Photonic Quantum Information Processing
OPTI 647: Lecture 7

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Plan for today

- Recap of squeezed states
- Phase space representations of states

Recap of Lecture 6

- Quadrature eigenkets $\hat{a}_1|\alpha_1\rangle_1 = \alpha_1|\alpha_1\rangle_1$, $\hat{a}_2|\alpha_2\rangle_2 = \alpha_2|\alpha_2\rangle_2$
  - Wavefunctions $\psi(\alpha_1) \equiv {}_1\langle\alpha_1|\psi\rangle$ and $\Psi(\alpha_2) \equiv {}_2\langle\alpha_2|\psi\rangle$ are a FT pair
    - Main result that led us to that: ${}_1\langle\alpha_1|\alpha_2\rangle_2 = e^{2j\alpha_2\alpha_1}/\sqrt{\pi}$
    - Proof involved "translation operators"

$$\hat{A}_1(\xi) \equiv \exp(-2j\xi\hat{a}_2);\ -\infty < \xi < \infty \quad ; \hat{A}_1(\xi)|\alpha_1\rangle_1 = |\alpha_1 + \xi\rangle_1$$

$$\hat{A}_2(\xi) \equiv \exp(2j\xi\hat{a}_1);\ -\infty < \xi < \infty \quad ; \hat{A}_2(\xi)|\alpha_2\rangle_2 = |\alpha_2 + \xi\rangle_2$$

  - Eigenkets $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$: infinite-energy (unphysical) states
- Minimum uncertainly product (MUP) states
  - We used equality condition in derivation of $\langle\Delta\hat{a}_2^2\rangle\langle\Delta\hat{a}_1^2\rangle \geq 1/16$ to derive most general form of a state that meets equality

$$\psi(\alpha_1) = \frac{\exp\left(\left[2j\langle\Delta\hat{a}_2\rangle\alpha_1 - j\langle\Delta\hat{a}_1\rangle\langle\Delta\hat{a}_2\rangle - (\alpha_1 - \langle\Delta\hat{a}_1\rangle)^2\right]/4\langle\Delta\hat{a}_1^2\rangle\right)}{(2\pi\langle\Delta\hat{a}_1^2\rangle)^{1/4}}$$

$$\Psi(\alpha_2) = \frac{\exp\left(\left[-2j\langle\Delta\hat{a}_1\rangle\alpha_2 + j\langle\Delta\hat{a}_1\rangle\langle\Delta\hat{a}_2\rangle - (\alpha_2 - \langle\Delta\hat{a}_2\rangle)^2\right]/4\langle\Delta\hat{a}_2^2\rangle\right)}{(2\pi\langle\Delta\hat{a}_2^2\rangle)^{1/4}}$$

Properties of $|\beta;\mu,\nu\rangle$ (Lecture 6)

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- Define $\hat{b} \equiv \mu\hat{a} + \nu\hat{a}^\dagger,\ \mu,\nu \in \mathbb{C},\ |\mu|^2 - |\nu|^2 = 1$
  - Verify that $[\hat{b},\hat{b}^\dagger] = 1$ still holds
  - $\hat{b}|\beta;\mu,\nu\rangle = \beta|\beta;\mu,\nu\rangle,\ \mu,\nu \in \mathbb{C},\ |\mu|^2 - |\nu|^2 = 1$
  - $\hat{N}_b|n;\mu,\nu\rangle = n|n;\mu,\nu\rangle,\ \mu,\nu \in \mathbb{C},\ |\mu|^2 - |\nu|^2 = 1$ CON basis states
  - Mean, $\langle\hat{a}\rangle = \langle\beta;\mu,\nu|\hat{a}|\beta;\mu,\nu\rangle = \mu^*\beta - \nu\beta^*$
- Prove that:
  - Mean photon number of the state $|\beta;\mu,\nu\rangle$ is given by:

    $$\langle\hat{N}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = |\langle\hat{a}\rangle|^2 + |\nu|^2$$

