Photonic Quantum Information Processing
OPTI 647: Lecture 9

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Announcements

- Topics to be covered today
  - Phase space picture of squeezed states
  - Heterodyne detection
  - Quantum description of beamsplitter and phase
  - Quantum enhanced phase estimation
  - Outlook of the next few lectures

Recap of Lecture 8: Characteristic functions versus probability distributions

$$\chi_W(\zeta^*,\zeta) = \mathrm{tr}(\hat\rho\, e^{-\zeta^*\hat a+\zeta\hat a^\dagger})$$
$$\chi_A(\zeta^*,\zeta) = \mathrm{tr}(\hat\rho\, e^{-\zeta^*\hat a}e^{\zeta\hat a^\dagger})$$
$$\chi_N(\zeta^*,\zeta) = \mathrm{tr}(\hat\rho\, e^{\zeta\hat a^\dagger}e^{-\zeta^*\hat a})$$

$$\chi_W(\zeta) = e^{|\zeta|^2/2}\chi_A(\zeta) = e^{-|\zeta|^2/2}\chi_N(\zeta)$$

Always a proper probability density function; pdf for ideal heterodyne detection OR dual homodyne detection →

$$\chi_A(\zeta) = \int Q(\alpha)e^{\zeta\alpha^*-\zeta^*\alpha}d^2\alpha$$
$$Q(\alpha) = \frac{1}{\pi^2}\int \chi_A(\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}d^2\zeta$$

May not be a proper probability density function. Negativity used to show a state is non-classical. Not all non-classical states have a negative Wigner function →

$$\chi_W(\zeta) = \int W(\alpha)e^{\zeta\alpha^*-\zeta^*\alpha}d^2\alpha$$
$$W(\alpha) = \frac{1}{\pi^2}\int \chi_W(\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}d^2\zeta$$

Always a proper probability density function *when it exists.* The states for which a proper P function exists are called classical states →

$$\chi_N(\zeta) = \int P(\alpha)e^{\zeta\alpha^*-\zeta^*\alpha}d^2\alpha$$
$$P(\alpha) = \frac{1}{\pi^2}\int \chi_N(\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}d^2\zeta$$

$$\langle n|\hat\rho|m\rangle = \frac{1}{\pi}\int \chi_A(\zeta)\langle n|e^{-\zeta\hat a^\dagger}e^{\zeta^*\hat a}|m\rangle d^2\zeta = \frac{1}{\pi}\int \chi_A(\zeta)\sqrt{\frac{n!}{m!}}(-\zeta)^{m-n}L_n^{(m-n)}(|\zeta|^2)d^2\zeta$$

Recap of Lecture 8

- Normally ordered form, $F(\hat{a}^\dagger, \hat{a}) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{nm} \hat{a}^{\dagger n} \hat{a}^m$
  - $\langle\alpha|F(a^\dagger, a)|\alpha\rangle = F^{(n)}(\alpha^*, \alpha)$
  - $\langle\alpha|F(a^\dagger, a)|\beta\rangle = F^{(n)}(\alpha^*, \beta)\langle\alpha|\beta\rangle$
- Anti-normally ordered form, $G(\hat{a}, \hat{a}^\dagger) \equiv \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{nm} \hat{a}^n \hat{a}^{\dagger m}$
- Characteristic function for coherent state, $\hat{\rho} = |\alpha\rangle\langle\alpha|$
  - $\chi_N(\zeta^*, \zeta) = \mathrm{tr}(\hat{\rho}\, e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}) = e^{\zeta\alpha^* - \zeta^*\alpha}$ is Gaussian
- Circularly symmetric $\equiv$ number diagonal $\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|$
- If a state is classical (has a proper P function),
  - Number detection statistics; $\langle\Delta\hat{N}^2\rangle \geq \langle\hat{N}\rangle$
  - Quadrature (homodyne) detection statistics; $\langle\Delta\hat{a}_1^2\rangle \geq 1/4$

Recap of Lecture 8

- Measurement of the $\hat{a}$ POVM, $\hat{\Pi}(\alpha) \equiv \dfrac{|\alpha\rangle\langle\alpha|}{\pi}, \quad \text{for } \alpha \in \mathcal{C}$
  - Distribution of output, $Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle/\pi \equiv \hat{\rho}^{(n)}(\alpha^*, \alpha)/\pi$
  - (classical) characteristic function of this distribution

$$M_{\alpha_1,\alpha_2}(jv_1, jv_2) = \int e^{jv_1\alpha_1 + jv_2\alpha_2} Q(\alpha) d^2\alpha \; = \chi_A(\zeta^*, \zeta)\big|_{\zeta = jv/2}$$

- $\hat{a}$ POVM on squeezed state $|\beta; \mu, \nu\rangle$, assume $\mu, \nu \in \mathbb{R}$
  - $\chi_A(\zeta^*, \zeta) = e^{(\zeta\mu + \zeta^*\nu)\beta^* - (\zeta^*\mu + \zeta\nu)\beta - |\zeta|^2\mu^2 - \mathrm{Re}(\zeta^2)\mu\nu}$: Gaussian
  - $M_{\alpha_1,\alpha_2}(jv_1, jv_2) = \chi_A(\zeta^*, \zeta)|_{\zeta = jv/2} = e^{jv_1(\mu-\nu)\beta_1 - v_1^2\sigma_1^2/2} e^{jv_2(\mu+\nu)\beta_2 - v_2^2\sigma_2^2/2}$
  - Recall, for $p_X(x) = \dfrac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$, $M_X(jv) = e^{jv\mu - v^2\sigma^2/2}$
  - Measurement outcomes (α_1, α_2): S.I. Gaussian with:

$$\langle\alpha_1\rangle = (\mu - \nu)\beta_1 \qquad \langle\alpha_2\rangle = (\mu + \nu)\beta_2$$

$$\langle\Delta\alpha_1^2\rangle \equiv \sigma_1^2 = \frac{(\mu-\nu)^2 + 1}{4} \qquad \langle\Delta\alpha_2^2\rangle \equiv \sigma_2^2 = \frac{(\mu+\nu)^2 + 1}{4}$$

  - Gaussian state: characteristic fns. are Gaussian. Mean fields and quadrature variances completely define the state

Statistics of measurement of $\hat{a}$ POVM on a squeezed state, $|\beta;\mu,\nu\rangle,\ \mu,\nu\in\mathbb{C}$

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- Mean, variances of measurement outcomes (α_1, α_2)
  - Measurement described by POVM

State	$\langle\alpha\rangle$
$\|n\rangle$	0
$\|\beta\rangle$	β
$\|\beta;\mu,\nu\rangle$	$\mu^*\beta-\nu\beta^*$

State	$\langle\Delta\alpha_1^2\rangle$	$\langle\Delta\alpha_2^2\rangle$
$\|n\rangle$	$(n+1)/2$	$(n+1)/2$
$\|\beta\rangle$	$1/2$	$1/2$
$\|\beta;\mu,\nu\rangle$	$(\|\mu-\nu\|^2+1)/4$	$(\|\mu+\nu\|^2+1)/4$

