Photonic Quantum Information Processing
OPTI 647: Lecture 8

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Plan for today

- Phase space representations of single-mode quantum states

Recap of Lecture 7

- Bogoliubov transformation $\hat{b} \equiv \mu\hat{a} + \nu\hat{a}^\dagger,\ \mu,\nu \in \mathbb{C},\ |\mu|^2 - |\nu|^2 = 1$
  - Squeezed state $|\beta;\mu,\nu\rangle$ is a coherent state of $\hat{b}$
  - Mean (field), $\langle\hat{a}\rangle = \langle\beta;\mu,\nu|\hat{a}|\beta;\mu,\nu\rangle = \mu^*\beta - \nu\beta^*$
  - Mean photon number, $\langle\hat{N}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = |\langle\hat{a}\rangle|^2 + |\nu|^2$
  - Quadrature variances, $\langle\Delta\hat{a}_1^2\rangle = \dfrac{|\mu-\nu|^2}{4},\ \langle\Delta\hat{a}_2^2\rangle = \dfrac{|\mu+\nu|^2}{4}$
  - Squeezed state is MUP, i.e., $\langle\Delta\hat{a}_1^2\rangle\langle\Delta\hat{a}_2^2\rangle = 1/16$ if $\mu^*\nu \in \mathbb{R}$
- Caves notation $|\alpha;r,\theta\rangle$
  - Relationship to the Yuen-Shapiro notation $\beta = \mu\alpha + \nu\alpha^*$
  - Mean field, $\langle\hat{a}\rangle = \alpha$ $\quad \mu = \cosh r$
  - Mean photon number, $\langle\hat{N}\rangle = |\alpha|^2 + \sinh^2(r)$ $\quad \nu = e^{i\theta}\sinh r.$
  - Quadrature variances when MUP, i.e., $\theta = 0,\pi$

    $\langle\hat{\Delta}a_1^2\rangle = \frac{1}{4}e^{-2r},\quad \langle\hat{\Delta}a_2^2\rangle = \frac{1}{4}e^{+2r}$ (or vice versa)

Recap of Lecture 7 (continued)

- Signal to noise ratio of real-quadrature homodyne detection, $\mathrm{SNR} \equiv \langle \hat{a}_1 \rangle^2 / \langle \Delta \hat{a}_1^2 \rangle$ under a mean photon number constraint, $\langle \hat{a}^\dagger \hat{a} \rangle \leq N$ is attained by squeezed state: $\beta = \sqrt{N(N+1)}, \mu = (N+1)/\sqrt{2N+1}, \nu = N/\sqrt{2N+1}$
- Binary phase shift keying (BPSK) modulation with squeezed states $|\psi_0\rangle = |\beta; \mu, \nu\rangle, |\psi_1\rangle = |-\beta; \mu, \nu\rangle$, with mean photon number N, but with β, μ, ν optimally chosen attains $P_{e,\min} = [1 - \sqrt{1 - e^{-4N(N+1)}}]/2 \sim e^{-4N^2}$, but with BPSK coherent states, $P_{e,\min} = [1 - \sqrt{1 - e^{-4N}}]/2 \sim e^{-4N}$

Recap of Lecture 7 (continued)

- Characteristic function = FT of probability density

$$M_X(jv) \equiv \langle e^{jvX} \rangle = \int_{-\infty}^{\infty} p_X(x) e^{jvx} dx \qquad p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_X(jv) e^{-jvx} dv$$

- Measuring $\hat{a}_1$ on $|\psi\rangle$: output is a random variable X_1
  - $M_{X_1}(jv) \equiv \langle e^{jvX_1} \rangle_{p_{X_1}(x)} = \langle e^{jv\hat{a}_1} \rangle_{|\psi\rangle} = \langle \psi | e^{jv\hat{a}_1} | \psi \rangle$
- Wigner characteristic function (of a state $|\psi\rangle$)

$$\chi_W(\zeta^*, \zeta) \equiv \chi_W(\zeta) = \langle e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} \rangle = \langle \psi | e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} | \psi \rangle, \ \zeta = \zeta_1 + j\zeta_2$$

  - Characteristic fn. of $\hat{a}_1$ meas. outcome, $M_{X_1}(jv) = \chi_W(\zeta)\big|_{\zeta = jv/2}$
  - Characteristic fn. of $\hat{a}_2$ meas. outcome, $M_{X_2}(jv) = \chi_W(\zeta)\big|_{\zeta = -v/2}$