    Problem 29

    - Hint: $\hat{a} = \mu^*\hat{b} - \nu\hat{b}^\dagger$
    - Even for $\langle\hat{a}\rangle = 0$, $\langle\hat{N}\rangle = |\nu|^2$
  - Second moment, $\langle\hat{a}^2\rangle = \langle\hat{a}^{\dagger 2}\rangle^* = \mu^{*2}\beta^2 + \nu^2\beta^{*2} - 2\mu^*\nu|\beta|^2 - \mu^*\nu$

    Problem 30

Quadrature variances of squeezed state

- MUP states are "squeezed" states
  - Bogoliubov transformation $\hat{b} \equiv \mu\hat{a} + \nu\hat{a}^\dagger,\ \mu,\nu \in \mathbb{C},\ \ |\mu|^2 - |\nu|^2 = 1$
    - Satisfies same commutation relation: $[\hat{b},\hat{b}^\dagger] = [\hat{a},\hat{a}^\dagger] = 1$
    - "Coherent states" of $\hat{b}$ satisfy MUP cond.: $\hat{b}|\beta;\mu,\nu\rangle = \beta|\beta;\mu,\nu\rangle,\ \beta \in \mathbb{C}$
    - "Number states": $\hat{N}_b \equiv \hat{b}^\dagger\hat{b}$, $\hat{N}_b|n;\mu,\nu\rangle = n|n;\mu,\nu\rangle$, $\langle m;\mu,\nu|n;\mu,\nu\rangle = \delta_{mn}$
    - Mean field, $\langle\hat{a}\rangle = \langle\beta;\mu,\nu|\hat{a}|\beta;\mu,\nu\rangle = \mu^*\beta - \nu\beta^*$
    - Mean photon number, $\langle\hat{N}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = |\langle\hat{a}\rangle|^2 + |\nu|^2$
    - Prove that: $\langle\Delta\hat{a}_1^2\rangle = \dfrac{|\mu-\nu|^2}{4},\ \langle\Delta\hat{a}_2^2\rangle = \dfrac{|\mu+\nu|^2}{4}$ **Problem 31**
- A few comments
  - Coherent state is a special case, $\mu = 1, \nu = 0$
  - Optical Parametric Amplifier (OPA): a device that realizes the Bogoliubov transformation (we will do a proper non-linear-optics EM theory derivation of it later)

When is a squeezed state MUP

- Quadrature variances, $\langle \Delta \hat{a}_1^2 \rangle = \dfrac{|\mu - \nu|^2}{4}, \; \langle \Delta \hat{a}_2^2 \rangle = \dfrac{|\mu + \nu|^2}{4}$
- $|\mu - \nu|^2 = |\mu|^2 + |\nu|^2 - 2\mathrm{Re}(\mu^* \nu)$
- If $\mu^* \nu > 0$ and real
  - $2\mathrm{Re}(\mu^* \nu) = 2|\mu^* \nu| = 2|\mu\nu|$
  - $|\mu - \nu|^2 = (|\mu| - |\nu|)^2$ and $|\mu + \nu|^2 = (|\mu| + |\nu|)^2$
  - $\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle = \dfrac{1}{16}\left((|\mu| - |\nu|)^2 (|\mu| + |\nu|)^2\right) = \dfrac{1}{16}(|\mu|^2 - |\nu|^2)^2 = \dfrac{1}{16}$
- If $\mu^* \nu < 0$ and real
  - $2\mathrm{Re}(\mu^* \nu) = -2|\mu^* \nu| = -2|\mu\nu|$
  - $|\mu - \nu|^2 = (|\mu| + |\nu|)^2$ and $|\mu + \nu|^2 = (|\mu| - |\nu|)^2$
  - $\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle = \dfrac{1}{16}\left((|\mu| + |\nu|)^2 (|\mu| - |\nu|)^2\right) = \dfrac{1}{16}(|\mu|^2 - |\nu|^2)^2 = \dfrac{1}{16}$
- Squeezed state is MUP iff $\mu^* \nu$ is real