- Mean and variances for measuring $\hat{a}_1$ OR $\hat{a}_2$
  - Both are projective measurements (but cannot be done simultaneously)

Means $\langle\hat{a}_1\rangle$ and $\langle\hat{a}_2\rangle$ are just real and imaginary parts of the $\hat{a}$ POVM measurement's mean

State	$\langle\Delta\hat{a}_1^2\rangle$	$\langle\Delta\hat{a}_2^2\rangle$
$\|n\rangle$	$(2n+1)/4$	$(2n+1)/4$
$\|\beta\rangle$	$1/4$	$1/4$
$\|\beta;\mu,\nu\rangle$	$\left(\|\mu-\nu\|^2\right)/4$	$\left(\|\mu+\nu\|^2\right)/4$

Yuen-Shapiro vs. Caves notation

- Squeezed state, $|\beta;\mu,\nu\rangle,\ \mu,\nu\in\mathbb{C},\ |\mu|^2-|\nu|^2=1$
  - $\langle\hat{a}\rangle=\mu^*\beta-\nu\beta^*$, $\langle\hat{a}_1\rangle=\mathrm{Re}(\langle\hat{a}_1\rangle)$, $\langle\hat{a}_2\rangle=\mathrm{Re}(\langle\hat{a}_2\rangle)$
  - $\langle\hat{N}\rangle\equiv\langle\hat{a}^\dagger\hat{a}\rangle=|\langle\hat{a}\rangle|^2+|\nu|^2$
  - $\langle\Delta\hat{a}_1^2\rangle=|\mu-\nu|^2/4,\ \langle\Delta\hat{a}_2^2\rangle=|\mu+\nu|^2/4$
  - Useful notation to think of squeezing as phase-sensitive amplification with "gain", $G=|\mu|^2>1, G-1=|\nu|^2$
- Caves notation, $|\alpha;r,\theta\rangle\equiv|\alpha;\xi\rangle,\ \xi=re^{j\theta}$
  - Define, $\mu^*\beta-\nu\beta^*\equiv\alpha$. This implies: $\beta=\mu\alpha+\nu\alpha^*$
  - Define, $\mu=\cosh(r),\nu=e^{j\theta}\sinh(r)$, with $\xi=re^{j\theta}$ being the (complex-valued) "squeezing parameter"
  - $\langle\hat{a}\rangle=\alpha\equiv\alpha_1+j\alpha_2,\ \langle\hat{a}_1\rangle=\alpha_1,\langle\hat{a}_2\rangle=\alpha_2$
  - $\langle\hat{N}\rangle\equiv\langle\hat{a}^\dagger\hat{a}\rangle=|\alpha|^2+\sinh^2(r)$
  - Useful notation to picture "squeezing a coherent state"

Squeezing in Caves' notation

- Quadrature variances: $\langle \Delta \hat{a}_1^2 \rangle = |\mu - \nu|^2/4, \ \langle \Delta \hat{a}_2^2 \rangle = |\mu + \nu|^2/4$

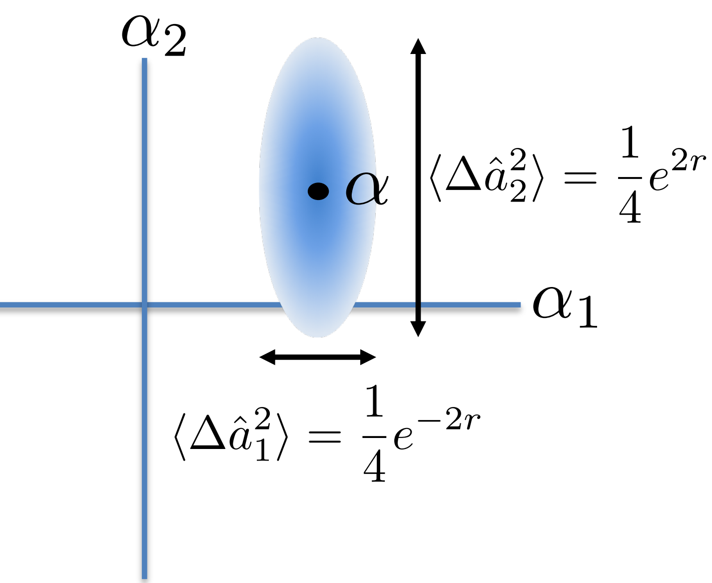
$$|\mu \pm \nu|^2 = |\mu|^2 + |\nu|^2 \pm 2\mathrm{Re}(\mu^* \nu)$$
$$= \sinh^2(r) + \cosh^2(r) \pm 2\mathrm{Re}(\mathrm{sinh(r)cosh(r)}e^{j\theta})$$

$$\frac{1}{4}|\mu \pm \nu|^2 = \frac{1}{4}\left[\cosh(2r) \pm \cos(\theta)\mathrm{sinhr}(2r)\right]$$

  - Recall condition for MUP: $\mu^* \nu \in \mathbb{R} \Rightarrow \sin(\theta) = 0 \Rightarrow \cos(\theta) = \pm 1$
  - If $\cos(\theta) = 1$, $\langle \Delta \hat{a}_1^2 \rangle = \frac{1}{4}e^{-2r}$, $\langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4}e^{2r}$
  - If $\cos(\theta) = -1, \ \langle \Delta \hat{a}_1^2 \rangle = \frac{1}{4}e^{2r}, \ \langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4}e^{-2r}$
  - Squeezing often measured in dB: $10\log_{10}\left(e^{2r}\right)$
    - Example: 3 dB squeezing: $e^{-2r} \approx 0.5$, $r = 0.3454$

Squeezed state in phase space

- Squeezed state, $|\beta;\mu,\nu\rangle \equiv |\alpha;\mu,\nu\rangle$
  - Coherent state is a special case, $\mu = 1, \nu = 0; r = 0, \theta = ?$
  - Wigner function $W(\alpha_1,\alpha_2)$ is a 2D Gaussian as shown:



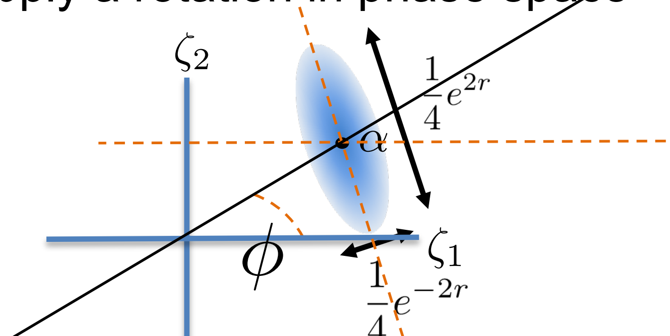
$$\mu = \cosh(r), \nu = \sinh(r), \theta = 0$$

$$\langle \hat{N} \rangle \equiv \langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2 + \sinh^2(r)$$

Let us redo everything we did, but with $\mu = e^{j\phi}\cosh(r),\ \ \nu = e^{j(\phi+\theta)}\sinh(r)$ (we had ignored one degree of freedom by setting $\phi = 0$ when going from Yuen-Shapiro to Caves notation)