Baker-Campbell-Hausdorff theorem

- Exponential of an operator $e^{\hat{C}} \equiv \sum_{n=0}^{\infty} \frac{\hat{C}^n}{n!}$
  - If $\left[\hat{A}, [\hat{A}, \hat{B}]\right] = \left[\hat{B}, [\hat{A}, \hat{B}]\right] = 0$
  - then, $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]/2}$
- Consider the Wigner c.f., and $\hat{A} = -\zeta^* \hat{a}, \hat{B} = \zeta \hat{a}^\dagger$
  - We then get, $[\hat{A}, \hat{B}] = -|\zeta|^2 [\hat{a}, \hat{a}^\dagger] = -|\zeta|^2$
- Define:
  - Antinormally-ordered characteristic function

$$\chi_A(\zeta) \equiv \langle e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger} \rangle$$

  - Normally-ordered characteristic function

$$\chi_N(\zeta) \equiv \langle e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} \rangle$$

  - Show that for any state, **Problem 36**

$$\chi_W(\zeta) = \chi_A(\zeta) e^{|\zeta|^2/2} = \chi_N(\zeta) e^{-|\zeta|^2/2}$$

Homodyne detection statistics on a number state

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- $M_{X_1}(jv) = \chi_W(\zeta)\big|_{\zeta=jv/2} = [\chi_N(\zeta)e^{-|\zeta|^2/2}]\big|_{\zeta=jv/2}$

$$= [\langle n|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|n\rangle e^{-|\zeta|^2/2}]\big|_{\zeta=jv/2}$$

- $M_{X_1}(jv) = \left[\left(\sum_{m=0}^{\infty}\frac{\zeta^m}{m!}\langle n|\hat{a}^{\dagger m}\right)\left(\sum_{k=0}^{\infty}\frac{(-\zeta^*)^k}{k!}\hat{a}^k|n\rangle\right)e^{-|\zeta|^2/2}\right]_{\zeta=jv/2}$

$$= \left(\sum_{m=0}^{n}\frac{(jv/2)^m}{m!}\sqrt{\frac{n!}{(n-m)!}}\langle n-m|\right)\left(\sum_{k=0}^{n}\frac{(jv/2)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle\right)e^{-v^2/8}$$

$$= \left(\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{(-v^2/4)^m}{m!}\right)e^{-v^2/8} = L_n(v^2/4)e^{-v^2/8},$$

where

$$L_n(z) \equiv \sum_{m=0}^{n}(-1)^m\frac{n!}{m!(n-m)!}\frac{z^m}{m!},$$

Homodyne detection statistics on a number state (continued)

- Probability distribution function,

$$p_{X_1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L_n(v^2/4)e^{-v^2/8}e^{-jvx}dx$$

$$= \frac{2}{\pi}\frac{e^{-2x^2}}{2^n n!}[H_n(\sqrt{2}x)]^2$$

  – where $H_n(z) = (-1)^n e^{z^2}\frac{d^n e^{-z^2}}{dz^n}$ is the n^{th} Hermite polynomial

The "Wigner function": a quasi-probability

- FT of the characteristic function of $\hat{a}_1$ measurement which is $M_{X_1}(jv) = \chi_W(\xi)\big|_{jv/2}$,gives us the pdf $p_{X_1}(x)$
- What if we took a (2D) FT of the full Wigner function. Will it give us some sort of a pdf of measuring both quadratures together? But we know it is not possible to measure them both together!
- Define $W(\alpha^*, \alpha) \equiv \int \frac{\mathrm{d}^2\zeta}{\pi^2} \chi_W(\zeta^*, \zeta) e^{\zeta^* \alpha - \zeta \alpha^*}$,

$$\int \frac{\mathrm{d}^2\zeta}{\pi^2} \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta_1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta_2}{\pi}, \quad \zeta^*\alpha - \zeta\alpha^* = 2j\zeta_1\alpha_2 - 2j\zeta_2\alpha_1$$

$$\chi_W(\zeta^*, \zeta) = \int \mathrm{d}^2\alpha \, W(\alpha^*, \alpha) e^{-\zeta^*\alpha + \zeta\alpha^*}$$