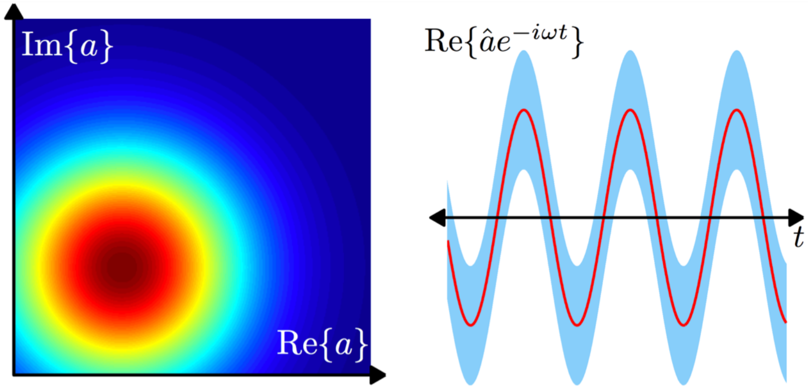
Time dependent annihilation operator

- So far, we have been ignoring the oscillatory term in the field operator $\hat{a}(t) = \hat{a}e^{-j\omega t}$
  - Coherent state can be thought of as a fixed complex number only when we station ourselves at a fixed "phase reference" (a phase modulo 2π)
  - $\langle \Delta \hat{a}_1^2(t) \rangle = \langle \Delta \hat{a}_2^2(t) \rangle = \frac{1}{4}, \ \forall t$: stays MUP *all the time*
- For a squeezed state, $|\beta; \mu, \nu\rangle$
  - $\langle \Delta \hat{a}_1^2(t) \rangle = \frac{|\mu - \nu e^{-2j\omega t}|^2}{4}$ and $\langle \Delta \hat{a}_2^2(t) \rangle = \frac{|\mu + \nu e^{-2j\omega t}|^2}{4}$
  - MUP only at times when $\mu^* \nu e^{-2j\omega t} \in \mathbb{R}$
- For most of the forthcoming development, we will station ourselves at a fixed phase reference and hence drop the time index