MUP condition is the same as before:
$\mu^*\nu \in \mathbb{R} \Rightarrow \sin(\theta) = 0 \Rightarrow \cos(\theta) = \pm 1$

Effect of this phase: $\hat{b} \rightarrow \hat{b}e^{j\phi}$, is to apply a rotation in phase space



Q function of squeezed state

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- Squeezed state, $|\beta;\mu,\nu\rangle \equiv |\alpha;\mu,\nu\rangle$
  - Consider $\mu=\cosh(r), \nu=\sinh(r), \theta=0$
  - We already know what to expect, for the distribution of the $\hat{a}$ POVM measurement we derived

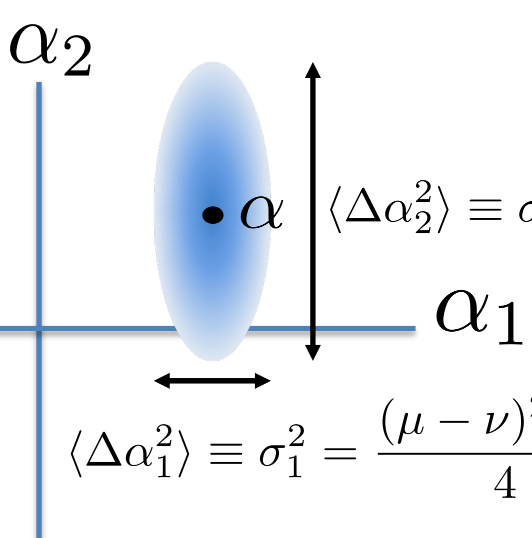
For $\mu,\nu\in\mathbb{R}$, we showed from first principles,

$$\chi_A(\zeta^*,\zeta) = e^{(\zeta\mu+\zeta^*\nu)\beta^*-(\zeta^*\mu+\zeta\nu)\beta-|\zeta|^2\mu^2-\mathrm{Re}(\zeta^2)\mu\nu}$$

$$= \exp\left[2j(\zeta_2\alpha_1-\zeta_1\alpha_2)-\zeta_1^2\left(\frac{1+e^{2r}}{2}\right)-\zeta_2^2\left(\frac{1+e^{-2r}}{2}\right)\right]$$

If we take the 2D fourier transform to obtain the Q function, we get a 2D Gaussian distribution with variances,

$$\sigma_1^2=\frac{1+e^{-2r}}{4}, \sigma_2^2=\frac{1+e^{2r}}{4}$$



$$\langle\Delta\alpha_2^2\rangle \equiv \sigma_2^2 = \frac{(\mu+\nu)^2+1}{4}$$

$$\langle\Delta\alpha_1^2\rangle \equiv \sigma_1^2 = \frac{(\mu-\nu)^2+1}{4}$$

Covariance matrix

- A Gaussian state can be completely determined from its first and second moments
  - First moment: $\langle \hat{a} \rangle = \langle \hat{a}_1 \rangle + j \langle \hat{a}_2 \rangle \in \mathbb{C}$ is the mean field
    - Note that: $\langle \hat{a} \rangle = \langle \alpha \rangle_P = \langle \alpha \rangle_Q = \langle \alpha \rangle_W$
  - Second moments form a **covariance matrix**:

$$V = \begin{pmatrix} \langle \hat{a}_1^2 \rangle & \langle \{\hat{a}_1, \hat{a}_2\} \rangle / 2 \\ \langle \{\hat{a}_1, \hat{a}_2\} \rangle / 2 & \langle \hat{a}_2^2 \rangle \end{pmatrix}$$

Poisson bracket

$$\left\{\hat{A}, \hat{B}\right\} = \hat{A}\hat{B} + \hat{B}\hat{A}$$

- Note that: $$V = \begin{pmatrix} \langle \alpha_1^2 \rangle_W & \langle \alpha_1 \alpha_2 \rangle_W \\ \langle \alpha_1 \alpha_2 \rangle_W & \langle \alpha_2^2 \rangle_W \end{pmatrix} = \begin{pmatrix} \langle \alpha_1^2 \rangle_Q & \langle \alpha_1 \alpha_2 \rangle_Q \\ \langle \alpha_1 \alpha_2 \rangle_Q & \langle \alpha_2^2 \rangle_Q \end{pmatrix} - I/4 = \begin{pmatrix} \langle \alpha_1^2 \rangle_P & \langle \alpha_1 \alpha_2 \rangle_P \\ \langle \alpha_1 \alpha_2 \rangle_P & \langle \alpha_2^2 \rangle_P \end{pmatrix} + I/4$$

Covariance matrix of a squeezed state

- For the squeezed state $|\beta;\mu,\nu\rangle,\ \mu,\nu\in\mathbb{C}$,
  - Mean field (first moment), $\langle\hat{a}\rangle = \mu^*\beta - \nu\beta^*$
  - Prove that the covariance matrix is given by:

$$V = \frac{1}{4}\begin{pmatrix} |\mu-\nu|^2 & -2\mathrm{Im}(\mu^*\nu) \\ -2\mathrm{Im}(\mu^*\nu) & |\mu+\nu|^2 \end{pmatrix}$$

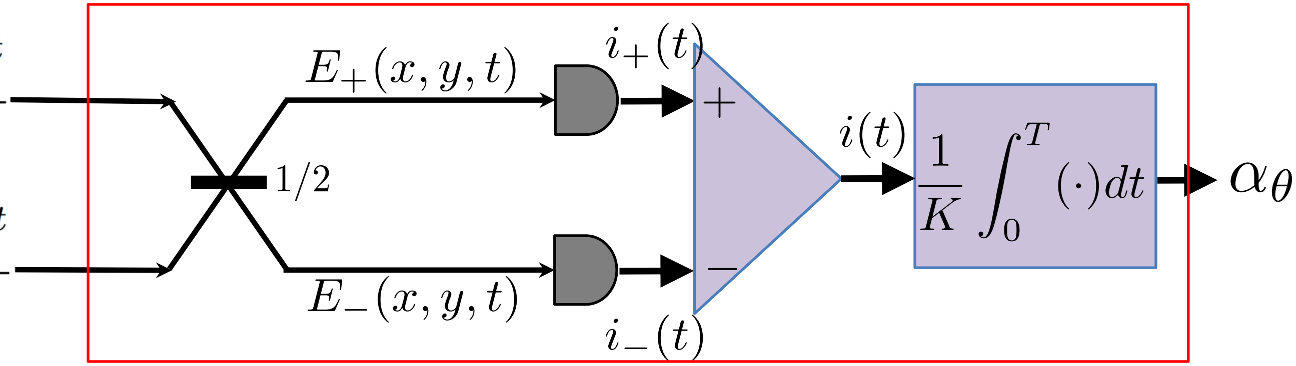
Diagonal terms are zero if the squeezed state is MUP