Wigner function of a coherent state

- Verify that $\int \mathrm{d}^2\alpha\, W(\alpha^*, \alpha) = 1$

- Consider coherent state $|\beta\rangle$

$$\chi_W(\zeta^*, \zeta) = \langle\beta|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|\beta\rangle e^{-|\zeta|^2/2} = e^{\zeta\beta^* - \zeta^*\beta}e^{-|\zeta|^2/2}$$

$$W(\alpha^*, \alpha) = \frac{e^{-2|\alpha-\beta|^2}}{\pi/2}$$

  – Two S.I. Gaussian random variables each with variance ¼ and means β_1 and β_2 respectively

Wigner function of a number state $|n\rangle$

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- Evaluate $W(\alpha^*,\alpha) = \int \frac{\mathrm{d}^2\zeta}{\pi^2} L_n(|\zeta|^2) e^{-|\zeta|^2/2} e^{\zeta^*\alpha - \zeta\alpha^*}$

$$W(\alpha^*,\alpha) = \frac{2}{\pi}\int_0^\infty \mathrm{d}r\, r L_n(r^2) e^{-r^2/2} J_0(2r|\alpha|) = (-1)^n \frac{2}{\pi} L_n(4|\alpha|^2) e^{-2|\alpha|^2}$$

- For number state $|0\rangle$, $L_0(z) = 1$

$$W(\alpha^*,\alpha) = \frac{2}{\pi} e^{-2|\alpha|^2}$$

- For number state $|1\rangle$, $L_1(z) = (1 - z)$

$$W(\alpha^*,\alpha) = \frac{2}{\pi}(4|\alpha|^2 - 1) e^{-2|\alpha|^2} \quad < 0, |\alpha| < \frac{1}{2}$$

- $W(\alpha) < 0$ *not* necessary for non-classicality

Normal and anti-normal ordered forms

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- Normal-ordered form: $F(\hat{a}^\dagger,\hat{a}) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{nm}\hat{a}^{\dagger n}\hat{a}^m$
- Anti-normal-ordered form: $G(\hat{a},\hat{a}^\dagger) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g_{nm}\hat{a}^{n}\hat{a}^{\dagger m}$
- A few simple exercises: **Problem 37**
  - Write normal-ordered form $\hat{F}^{(n)}(\hat{a}^\dagger,\hat{a})$ of operator $\hat{F} = \hat{a}\hat{a}^\dagger\hat{a}$
  - Prove that for any operator F, $\langle\alpha|\hat{F}|\alpha\rangle = F^{(n)}(\alpha^*,\alpha)$
- Q function
  - In a previous problem, we showed that the normal-order form of the density operator, $\rho^{(n)}(\alpha^*,\alpha) \equiv \langle\alpha|\hat{\rho}|\alpha\rangle$ scaled by π, i.e., $Q(\alpha) = \langle\alpha|\hat{\rho}|\alpha\rangle/\pi$ is a proper pdf

Characteristic fns. of a coherent state

- For a coherent state, $\hat{\rho} = |\alpha\rangle\langle\alpha|$
  - Normal-ordered characteristic function,

$$\begin{aligned}\chi_N^\rho(\zeta^*, \zeta) &\equiv \mathrm{tr}\Big(\hat{\rho} e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) = \mathrm{tr}\Big(|\alpha\rangle\langle\alpha| e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) \\ &= \langle\alpha| e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} |\alpha\rangle = e^{\zeta\alpha^* - \zeta^*\alpha}.\end{aligned}$$

  - Wigner characteristic function,

$$\chi_W^\rho(\zeta^*, \zeta) = \chi_N^\rho(\zeta^*, \zeta) e^{-|\zeta|^2/2} = e^{\zeta\alpha^* - \zeta^*\alpha - |\zeta|^2/2}$$

  - Anti-normally-ordered characteristic function,

$$\chi_A^\rho(\zeta^*, \zeta) = \chi_W^\rho(\zeta^*, \zeta) e^{-|\zeta|^2/2} = e^{\zeta\alpha^* - \zeta^*\alpha - |\zeta|^2}$$