Physical meaning of squeezed states

Figure courtesy, Dr. Baris Erkmen, MIT Ph.D. 2008

Coherent state



(a) Wigner distribution **(b)** Real quadrature

$$|\beta;\mu,\nu\rangle \quad \text{vs,.} \quad |\alpha;r,\theta\rangle$$

Jeff Shapiro and Horace Yuen

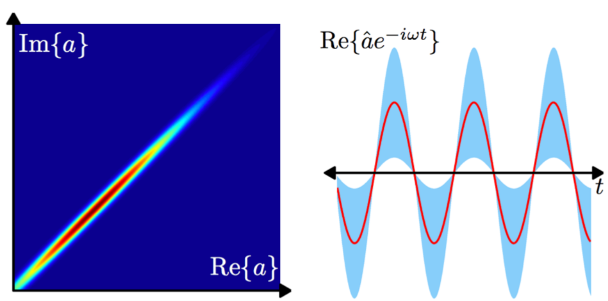
Carl Caves

$$\beta = \mu\alpha + \nu\alpha^*$$

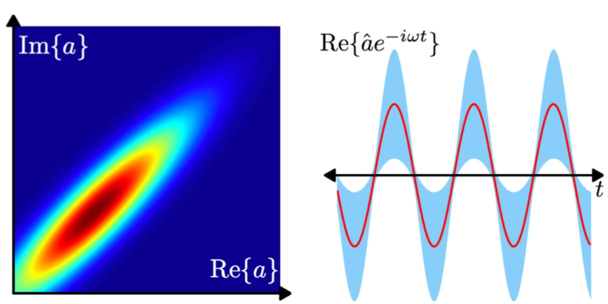
$$\mu = \cosh r$$

$$\nu = e^{i\theta}\sinh r.$$

Phase-squeezed state $\theta = \pi/2$

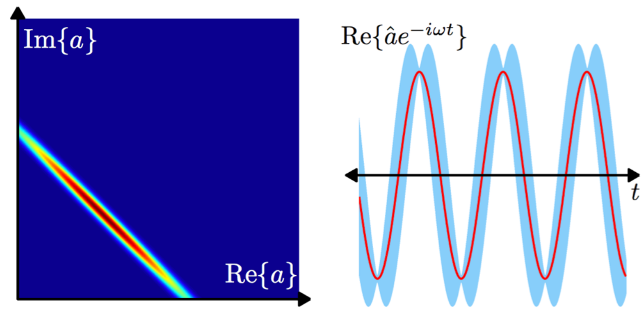


(a) Wigner distribution **(b)** Real quadrature

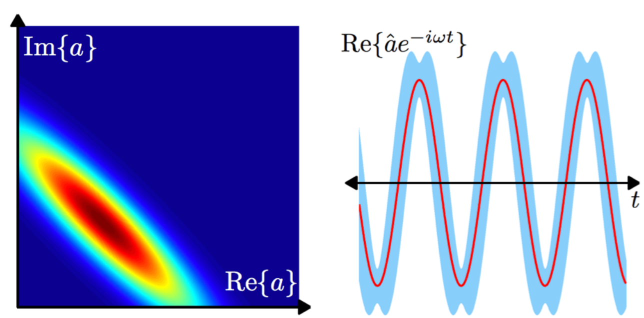


(c) Wigner distribution **(d)** Real quadrature

Amplitude-squeezed state $\theta = 3\pi/2$



(a) Wigner distribution **(b)** Real quadrature



(c) Wigner distribution **(d)** Real quadrature

SNR optimal state under quadrature measurements

- Recall: SNR of quadrature and number measurement on coherent state $|\alpha\rangle = |\sqrt{N}e^{j\theta}\rangle$

$$\mathrm{SNR}_{\mathrm{quadrature}} = \frac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} = \frac{\left(\mathrm{Re}(\alpha e^{j\theta})\right)^2}{1/4} = 4N\cos^2\theta$$

$$\mathrm{SNR}_{\mathrm{number}} \equiv \frac{\langle \hat{N} \rangle^2}{\langle \Delta \hat{N}^2 \rangle} = |\alpha|^2 = N$$

- Let us derive what state has the highest SNR for $\hat{a}_1$ quadrature measurement

$$\mathrm{SNR} \equiv \frac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle}$$

  - under a mean photon number constraint $\langle \hat{a}^\dagger \hat{a} \rangle \leq N$

Derivation of the optimal SNR from HUP

- Express mean photon number as

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle (\hat{a}_1 - j\hat{a}_2)(\hat{a}_1 + j\hat{a}_2) \rangle = \langle \hat{a}_1^2 \rangle + \langle \hat{a}_2^2 \rangle - \frac{1}{2}$$

$$= \langle \Delta \hat{a}_1^2 \rangle + \langle \hat{a}_1 \rangle^2 + \langle \Delta \hat{a}_2^2 \rangle + \langle \hat{a}_2 \rangle^2 - \frac{1}{2} \leq N$$

- Rearranging terms,

$$\mathrm{SNR} \equiv \frac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} \leq \frac{N + 1/2 - \langle \Delta \hat{a}_2^2 \rangle - \langle \hat{a}_2 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} - 1$$

  - with equality when $\langle \hat{a}^\dagger \hat{a} \rangle = N$
  - Also, SNR is maximized if $\langle \hat{a}_2 \rangle = 0$
  - So, we have: $\mathrm{SNR} = \dfrac{N + 1/2 - \langle \Delta \hat{a}_2^2 \rangle}{\langle \Delta \hat{a}_1^2 \rangle} - 1$

Optimal SNR derivation (continued)

- For fixed N and $\langle \Delta \hat{a}_1^2 \rangle$, SNR is maximum if $\langle \Delta \hat{a}_2^2 \rangle$ is as small as possible, i.e., $\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle = 1/16$
  - Setting $\langle \Delta \hat{a}_1^2 \rangle = x$
  - $\mathrm{SNR} = \dfrac{N+1/2}{x} - \dfrac{1}{16x^2} - 1$
  - $\dfrac{d\mathrm{SNR}}{dx} = -\dfrac{N+1/2}{x^2} + \dfrac{1}{8x^3} = 0, \text{ at } x = 1/8(N+1/2)$
  - $\dfrac{d^2\mathrm{SNR}}{dx^2} = \dfrac{2N+1}{x^3} - \dfrac{3}{8x^4} = -64(N+1/2) < 0$
  - Resulting maximum SNR = $4N(N+1)$
- What state would attain this? It must be MUP (since we used that to attain the equality condition)