Problem 43

Revisiting Homodyne Detection: semi-classical theory, coherent state input

$$E_S(x,y,t) = \frac{a_S e^{-j\omega t}}{\sqrt{AT}}$$

$$E_{\rm LO}(x,y,t) = \frac{a_{\rm LO} e^{-j\omega t}}{\sqrt{AT}}$$



Flat-top modes in [0, T] and spatial area A;
unit: $\sqrt{\text{photon}/\text{m}^2\,\text{sec}}$

$a_{\rm LO} = \sqrt{N_{\rm LO}}\, e^{j\theta}$

$|a_S| \ll |a_{\rm LO}|$

$$E_\pm(x,y,t) = \frac{a_\pm e^{-j\omega t}}{\sqrt{AT}} \quad \text{where} \quad a_\pm \equiv \frac{a_S \pm a_{\rm LO}}{\sqrt{2}}$$

$\alpha_\theta = (qN_+ - qN_-)/K$ with $K = 2q\sqrt{N_{\rm LO}}$

$N_\pm \sim \text{Poisson}(|a_\pm|^2)$ S.I. random variables

$$\begin{aligned} M_{\alpha_\theta}(jv) &= \langle e^{jv(qN_+ - qN_-)/K}\rangle = \langle e^{jvqN_+/K}\rangle\langle e^{-jvqN_-/K}\rangle \\ &= \exp[|a_+|^2(e^{jvq/K}-1)]\exp[|a_-|^2(e^{-jvq/K}-1)] \end{aligned}$$

As $N_{\rm LO}\to\infty$ we have $K\to\infty$, $e^{\pm jvq/K}-1 \approx \pm jvq/K - v^2q^2/2K^2$ to second order

Also, $|a_\pm|^2 = \dfrac{N_{\rm LO} \pm 2{\rm Re}(a_S\sqrt{N_{\rm LO}}\,e^{-j\theta}) + |a_S|^2}{2} \approx \dfrac{N_{\rm LO} \pm 2{\rm Re}(a_S\sqrt{N_{\rm LO}}\,e^{-j\theta})}{2}$

$$M_{\alpha_\theta}(jv) = e^{jv{\rm Re}(a_S e^{-j\theta}) - v^2/8} \quad \Rightarrow \alpha_\theta \sim \mathcal{N}\left({\rm Re}(a_S e^{-j\theta}), 1/4\right)$$

Heterodyne detection

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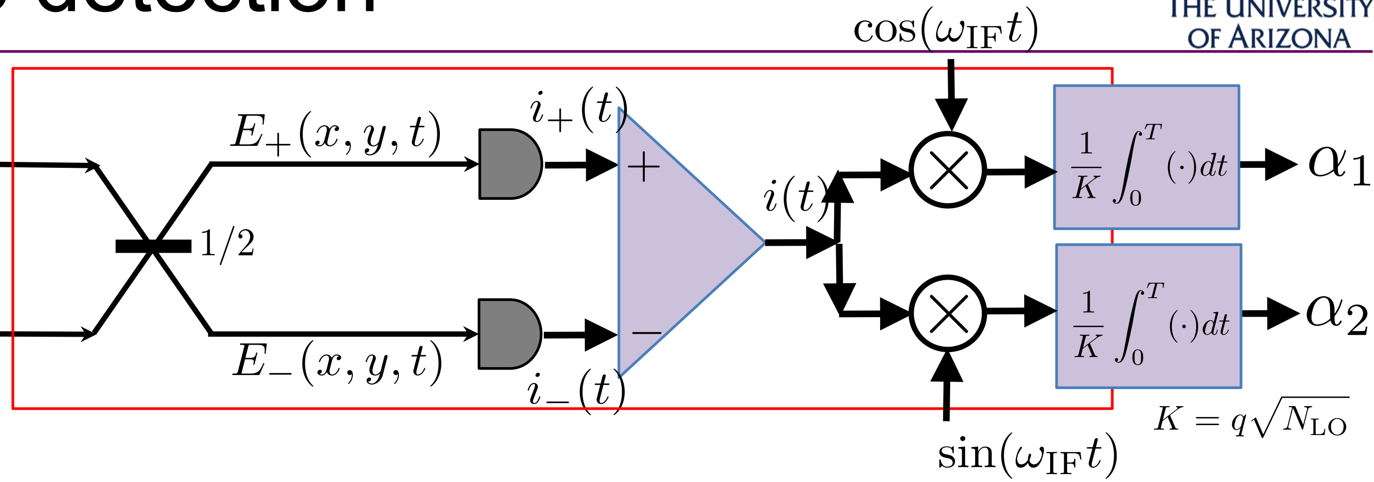
$$E_S(x,y,t) = \frac{a_S e^{-j\omega t}}{\sqrt{AT}}$$

$$E_{\mathrm{LO}}(x,y,t) = \frac{a_{\mathrm{LO}} e^{-j(\omega-\omega_{\mathrm{IF}})t}}{\sqrt{AT}}$$



Intermediate frequency: ω_{IF} low enough to be handled by post-photodetection electronics

$$\begin{aligned}\langle i_\pm(t)\rangle &= q\int_{\mathcal{A}} \mathrm{d}x\,\mathrm{d}y\,|E_\pm(x,y,t)|^2 = \frac{q}{2T}|a_S e^{-j\omega t} \pm a_{\mathrm{LO}} e^{-j(\omega-\omega_{\mathrm{IF}})t}|^2 \\ &= \frac{q}{2T}[|a_S|^2 + |a_{\mathrm{LO}}|^2 \pm 2\mathrm{Re}(a_S a_{\mathrm{LO}}^* e^{-j\omega_{\mathrm{IF}} t})],\end{aligned}$$

$$\lim_{N_{\mathrm{LO}}\to\infty} \langle[\langle i_+(t) - i_-(t)]\rangle / q\sqrt{N_{\mathrm{LO}}} = \frac{2\mathrm{Re}(a_S e^{-j\omega_{\mathrm{IF}} t})}{T} \quad \Rightarrow \langle \alpha_k\rangle = a_{S_k}, \quad \text{for } k = 1,2$$

Complete the proof, using a characteristic function derivation, to show that:

$$p(\alpha_1,\alpha_2) = \frac{e^{-|\alpha-\alpha_S|}}{\pi}, \;\; \alpha = \alpha_1 + j\alpha_2$$ i.e. $$\begin{aligned}\alpha_1 &\sim \mathcal{N}(\mathrm{Re}(\alpha_S), 1/2) \\ \alpha_2 &\sim \mathcal{N}(\mathrm{Im}(\alpha_S), 1/2)\end{aligned}$$ and they are S.I.