- All the characteristic functions are Gaussian

Anti-normal-ordered characteristic function and Q function are FT pairs

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- Consider,

$$\chi_A^{\hat\rho}(\zeta^*,\zeta) \equiv \mathrm{tr}\left(\hat\rho e^{-\zeta^*\hat a} e^{\zeta \hat a^\dagger}\right)$$

$$\hat I = \int_{\mathbb{C}} \frac{|\alpha\rangle\langle\alpha|}{\pi} d^2\alpha$$

$$\begin{aligned}
\chi_A^{\rho}(\zeta^*,\zeta) &= \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\left(\hat\rho e^{-\zeta^*\hat a}|\alpha\rangle\langle\alpha| e^{\zeta\hat a^\dagger}\right)\\
&= \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\left(\hat\rho|\alpha\rangle\langle\alpha| e^{-\zeta^*\alpha+\zeta\alpha^*}\right)\\
&= \int \frac{d^2\alpha}{\pi}\,\langle\alpha|\hat\rho|\alpha\rangle e^{-\zeta^*\alpha+\zeta\alpha^*}\\
&= \iint d\alpha_1\, d\alpha_2\, \rho^{(n)}(\alpha^*,\alpha)\frac{e^{2j\zeta_2\alpha_1-2j\zeta_1\alpha_2}}{\pi}\\
&= \left.\frac{\mathcal{F}[\rho^{(n)}(\alpha^*,\alpha)]}{\pi}\right|_{\substack{f_1=-\zeta_2/\pi\\ f_2=\zeta_1/\pi}},
\end{aligned}$$

- 2D Fourier Transform

$$X(f_1,f_2)=\mathcal{F}[x(t_1,t_2)] \equiv \iint dt_1\, dt_2\, x(t_1,t_2)e^{-j2\pi(f_1t_1+f_2t_2)}$$

$$x(t_1,t_2)=\mathcal{F}^{-1}[X(f_1,f_2)] \equiv \iint df_1\, df_2\, X(f_1,f_2)e^{j2\pi(f_1t_1+f_2t_2)}$$

Characteristic functions and Probability distributions are FT pairs

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- Anti-normal ordered:

$$\chi_A(\zeta) = \int Q(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

Always a proper probability density function; pdf for ideal heterodyne detection

$$Q(\alpha) = \frac{1}{\pi^2} \int \chi_A(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

- Wigner:

$$\chi_W(\zeta) = \int W(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

May not be a proper probability density function. Negativity used to show a state is non-classical

$$W(\alpha) = \frac{1}{\pi^2} \int \chi_W(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

Recall that: $\chi_W(\zeta) = e^{|\zeta|^2/2} \chi_A(\zeta)$

Normally ordered characteristic function

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This step holds ONLY IF $\hat{\rho}$ is a classical state, $\hat{\rho} = \int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$

- $$\begin{aligned} \chi_N^\rho(\zeta^*,\zeta) &\equiv \mathrm{tr}\left(\hat{\rho}e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}\right) \\ &= \int d^2\alpha\, P(\alpha,\alpha^*)\mathrm{tr}\left(|\alpha\rangle\langle\alpha|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}\right) \\ &= \int d^2\alpha\, P(\alpha,\alpha^*)e^{\zeta\alpha^*-\zeta^*\alpha} \\ &= \int d^2\alpha\, P(\alpha,\alpha^*)e^{2j\zeta_2\alpha_1-2j\zeta_1\alpha_2} \end{aligned}$$

- Normally-ordered:

$$\chi_N(\zeta) = \int P(\alpha)e^{\zeta\alpha^*-\zeta^*\alpha}d^2\alpha$$

Always a proper probability density function *when it exists.* The states for which a proper P function exists are called classical states →

$$P(\alpha) = \frac{1}{\pi^2}\int \chi_N(\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}d^2\zeta$$

Characteristic functions versus probability (or quasi-probability) distributions

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Always a proper probability density function; pdf for ideal heterodyne detection

$$\chi_A(\zeta) = \int Q(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

$$Q(\alpha) = \frac{1}{\pi^2} \int \chi_A(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

May not be a proper probability density function. Negativity used to show a state is non-classical

$$\chi_W(\zeta) = \int W(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

$$W(\alpha) = \frac{1}{\pi^2} \int \chi_W(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

Always a proper probability density function *when it exists.* The states for which a proper P function exists are called classical states

$$\chi_N(\zeta) = \int P(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

$$P(\alpha) = \frac{1}{\pi^2} \int \chi_N(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

Few problems

- Prove the following:
  - $$W(\alpha) = \frac{2}{\pi}\int P(\beta)e^{-2|\beta-\alpha|^2}d^2\beta$$ **Problem 38**
  - $$\langle \hat{a}^n \hat{a}^{\dagger m}\rangle = \int \alpha^n \alpha^{*m} Q(\alpha) d^2\alpha$$ **Problem 39**

Retrieve density operator from characteristic functions (coming back the full circle!)