SNR-optimal state

- Consider squeezed state $|\beta;\mu,\nu\rangle$
  - $\langle\hat{a}_1\rangle = \mathrm{Re}(\mu^*\beta - \nu\beta^*)$
  - $\langle\Delta\hat{a}_1^2\rangle = \dfrac{|\mu-\nu|^2}{4}$
  - $\langle\hat{a}^\dagger\hat{a}\rangle = |\mu^*\beta - \nu\beta^*|^2 + |\nu|^2$
  - Take a squeezed state with following real parameters

$$\beta = \sqrt{N(N+1)}, \mu = (N+1)/\sqrt{2N+1}, \nu = N/\sqrt{2N+1}$$

- Show that it achieves, $\mathrm{SNR} = 4N(N+1)$
- Find the mean, $\langle\hat{a}_1\rangle$ and compare with mean $\langle\hat{a}_1\rangle$ of a coherent state of same mean photon number N

Problem 32

Binary phase modulation for optical communications

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- BPSK coherent states $|\psi_0\rangle = |\alpha\rangle, |\psi_1\rangle = |-\alpha\rangle$
  - Assume real α, and mean photon number, $N = \alpha^2$
  - Inner product $\langle\psi_0|\psi_1\rangle \equiv \sigma = e^{-2\alpha^2} = e^{-2N}$
  - Minimum error probability of discrimination (equal priors)

$$P_{e,\min} = \frac{1-\sqrt{1-|\sigma|^2}}{2} = \frac{1-\sqrt{1-e^{-4N}}}{2} \quad \approx \frac{1}{4}e^{-4N}, N \gg 1$$

- BPSK squeezed states, $|\psi_0\rangle = |\beta;\mu,\nu\rangle, |\psi_1\rangle = |-\beta;\mu,\nu\rangle$
  - Assume real β, μ, ν
  - Inner product, $\langle\psi_0|\psi_1\rangle \equiv \sigma = e^{-2\beta^2}$
  - Find β, μ, ν s.t. σ is minimized for given N and show tha

$$P_{e,\min} = \frac{1-\sqrt{1-|\sigma|^2}}{2} = \frac{1-\sqrt{1-e^{-4N(N+1)}}}{2} \approx \frac{1}{4}e^{-4N^2}, N \gg 1$$

Problem 33

Measurement statistics: summary so far

- Mean field

State	$\langle \hat{a}(t) \rangle$
$\lvert n \rangle$	0
$\lvert \alpha \rangle$	$\alpha e^{-j\omega t}$
$\lvert \beta; \mu, \nu \rangle$	$(\mu^* \beta - \nu \beta^*) e^{-j\omega t}$

- Variance

State	$\langle \Delta \hat{a}_1^2(t) \rangle$	$\langle \Delta \hat{a}_2^2(t) \rangle$
$\lvert n \rangle$	$(2n+1)/4$	$(2n+1)/4$
$\lvert \alpha \rangle$	$1/4$	$1/4$
$\lvert \beta; \mu, \nu \rangle$	$\lvert \mu - \nu e^{-2j\omega t} \rvert^2/4$	$\lvert \mu + \nu e^{-2j\omega t} \rvert^2/4$

Characteristic functions

- Random variable $X \in \mathbb{R}$ with probability distribution function (pdf) $p_X(x), x \in \mathbb{R}$
- Characteristic function of X

$$M_X(jv) \equiv \langle e^{jvX} \rangle = \int_{-\infty}^{\infty} p_X(x) e^{jvx} dx$$

- Inverse relation

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_X(jv) e^{-jvx} dv$$

- Differentiating $M_X(jv)$ w.r.t. v repeatedly can be used to calculate moments of X, i.e., $E[X^n]$

A few quick exercises…

- Gaussian r.v. X $p_X(x) = \dfrac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$ $, x \in \mathbb{R}$
  - Prove that $M_X(jv) = e^{jv\mu - v^2\sigma^2/2}$

    Problem 34(a)
  - Differentiate M_X (once, and then twice) to verify the mean and variance come out as μ and σ^2
- Poisson random variable N, $P_X[n] = \dfrac{e^{-\mu}\mu^n}{n!}, n = 0, 1, \ldots$
  - Prove that $M_X(jv) \equiv \langle e^{jvX} \rangle = e^{[\mu(e^{jv}-1)]}$
  - Differentiate M_X (once, and then twice) to verify the mean and variance both come out as μ