Advanced problem 1

Heterodyne detection is $\hat{a}$ POVM

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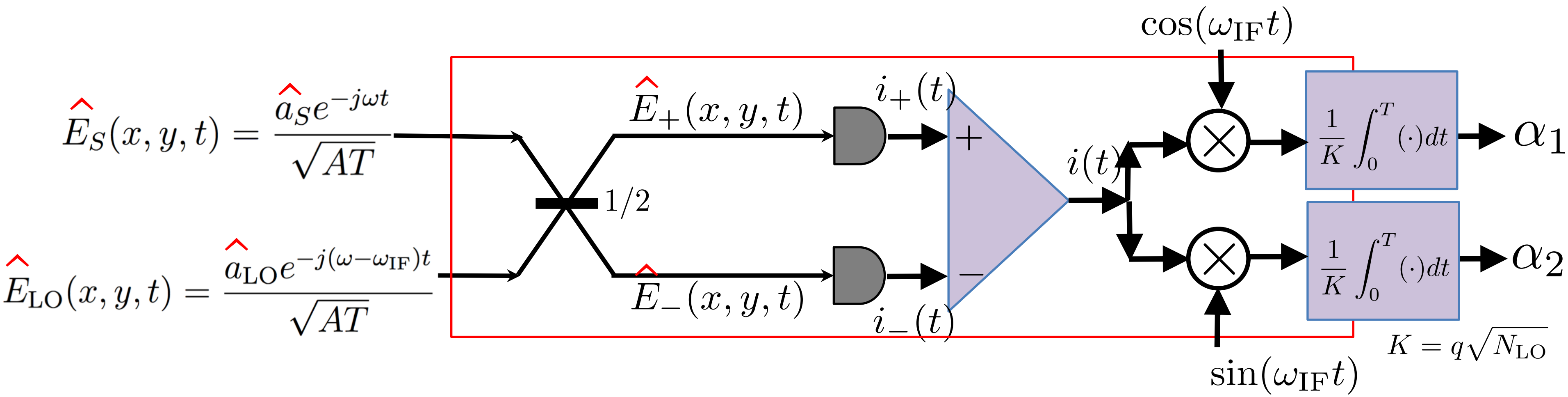

- Heterodyne detection is the measurement of the $\hat{a}_S$ POVM (i.e., no matter what is the quantum state of the $\hat{a}_S$ mode is), i.e., measurement in the basis of coherent states of the $\hat{a}_S$ mode

Heisenberg vs. Schrodinger interpretation

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- Schrodinger picture of QM:
  - States evolve under a unitary $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, $\hat{U}(t) = e^{i\hat{H}t}$
  - Hamiltonian of the evolution, $\hat{H}$
  - Hermitian operator for observable $\hat{A}$ stays constant
- Heisenberg picture:
  - State does not evolve, it stays $|\psi(0)\rangle \equiv |\psi\rangle$
  - Observable evolves as: $\hat{A}(t) = \hat{U}(t)^\dagger \hat{A}\hat{U}(t)$
- We will often find it convenient to use the Heisenberg picture to evolve states in optical transformations
  - We will evolve the field operators, and from the moments of the evolved field operators, deduce the output states

Field transformations in optics

- Linear (classical) transformations
  - Reversible (unitary)

$$E_{\text{in}}(x, y, t) \rightarrow E_{\text{out}}(x, y, t) = U E_{\text{in}}(x, y, t)$$

$U^* U = I$ Complex-valued unitary matrix

- Non-linear (classical) transformations
  - May not be reversible
    - Many examples from classical non-linear optics

$$E_{\text{in}}(x, y, t) \rightarrow E_{\text{out}}(x, y, t)$$

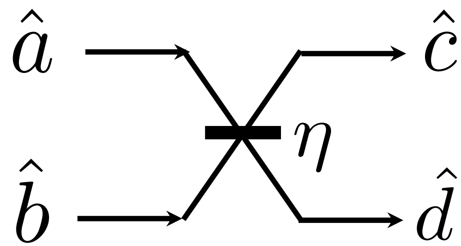
- Quantum transformations
  - Unitary (reversible)
  - Non-unitary (non-reversible)

$$\hat{E}_{\text{in}}(x, y, t) \rightarrow \hat{E}_{\text{out}}(x, y, t)$$

Cannot even talk in terms of a transformation on the “field”

Beam splitter as a quantum two-mode (unitary) transformation

Annihilation operators corresponding to two orthogonal modes



$$\begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = U \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}$$

Transmissivity, $\eta = \cos^2\theta \in [0, 1)$

$\theta \in (0, \pi/2]$

Phase, $\phi \in (0, 2\pi]$

$$\hat{c} = \sqrt{\eta}\hat{a} + e^{i\phi}\sqrt{1-\eta}\hat{b}$$

$$\hat{d} = \sqrt{1-\eta}\hat{a} - e^{i\phi}\sqrt{\eta}\hat{b}$$

$$U(\theta, \phi) = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ \sin\theta & -e^{i\phi}\cos\theta \end{pmatrix} \quad U^*U = UU^* = I$$

Unitary matrix

$$\chi_N^\rho(\zeta) = \langle e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} \rangle = e^{\zeta\alpha^* - \zeta^*\alpha}, \text{ for } \hat{\rho} = |\alpha\rangle\langle\alpha|$$

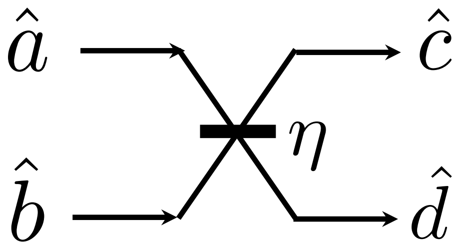
Let us consider the input modes in coherent states $|\alpha\rangle$ and $|\beta\rangle$; set $\phi = 0$

$$\chi_N^{\rho_c}(\zeta) = \langle e^{\zeta \hat{c}^\dagger} e^{-\zeta^* \hat{c}} \rangle = \langle e^{\zeta(\sqrt{\eta}\hat{a}^\dagger + \sqrt{1-\eta}\hat{b}^\dagger)} e^{-\zeta^*(\sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b})} \rangle$$

$$= \chi_N^{\rho_a}(\sqrt{\eta}\zeta)\, \chi_N^{\rho_b}(\sqrt{1-\eta}\zeta) = e^{\zeta(\sqrt{\eta}\alpha^* + \sqrt{1-\eta}\beta^*) - \zeta^*(\sqrt{\eta}\alpha + \sqrt{1-\eta}\beta)}$$

Therefore, state of the c mode is: $\hat{\rho}_c = |\gamma\rangle\langle\gamma|$, with $\gamma = \sqrt{\eta}\alpha + \sqrt{1-\eta}\beta$

Squeezed vacuum injection



$$\hat{c} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b}$$

$$\hat{d} = \sqrt{1-\eta}\hat{a} - \sqrt{\eta}\hat{b}$$

- Take input mode $\hat{a}$ in coherent state $|\alpha\rangle$ and $\langle\hat{b}\rangle = 0$

$$\mathrm{SNR}_a = \frac{\langle\hat{a}_1\rangle^2}{\langle\Delta\hat{a}_1^2\rangle} = 4|\alpha|^2$$

$$\mathrm{SNR}_c = \frac{\langle\hat{c}_1\rangle^2}{\langle\Delta\hat{c}_1^2\rangle} = \frac{4\eta|\alpha|^2}{\eta + 4(1-\eta)\langle\Delta\hat{b}_1^2\rangle} \qquad \mathrm{SNR}_d = \frac{\langle\hat{d}_1\rangle^2}{\langle\Delta\hat{d}_1^2\rangle} = \frac{4(1-\eta)|\alpha|^2}{1-\eta+4\eta\langle\Delta\hat{b}_1^2\rangle}$$