- Start with: $\dfrac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi} = Q(\alpha) = \dfrac{1}{\pi^2}\displaystyle\int \chi_A(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$
- Therefore, $\langle\alpha|\hat{\rho}|\alpha\rangle = \dfrac{1}{\pi}\displaystyle\int \chi_A(\zeta)\langle\alpha|e^{-\zeta\hat{a}^\dagger}e^{\zeta^*\hat{a}}|\alpha\rangle d^2\zeta$
  - Recall that the density operator is completely determined by its coherent state elements. Hence,

$$\hat{\rho} = \frac{1}{\pi}\int \chi_A(\zeta) e^{-\zeta\hat{a}^\dagger} e^{\zeta^*\hat{a}} d^2\zeta$$

  - We already know: $\chi_A(\zeta) = \mathrm{Tr}\left(\hat{\rho}e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger}\right)$
  - The above two relationships are an operator FT
  - $\langle n|\hat{\rho}|m\rangle = \dfrac{1}{\pi}\displaystyle\int \chi_A(\zeta)\langle n|e^{-\zeta\hat{a}^\dagger}e^{\zeta^*\hat{a}}|m\rangle d^2\zeta = \dfrac{1}{\pi}\displaystyle\int \chi_A(\zeta)\sqrt{\dfrac{n!}{m!}}(-\zeta)^{m-n}L_n^{(m-n)}(|\zeta|^2)d^2\zeta$

Circularly-symmetric in the phase space $\equiv$ diagonal in number basis

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- The phase-space representation of a state
  - Wigner $W(\alpha)$, Q-function $Q(\alpha)$, P-function $P(\alpha)$
  - P function may or may not exist
- If the Q function is circulatly symmetric, i.e., just a function of $|\alpha|, \alpha = \alpha_1 + j\alpha_2$
  - The antinormally-ordered characteristic function must also be circularly symmetric $\chi_A(\zeta)$ (FT pair)
  - If so, prove that the state is diagonal in the number basis, i.e., $\langle n|\hat{\rho}|m\rangle = \langle n|\hat{\rho}|n\rangle\, \delta_{mn}$ **Problem 40**
- All number diagonal states, $\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|$ are circularly-symmetric states
  - The number states $\{|n\rangle\}, n = 0, 1, \ldots$ are the only circularly-symmetric pure states

Number detection on a classical state

- Consider a classical state, $\hat{\rho} = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|$

$$\Pr(\,\hat{N}\text{ outcome } = n \mid \text{state is } |\alpha\rangle\,) = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

$$\Pr(\hat{N}\text{ outcome } = n) = \int d^2\alpha\, P(\alpha, \alpha^*)\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

$$\begin{aligned}\langle\Delta\hat{N}^2\rangle &\geq \int d^2\alpha\, P(\alpha, \alpha^*)\text{var}(\,\hat{N}\text{ measurement } \mid \text{ state is } |\alpha\rangle\,)\\ &= \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha|^2,\end{aligned}$$

$$\langle\hat{N}\rangle = \int d^2\alpha\, P(\alpha, \alpha^*)E(\,\hat{N}\text{ measurement } \mid \text{ state is } |\alpha\rangle\,) = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha|^2$$

Therefore, for a state that admits a proper P-function representation, $\langle\Delta\hat{N}^2\rangle \geq \langle\hat{N}\rangle$

$$\text{number state } |n\rangle \text{ has } \langle\hat{N}\rangle = n \text{ and } \langle\Delta\hat{N}^2\rangle = 0$$

$$\hat{\rho} = |n\rangle\langle n| \text{ does } \textit{not} \text{ have a proper } P\text{-representation for } n > 0$$