Problem 34(b)

(Classical) characteristic function of a (quantum) quadrature measurement

- Suppose we measure $\hat{a}_1$ on state $|\psi\rangle$
  - Call the random variable associated with the (random) measurement outcome, as X_1
  - $M_{X_1}(jv) \equiv \langle e^{jvX_1}\rangle = \int_{-\infty}^{\infty} p_{X_1}(x) e^{jvx}\, dx$

$$= \langle\psi| \left(\int_{-\infty}^{\infty} dx\, e^{jvx} |x\rangle_{1\,1}\langle x| \right) |\psi\rangle$$

$$= \langle\psi| \left(\int_{-\infty}^{\infty} dx \sum_{n=0}^{\infty} \frac{(jv)^n}{n!} x^n |x\rangle_{1\,1}\langle x| \right) |\psi\rangle$$

$$= \langle\psi| \left(\sum_{n=0}^{\infty} \frac{(jv)^n}{n!} \hat{a}_1^n \right) |\psi\rangle = \langle e^{jv\hat{a}_1}\rangle$$

Wigner characteristic function

- Let us introduce the Wigner characteristic function

$$\chi_W(\zeta^*, \zeta) \equiv \langle e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} \rangle$$

  – $\zeta = \zeta_1 + j\zeta_2$ is a complex argument
  – Unless it is unclear, we use simply use $\chi_W(\zeta)$
- Prove that the characteristic function $M_{X_1}(jv)$ of the quadrature moment can be obtained from the Wigner characteristic function $\chi_W(\zeta)$ at $\zeta = jv/2$
  – Hint: first show that $\chi_W(\zeta) = \langle e^{-2j\mathrm{Im}[\zeta^* \hat{a}]} \rangle$

Problem 35

  – So, … an appropriate slice of the Wigner c.f. gives us the quadrature distribution (c.f.) for measuring $\hat{a}_1 e^{j\theta}$ for any θ

Baker-Campbell-Hausdorff theorem

- Exponential of an operator $e^{\hat{C}} \equiv \sum_{n=0}^{\infty} \frac{\hat{C}^n}{n!}$
  - If $\left[\hat{A}, [\hat{A}, \hat{B}]\right] = \left[\hat{B}, [\hat{A}, \hat{B}]\right] = 0$
  - then, $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]/2}$
- Consider the Wigner c.f., and $\hat{A} = -\zeta^* \hat{a}, \hat{B} = \zeta \hat{a}^\dagger$
  - We then get, $[\hat{A}, \hat{B}] = -|\zeta|^2 [\hat{a}, \hat{a}^\dagger] = -|\zeta|^2$
- Define:
  - Antinormally-ordered characteristic function

$$\chi_A(\zeta) \equiv \langle e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger} \rangle$$

  - Normally-ordered characteristic function

$$\chi_N(\zeta) \equiv \langle e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} \rangle$$

  - Show that for any state, **Problem 36**

$$\chi_W(\zeta) = \chi_A(\zeta) e^{|\zeta|^2/2} = \chi_N(\zeta) e^{-|\zeta|^2/2}$$

Homodyne detection statistics on a number state

- $$M_{X_1}(jv) = \chi_W(\zeta)\big|_{\zeta=jv/2} = \left[\chi_N(\zeta)e^{-|\zeta|^2/2}\right]\Big|_{\zeta=jv/2}$$
$$= \left[\langle n|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|n\rangle e^{-|\zeta|^2/2}\right]\Big|_{\zeta=jv/2}$$