  - If $\hat{b}$ mode is in vacuum state $|0\rangle$

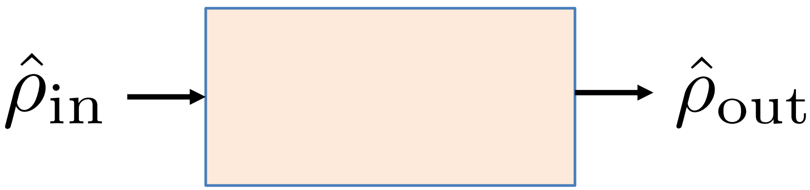
$$\mathrm{SNR}_c = 4\eta|\alpha|^2,\ \mathrm{SNR}_d = 4(1-\eta)|\alpha|^2$$
$$\mathrm{SNR}_a = \mathrm{SNR}_c + \mathrm{SNR}_d$$

  - If $\hat{b}$ mode is in squeezed vacuum $|0;\mu,\nu\rangle$ with $\langle\Delta\hat{b}_1^2\rangle = \frac{1}{4}e^{-2r}$
    - “SNR conservation” is not preserved!

$$\mathrm{SNR}_c \approx \mathrm{SNR}_d \approx \mathrm{SNR}_a = 4|\alpha|^2$$

Single-mode Phase

Schroedinger picture: state evolves under a unitary operation



$$U = e^{i\theta \hat{a}^\dagger \hat{a}}$$

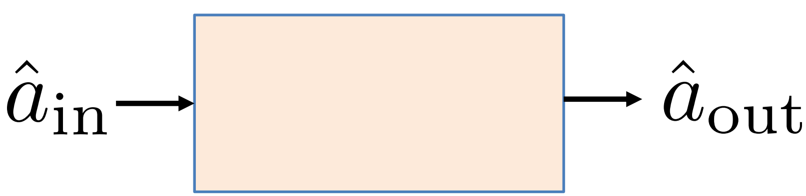
$$|\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$$

$$\hat{\rho}_{\text{out}} = U \hat{\rho}_{\text{in}} U^\dagger$$

Useful when calculating the output state directly

Example: input coherent state: $U|\alpha\rangle = |\alpha e^{i\theta}\rangle$

Heisenberg picture: operators evolve under a unitary operation; check to see, commutators preserved



$$\hat{a}_{\text{out}} = \hat{a}_{\text{in}} e^{i\theta} = U^\dagger \hat{a}_{\text{in}} U$$

Useful in transforming characteristic functions

Example: input coherent state:

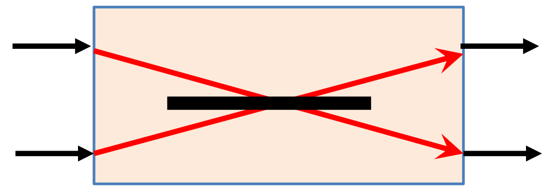
$$\chi_N^{\text{in}}(\zeta) = \langle e^{\zeta \hat{a}_{\text{in}}^\dagger} e^{-\zeta^* \hat{a}_{\text{in}}} \rangle = e^{\zeta \alpha^* - \zeta^* \alpha}$$

$$\chi_N^{\text{out}}(\zeta) = \langle e^{\zeta \hat{a}_{\text{out}}^\dagger} e^{-\zeta^* \hat{a}_{\text{out}}} \rangle = e^{\zeta e^{-i\theta} \alpha^* - \zeta^* e^{i\theta} \alpha}$$

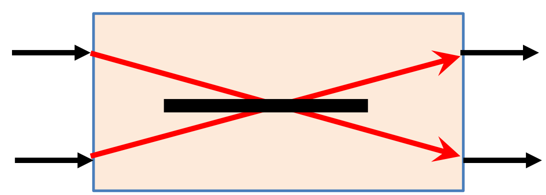
Two-mode Beamsplitter

$$U = e^{-\left[\arctan\sqrt{\eta^{-1}-1}\right](\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})}$$

Schroedinger

$\hat{\rho}_{\text{in}}$  $\hat{\rho}_{\text{out}}$

Heisenberg

$\hat{a}_{\text{in}}$, $\hat{b}_{\text{in}}$  $\hat{a}_{\text{out}}$, $\hat{b}_{\text{out}}$

$$\hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \sqrt{\eta}\,\hat{a}_{\text{in}} + \sqrt{1-\eta}\,\hat{b}_{\text{in}}$$
$$\hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1-\eta}\,\hat{a}_{\text{in}} + \sqrt{\eta}\,\hat{b}_{\text{in}}$$

$$\chi_A^{a_{\text{out}}}(\zeta) = \langle e^{-\zeta^* \hat{a}_{\text{out}}} e^{\zeta \hat{a}^\dagger_{\text{out}}} \rangle = \chi_A^{a_{\text{in}}}(\sqrt{\eta}\zeta)\,\chi_A^{b_{\text{in}}}(\sqrt{1-\eta}\zeta)$$

Phase sensing (phase conjugate MZI)