Homodyne detection on a classical state

- Consider a classical state, $\hat{\rho} = \int d^2\alpha \, P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|$
  - If $\hat{a}_1$ measurement is done on coherent state $|\alpha\rangle$, we get a Gaussian r.v. with mean $\alpha_1 = \mathrm{Re}(\alpha)$ and variance $1/4$
  - So, $p(\hat{a}_1 \text{ outcome } = a_1) = \int d^2\alpha \, P(\alpha, \alpha^*) \frac{\exp[-2(a_1 - \alpha_1)^2]}{\sqrt{\pi/2}}$
  - and, $\langle \Delta \hat{a}_1^2 \rangle \geq \int d^2\alpha \, P(\alpha, \alpha^*) \mathrm{var}(\, \hat{a}_1 \text{ measurement } \mid \text{ state is } |\alpha\rangle \,)$
    $$= \int d^2\alpha \, P(\alpha, \alpha^*) 1/4 = 1/4$$
  - The squeezed state $|\beta; \mu, \nu\rangle$ with $\mu, \nu > 0$ has: $\langle \Delta \hat{a}_1^2 \rangle = (\mu - \nu)^2/4 < 1/4$
  - So, the density operator $\hat{\rho} = |\beta; \mu, \nu\rangle\langle\beta; \mu, \nu|$ does not admit a proper P function representation

The $\hat{a}$ measurement as a POVM

- POVM elements can have a continuous spectrum
  - $\hat{\Pi}^\dagger(x) = \hat{\Pi}(x), \quad \langle\psi|\hat{\Pi}(x)|\psi\rangle \geq 0 \text{ for all } |\psi\rangle, \quad \hat{I} = \int dx\, \hat{\Pi}(x)$
  - $p(x) = \langle\psi|\hat{\Pi}(x)|\psi\rangle$
  - Projective measurement iff $\hat{\Pi}(x)\hat{\Pi}(y) = \hat{\Pi}(x)\delta(x-y)$
- Consider the operators,

$$\hat{\Pi}(\alpha) \equiv \frac{|\alpha\rangle\langle\alpha|}{\pi}, \quad \text{for } \alpha \in \mathcal{C}$$

  - This forms a POVM (check the conditions)
  - They do NOT form a projective measurement, since

$$\hat{\Pi}(\alpha)\hat{\Pi}(\beta) = \frac{(\langle\alpha|\beta\rangle)|\alpha\rangle\langle\beta|}{\pi^2} = \frac{e^{-|\alpha|^2/2-|\beta|^2/2+\alpha^*\beta}|\alpha\rangle\langle\beta|}{\pi^2} \neq \hat{\Pi}(\alpha)\delta(\alpha-\beta)$$

- where, $\delta(\alpha-\beta) \equiv \delta(\alpha_1-\beta_1)\delta(\alpha_2-\beta_2)$

Statistics of the $\hat{a}$ POVM

- Probability distribution of measurement outcome
  - Given by, $p(\alpha) = \langle\psi|\hat{\Pi}(\alpha)|\psi\rangle = \dfrac{|\langle\alpha|\psi\rangle|^2}{\pi}, \quad \text{for } \alpha \equiv \alpha_1 + j\alpha_2 \in \mathcal{C}$
  - This is same as the Q function (a proper pdf)

$$Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi}, \ \alpha = \alpha_1 + j\alpha_2$$

  - We can therefore define a (classical) characteristic fn.

$$M_{\alpha_1,\alpha_2}(jv_1, jv_2) \equiv \int \mathrm{d}^2\alpha \, e^{jv_1\alpha_1 + jv_2\alpha_2} p(\alpha)$$

  - Prove that: $M_{\alpha_1,\alpha_2}(jv_1, jv_2) = \chi_A(\zeta^*, \zeta)|_{\zeta = jv/2}, \quad \text{where } v \equiv v_1 + jv_2$

Problem 41

Measurement of $\hat{a}$ on a squeezed state

- Consider $|\beta;\mu,\nu\rangle$ with $\mu,\nu\in\mathbb{R}$ for simplicity
  - $\chi_A(\zeta)\equiv\langle e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger}\rangle=\langle\beta;\mu,\nu|e^{-\zeta^*(\mu\hat{b}-\nu\hat{b}^\dagger)}e^{\zeta(\mu\hat{b}^\dagger-\nu\hat{b})}|\beta;\mu,\nu\rangle$
  - Repeated use of the BCH identity gives us:

$$
\begin{aligned}
\chi_A(\zeta^*,\zeta) &= \langle\beta;\mu,\nu|e^{\zeta^*\nu\hat{b}^\dagger}e^{-\zeta^*\mu\hat{b}}e^{-\zeta^{*2}\mu\nu/2}e^{\zeta\mu\hat{b}^\dagger}e^{-\zeta\nu\hat{b}}e^{-\zeta^2\mu\nu/2}|\beta;\mu,\nu\rangle \\
&= \langle\beta;\mu,\nu|e^{(\zeta\mu+\zeta^*\nu)\hat{b}^\dagger}e^{-(\zeta^*\mu+\zeta\nu)\hat{b}}|\beta;\mu,\nu\rangle e^{-|\zeta|^2\mu^2-\mathrm{Re}(\zeta^2)\mu\nu}.
\end{aligned}
$$

  - Using the fact that $|\beta;\mu,\nu\rangle$ is an eigenket of $\hat{b}$

$$\chi_A(\zeta^*,\zeta)=e^{(\zeta\mu+\zeta^*\nu)\beta^*-(\zeta^*\mu+\zeta\nu)\beta-|\zeta|^2\mu^2-\mathrm{Re}(\zeta^2)\mu\nu}$$

  - $M_{\alpha_1,\alpha_2}(jv_1,jv_2)=\chi_A(\zeta^*,\zeta)|_{\zeta=jv/2}=e^{jv_1(\mu-\nu)\beta_1-v_1^2\sigma_1^2/2}e^{jv_2(\mu+\nu)\beta_2-v_2^2\sigma_2^2/2}$
  - Measurement variances of outcomes (α_1,α_2)

$$\sigma_1^2\equiv\frac{\mu^2-\mu\nu}{2}=\frac{(\mu-\nu)^2+1}{4}\quad\text{and}\quad\sigma_2^2\equiv\frac{\mu^2+\mu\nu}{2}=\frac{(\mu+\nu)^2+1}{4}$$

Statistics of measurement of $\hat{a}$ POVM

- Mean, variances of measurement outcomes (α_1, α_2)
  - Measurement described by POVM

State	$\langle \alpha \rangle$
$\lvert n \rangle$	0
$\lvert \beta \rangle$	β
$\lvert \beta; \mu, \nu \rangle$	$\mu^* \beta - \nu \beta^*$

State	$\langle \Delta \alpha_1^2 \rangle$	$\langle \Delta \alpha_2^2 \rangle$
$\lvert n \rangle$	$(n+1)/2$	$(n+1)/2$
$\lvert \beta \rangle$	$1/2$	$1/2$
$\lvert \beta; \mu, \nu \rangle$	$(\lvert \mu - \nu \rvert^2 + 1)/4$	$(\lvert \mu + \nu \rvert^2 + 1)/4$

- Recall variances for measuring $\hat{a}_1$ OR $\hat{a}_2$
  - Both are projective measurements (but cannot be done simultaneously)

State	$\langle \Delta \hat{a}_1^2(t) \rangle$	$\langle \Delta \hat{a}_2^2(t) \rangle$
$\lvert n \rangle$	$(2n+1)/4$	$(2n+1)/4$
$\lvert \alpha \rangle$	$1/4$	$1/4$
$\lvert \beta; \mu, \nu \rangle$	$\lvert \mu - \nu e^{-2j\omega t} \rvert^2/4$	$\lvert \mu + \nu e^{-2j\omega t} \rvert^2/4$

Gaussian state

- The phase space picture of a squeezed state, i.e., its Q function is Gaussian
- Gaussian states $\hat{\rho}$
  - whose characteristic functions (all three) are Gaussian
  - Equivalently, whose Q function is Gaussian

- Evaluate and plot the Wigner function and Q function of the cat state, $|\psi\rangle = \mathcal{N}_{\pm}\left(|\alpha\rangle \pm |-\alpha\rangle\right)$. Use the normalization constants $\mathcal{N}_{\pm}$ you found earlier. Are the cat states Gaussian?

Problem 42

Upcoming topics…

- Single mode quantum optics, continued
  - Phase space picture of squeezed states (conclusion)
  - Measurement of the $\hat{a}$ operator POVM: Heterodyne
  - Unitary **transformation** of single-mode and multi-mode states
    - Heisenberg vs. Schroedinger picture of quantum mechanics
    - General form of unitary transformation on a single mode
    - Displacement, Squeezing, Beamsplitters