- $$M_{X_1}(jv) = \left[\left(\sum_{m=0}^{\infty}\frac{\zeta^m}{m!}\langle n|\hat{a}^{\dagger m}\right)\left(\sum_{k=0}^{\infty}\frac{(-\zeta^*)^k}{k!}\hat{a}^k|n\rangle\right)e^{-|\zeta|^2/2}\right]_{\zeta=jv/2}$$
$$= \left(\sum_{m=0}^{n}\frac{(jv/2)^m}{m!}\sqrt{\frac{n!}{(n-m)!}}\langle n-m|\right)\left(\sum_{k=0}^{n}\frac{(jv/2)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle\right)e^{-v^2/8}$$
$$= \left(\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{(-v^2/4)^m}{m!}\right)e^{-v^2/8} = L_n(v^2/4)e^{-v^2/8},$$

where

$$L_n(z) \equiv \sum_{m=0}^{n}(-1)^m\frac{n!}{m!(n-m)!}\frac{z^m}{m!},$$

Homodyne detection statistics on a number state (continued)

- Probability distribution function,

$$p_{X_1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L_n(v^2/4)e^{-v^2/8}e^{-jvx}dx$$

$$= \frac{2}{\pi}\frac{e^{-2x^2}}{2^n n!}[H_n(\sqrt{2}x)]^2$$

  – where $H_n(z) = (-1)^n e^{z^2}\frac{d^n e^{-z^2}}{dz^n}$ is the n$^{\text{th}}$ Hermite polynomial

The "Wigner function": a quasi-probability

- FT of the characteristic function of $\hat{a}_1$ measurement which is $M_{X_1}(jv) = \chi_W(\xi)\big|_{jv/2}$,gives us the pdf $p_{X_1}(x)$
- What if we took a (2D) FT of the full Wigner function. Will it give us some sort of a pdf of measuring both quadratures together? But we know it is not possible to measure them both together!
- Define $W(\alpha^*, \alpha) \equiv \int \frac{\mathrm{d}^2\zeta}{\pi^2} \chi_W(\zeta^*, \zeta) e^{\zeta^*\alpha - \zeta\alpha^*}$,

$$\int \frac{\mathrm{d}^2\zeta}{\pi^2} \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta_1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta_2}{\pi}, \quad \zeta^*\alpha - \zeta\alpha^* = 2j\zeta_1\alpha_2 - 2j\zeta_2\alpha_1$$

$$\chi_W(\zeta^*, \zeta) = \int \mathrm{d}^2\alpha \, W(\alpha^*, \alpha) e^{-\zeta^*\alpha + \zeta\alpha^*}$$

Wigner function of a coherent state

- Verify that $\int \mathrm{d}^2\alpha\, W(\alpha^*, \alpha) = 1$

- Consider coherent state $|\beta\rangle$

$$\chi_W(\zeta^*, \zeta) = \langle\beta|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|\beta\rangle e^{-|\zeta|^2/2} = e^{\zeta\beta^* - \zeta^*\beta}e^{-|\zeta|^2/2}$$

$$W(\alpha^*, \alpha) = \frac{e^{-2|\alpha-\beta|^2}}{\pi/2}$$

  – Two S.I. Gaussian random variables each with variance ¼ and means β_1 and β_2 respectively

Wigner function of a number state $|n\rangle$

- Evaluate $W(\alpha^*,\alpha) = \int \frac{d^2\zeta}{\pi^2} L_n(|\zeta|^2) e^{-|\zeta|^2/2} e^{\zeta^*\alpha - \zeta\alpha^*}$

$$W(\alpha^*,\alpha) = \frac{2}{\pi}\int_0^\infty dr\, r L_n(r^2) e^{-r^2/2} J_0(2r|\alpha|) = (-1)^n \frac{2}{\pi} L_n(4|\alpha|^2) e^{-2|\alpha|^2}$$

- For number state $|0\rangle$, $L_0(z) = 1$

$$W(\alpha^*,\alpha) = \frac{2}{\pi} e^{-2|\alpha|^2}$$

- For number state $|1\rangle$, $L_1(z) = (1 - z)$

$$W(\alpha^*,\alpha) = \frac{2}{\pi}(4|\alpha|^2 - 1) e^{-2|\alpha|^2} < 0, |\alpha| < \frac{1}{2}$$

- $W(\alpha) < 0$ *not* necessary for non-classicality

Upcoming topics…

- Single mode quantum optics, continued
  - More on Characteristic functions and Wigner functions
  - Measurement of the $\hat{a}$ operator
  - Some more examples and applications
  - Gaussian vs. non-Gaussian states