Assume, $|\phi| \ll 1$

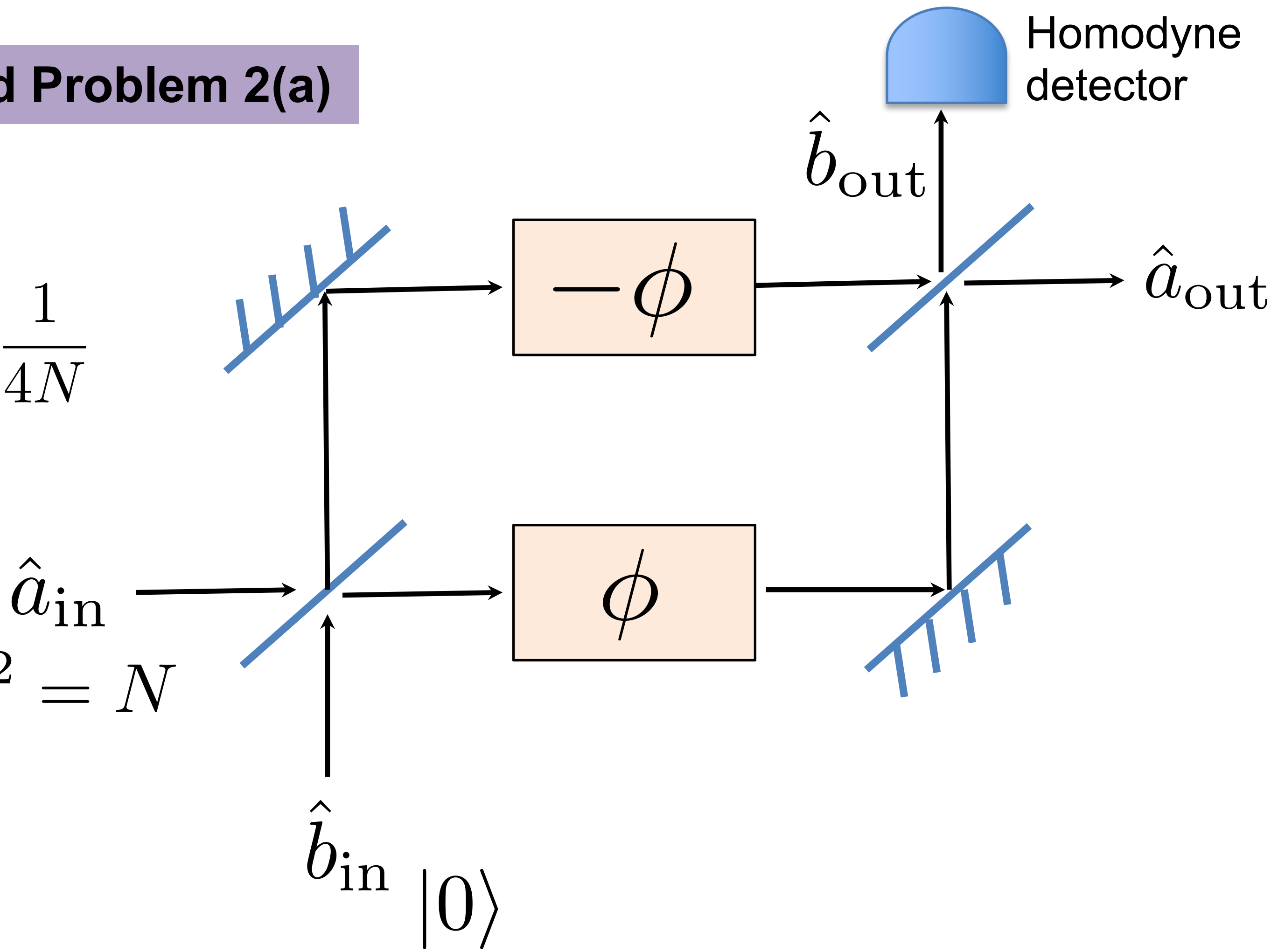
$$\hat{\phi} := -\mathrm{Im}(\hat{b}_{\mathrm{out}})/\sqrt{N}$$

Advanced Problem 2(a)

Show that:

$$\langle \hat{\phi} \rangle = \phi$$

$$\langle \Delta \hat{\phi}^2 \rangle = \frac{1}{4N}$$



Phase sensing with squeezed states

Assume, $|\phi| \ll 1$

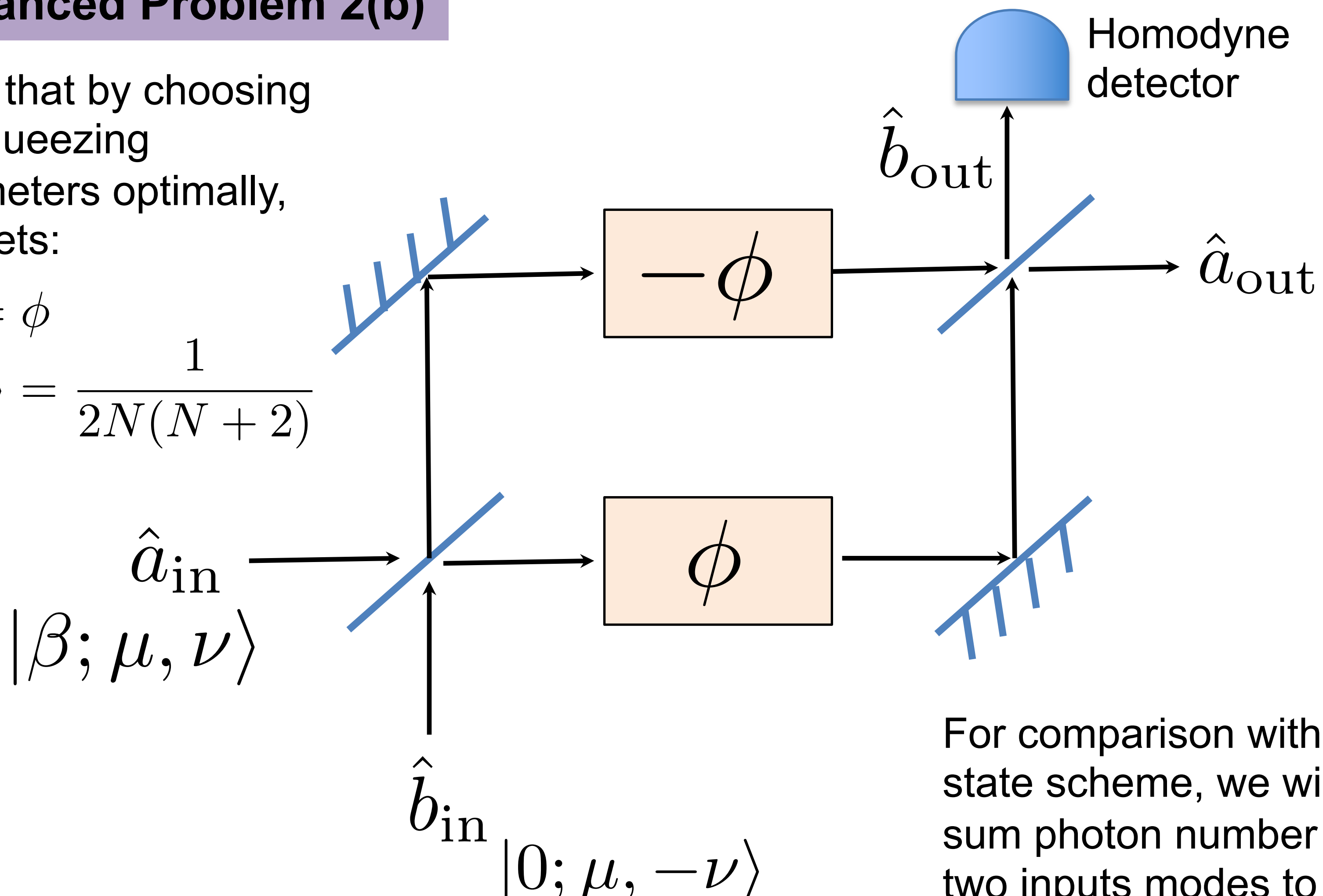
$$\hat{\phi} := -\mathrm{Im}(\hat{b}_{\mathrm{out}})/\sqrt{N - 2\nu^2}$$

Advanced Problem 2(b)

Show that by choosing the squeezing parameters optimally, one gets:

$$\langle \hat{\phi} \rangle = \phi$$

$$\langle \Delta \hat{\phi}^2 \rangle = \frac{1}{2N(N+2)}$$



For comparison with coherent state scheme, we will take the sum photon number over the two inputs modes to be N

Single-mode Squeezing (preview)

