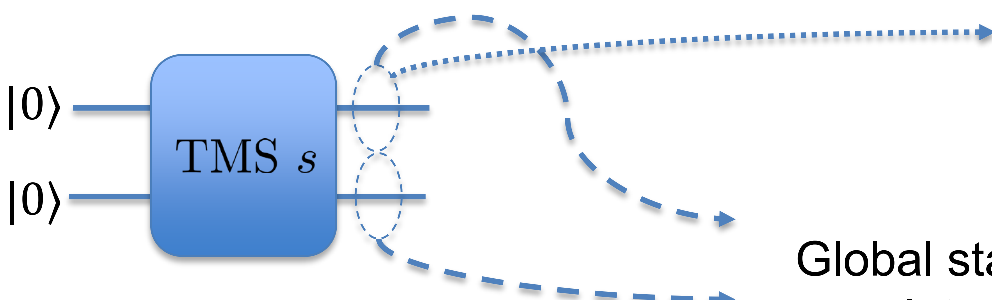
$$\lambda = -\tfrac{s}{|s|}\tanh|s|$$

The density operator is: $$\rho = (1-|\lambda|^2)\sum_{n,m=0}^{\infty}\lambda^n\lambda^{m*}|n\underline{n}\rangle\langle m\underline{m}|$$

Mode to be traced out
(let's call it mode B)

$$\rho_A = \mathrm{tr}_{\mathrm{B}}(\rho) = (1-|\lambda|^2)\sum_{n,m=0}^{\infty}\lambda^n\lambda^{m*}\delta_{nm}|n\rangle\langle m|$$

$$\boxed{\rho_A = (1-|\lambda|^2)\sum_{n=0}^{\infty}|\lambda|^{2n}|n\rangle\langle n|}$$ Thermal state



Reduced state is a thermal state

Global state is maximally correlated

Thermal state on coherent state basis

Let's change basis and go from Fock space to coherent state basis:

$$\rho_A = (1-|\lambda|^2)\sum_{n=0}^{\infty}|\lambda|^{2n}|n\rangle\langle n| \qquad I = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|$$

$$\rho_A = \frac{1-|\lambda|^2}{\pi^2}\sum_{n=0}^{\infty}\int d^2\alpha\int d^2\beta|\lambda|^{2n}\langle\beta|n\rangle\langle n|\alpha\rangle|\beta\rangle\langle\alpha| \qquad \langle n|\alpha\rangle = e^{-|\alpha|^2/2}a^n$$

$$\rho_A = \frac{1-|\lambda|^2}{\pi^2}\sum_{n=0}^{\infty}\int d^2\alpha\int d^2\beta\exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}\right)\frac{|\lambda|^{2n}}{n!}a^n\beta^{*n}|\beta\rangle\langle\alpha|$$

$$\rho_A = \frac{1-|\lambda|^2}{\pi^2}\int d^2\alpha\int d^2\beta\exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+|\lambda|^2\beta^*\alpha\right)|\beta\rangle\langle\alpha|$$

This is not the Glauber P representation! This is a proper representation on an (overcomplete) basis.

Thermal state on coherent state basis

$$\rho_A = \frac{1-|\lambda|^2}{\pi^2}\int d^2\alpha \int d^2\beta \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+|\lambda|^2\beta^*\alpha\right)|\beta\rangle\langle\alpha|$$

$|\lambda| = \tanh|s|$ If we take $\lambda \to 0$ ($|s| \to 0$, i.e., no squeezing):

$$\rho_A = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta \exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}\right)|\beta\rangle\langle\alpha| =$$

$$= \left(\frac{1}{\pi}\int d^2\beta \exp\left(-\frac{|\beta|^2}{2}\right)|\beta\rangle\right)\left(\frac{1}{\pi}\int d^2\alpha \exp\left(-\frac{|\alpha|^2}{2}\right)\langle\alpha|\right) = |0\rangle\langle 0|$$

Which means, that when we set λ=0, we go from a thermal state to the vacuum state, therefore λ must be related to the temperature of the thermal state.

We saw that if we squeeze a 2-mode vacuum state (pure state) and we trace out one of the modes we end up with a thermal state (mixed state), which means that we lost information encoded in the correlations. Therefore this procedure is not reversible.

TMS as a linear amplifier

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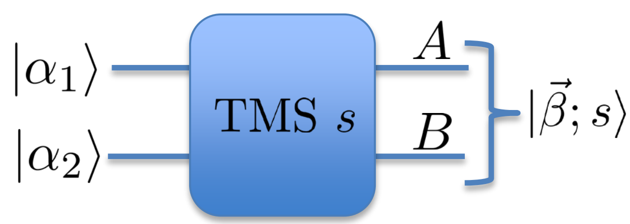
$$S(s)D(\vec{\alpha},\vec{\alpha}^*)|00\rangle = S(s)D(\vec{\alpha},\vec{\alpha}^*)S^\dagger(s)S(s)|00\rangle = D(\vec{\beta},\vec{\beta}^*)S(s)|00\rangle =$$

$$= D(\vec{\beta},\vec{\beta}^*)|00;s\rangle = |\vec{\beta};s\rangle$$

$$\vec{\beta} = (\beta_1,\beta_2) = (\alpha_1 \cosh|s| + \alpha_2^* \tfrac{s}{|s|}\sinh|s|, \alpha_2 \cosh|s| + \alpha_1^* \tfrac{s}{|s|}\sinh|s|)$$

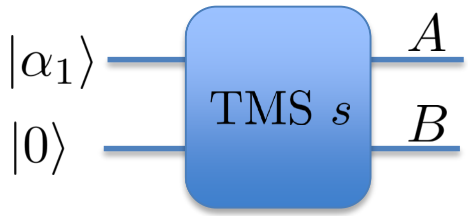
This is basically Problem 51, and quite similar to what we did for the SMS state.



The global state will be a 2-mode squeezed and displaced state. The reduced state in output mode A will be a **displaced** thermal state:

Problem 52: derive this expression $\rho_{A,d} = \mathrm{tr}_B(|\vec{\beta};s\rangle\langle\vec{\beta};s|) = D(\beta_1,\beta_1^*)\rho_A D^\dagger(\beta_1,\beta_1^*)$

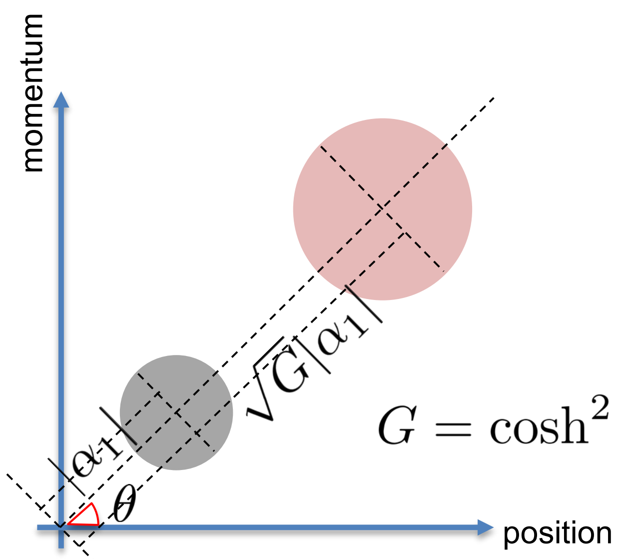
TMS as a linear amplifier



$$\rho_{A,d} = \mathrm{tr}_B(|\vec{\beta}; s\rangle\langle\vec{\beta}; s|) = D(\beta_1, \beta_1^*)\rho_A D^\dagger(\beta_1, \beta_1^*)$$

$$\vec{\beta} = (\beta_1, \beta_2) = (\alpha_1 \cosh|s|, \alpha_1^* \tfrac{s}{|s|} \sinh|s|)$$

If we inject into the TMS a coherent state (signal) and vacuum: We amplify the amplitude of the signal but at the same time we don't end up with a coherent state in mode A. We end up with a thermal state. Quantum mechanics prohibit deterministic amplification without introducing noise. This is called the **quantum limited amplifier**.



$$G = \cosh^2|s| > 1 : \text{Gain}$$

Note that the angle with respect to the position axis remains the same after amplification.

We will describe the quantum amplifier excessively, when we will introduce the symplectic transformations and quantum channels.

Why noiseless amplification is impossible

Noiseless amplification would mean there's a valid transformation:

$$\hat{b}_1 = \sqrt{G}\hat{a}_1, \; G > 1$$

The commutation relations would be: $[\hat{b}_1, \hat{b}_1^\dagger] = G[\hat{a}_1, \hat{a}_1^\dagger] = G > 1$

Therefore, it's impossible. As we shall see though in the following lectures, it can be done in a probabilistic fashion.

As we saw, we need a valid Bogoliubov transformation, i.e., the transformed field must obey the commutation relations. We saw that this transformation is the TMS:

$$\hat{b}_1 = \sqrt{G}\hat{a}_1 + \underbrace{\sqrt{G-1}\hat{a}_2^\dagger}_{\text{Noise that goes into the signal}} \rightarrow [\hat{b}_1, \hat{b}_1^\dagger] = G - G + 1 = 1$$

Noise is of fundamental nature: it is traced back to the commutation relations

Upcoming topics

Next lecture (do NOT miss):

1. Symplectic transformations, covariance matrices, i.e., full phase space description. Which are the most essential tools in CV quantum information.

2. Applications such as CV teleportation.