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Schroedinger

$$\hat{\rho}_{\rm in} \longrightarrow \boxed{} \longrightarrow \hat{\rho}_{\rm out}$$

$$U = \exp\left[\frac{r}{2}\left(e^{-i\theta}\hat{a}^2 - e^{i\theta}\hat{a}^{\dagger 2}\right)\right]$$

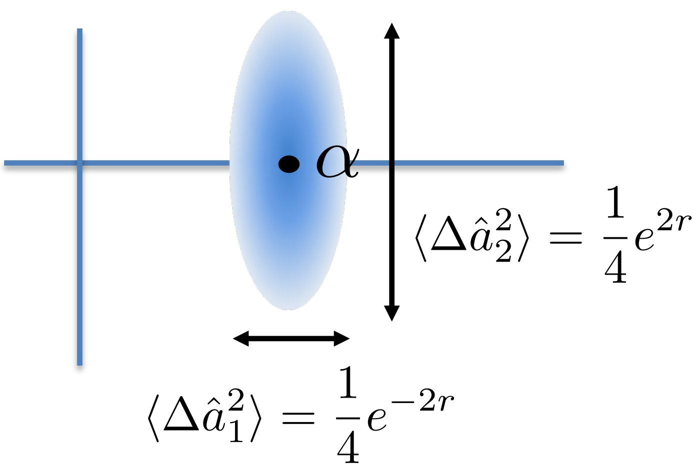
$$\equiv \hat{S}(z),\ z = re^{i\theta}$$

$$|\alpha; r, \theta\rangle = \hat{D}(\alpha)\hat{S}(z)|0\rangle = \hat{S}(z)\hat{D}(\beta)|0\rangle$$

$$\beta = \mu\alpha + \nu\alpha^*$$

$$\mu = \cosh r$$

$$\nu = e^{i\theta}\sinh r.$$



$$\theta = 0, \alpha \in \mathbb{R}$$

$$\beta = \alpha e^r$$

Mean photon number,

$$N = |\alpha|^2 + \sinh^2(r)$$

Heisenberg

$$\hat{a}_{\rm in} \longrightarrow \boxed{} \longrightarrow \hat{a}_{\rm out}$$

Squeezing is not passive. It adds photons

$$\hat{a}_{\rm out} = \hat{S}(z)\hat{a}_{\rm in}\hat{S}(z)^\dagger = \mu\hat{a}_{\rm in} + \nu\hat{a}_{\rm in}^\dagger$$

Single-mode Displacement (preview)

Schroedinger

$$\hat{\rho}_{\text{in}} \longrightarrow \boxed{} \longrightarrow \hat{\rho}_{\text{out}}$$

$$U = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \equiv \hat{D}(\alpha)$$

$$|\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$$

$$\hat{\rho}_{\text{out}} = U \hat{\rho}_{\text{in}} U^\dagger$$

Example: input coherent state: $U|\beta\rangle = |\beta + \alpha\rangle$

Heisenberg

$$\hat{a}_{\text{in}} \longrightarrow \boxed{} \longrightarrow \hat{a}_{\text{out}}$$

$$\hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \hat{a}_{\text{in}} + \alpha$$

$$\hat{a}^\dagger_{\text{out}} = U^\dagger \hat{a}^\dagger_{\text{in}} U = \hat{a}^\dagger_{\text{in}} + \alpha^*$$

$$\chi_N^{\text{in}}(\zeta) = \langle e^{\zeta \hat{a}^\dagger_{\text{in}}} e^{-\zeta^* \hat{a}_{\text{in}}} \rangle = e^{\zeta \beta^* - \zeta^* \beta}$$

$$\chi_N^{\text{out}}(\zeta) = \langle e^{\zeta \hat{a}^\dagger_{\text{out}}} e^{-\zeta^* \hat{a}_{\text{out}}} \rangle = \langle e^{\zeta (\hat{a}^\dagger_{\text{in}} + \alpha^*)} e^{-\zeta^* (\hat{a}_{\text{in}} + \alpha)} \rangle = e^{\zeta (\beta + \alpha)^* - \zeta^* (\beta + \alpha)}$$

Gaussian transformations

- Phase (1 → 1)
  - $U_{\text{phase}}(\theta), \theta \in [0, 2\pi)$
- Beam splitter (2 → 2)
  - $U_{\text{beamsplitter}}(\eta), \eta \in [0, 1)$
- Squeezing (1 → 1)
  - $U_{\text{squeezing}}(r) = \hat{S}(z), z = r \in [0, \infty)$
- Displacement (1 → 1)
  - $U_{\text{disp}}(\alpha) = \hat{D}(\alpha), \alpha \in \mathbb{C}$

general n-mode passive linear optical transformation

General zero-mean Gaussian unitary (an n-mode bogoliubov transformation)

General Gaussian transformation

Gaussian transformations not universal. Need any one non-Gaussian unitary

- Phase (1 → 1)
  - $U_{\text{phase}}(\theta), \theta \in [0, 2\pi)$
- Beam splitter (2 → 2)
  - $U_{\text{beamsplitter}}(\eta), \eta \in [0, 1)$
- Squeezing (1 → 1)
  - $U_{\text{squeezing}}(r) = \hat{S}(z), z = r \in [0, \infty)$
- Displacement (1 → 1)
  - $U_{\text{disp}}(\alpha) = \hat{D}(\alpha), \alpha \in \mathbb{C}$
- Self-Kerr (1 → 1)
  - $U(\kappa) = e^{i\kappa(\hat{a}^\dagger \hat{a})^2}$

general n-mode passive linear optical transformation

General zero-mean Gaussian unitary (an n-mode bogoliubov transformation)

General Gaussian transformation

General unitary transformation on n modes: universal quantum processing

Gaussian transformations not universal. Need any one non-Gaussian unitary

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- Phase (1 → 1)
  - $U_{\text{phase}}(\theta), \theta \in [0, 2\pi)$
- Beam splitter (2 → 2)
  - $U_{\text{beamsplitter}}(\eta), \eta \in [0, 1)$
- Squeezing (1 → 1)
  - $U_{\text{squeezing}}(r) = \hat{S}(z), z = r \in [0, \infty)$
- Displacement (1 → 1)
  - $U_{\text{disp}}(\alpha) = \hat{D}(\alpha), \alpha \in \mathbb{C}$
- Cubic phase (1 → 1)
  - $U(\gamma) = e^{i\gamma\hat{q}^3}, \hat{q} = \dfrac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$

general n-mode passive linear optical transformation

General zero-mean Gaussian unitary (bogoliubov transformation)

General Gaussian transformation

General unitary transformation on n modes: universal quantum processing

Can you name a 1-mode quantum state in each quadrant?

Classical Gaussian	Classical Non-Gaussian
Non-Classical Gaussian	Non-Classical Non-Gaussian

Upcoming topics…

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- Christos Gagatsos will teach a few lectures:
- Unitary transformation of bosonic states
  - Gaussian unitary on n modes: Phase, Beamsplitter, Displacement, Squeezing
  - Non-Gaussian state engineering using PNR detection
  - Gaussian Boson Sampling
- Gaussian measurements on n modes