Photonic Quantum Information Processing
OPTI 647: Lecture 5

Saikat Guha
September 10, 2019

Associate Professor
College of Optical Sciences
Meinel 523

Plan for today

- Recap: observables in quantum mechanics
- Heisenberg Uncertainly Principle (HUP) and Minimum uncertainty product (MUP) states
- Homodyne detection

Observables in quantum mechanics

- Observable: Hermitian operator $\hat{A}$, i.e., $\hat{A}^\dagger = \hat{A}$
- $\hat{A}|a_k\rangle = a_k|a_k\rangle$
- Eigenvalues a_k are real valued
- “Measuring observable $\hat{A}$” = von Neumann measurement described by, $\{\Pi_k = |a_k\rangle\langle a_k|\}$
- If $\hat{A}$ measured on the state $\hat{\rho}$,
  - probability of outcome k, $P(k) = \langle a_k|\hat{\rho}|a_k\rangle$
  - post-measurement state, $|a_k\rangle$
  - Average value of measurement outcome

$$\langle \hat{A} \rangle_{\hat{\rho}} = \sum_k a_k P(k) = \mathrm{Tr}(\rho \hat{A})$$

Heisenberg uncertainty principle (HUP)

THE UNIVERSITY OF ARIZONA

- Non-commuting observables, $\hat{A}$ and $\hat{B}$
  - $[\hat{A}, \hat{B}] = j\hat{C}$
  - Define $\langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}\rho) = \langle\psi|\hat{A}|\psi\rangle, \text{if}\, \rho = |\psi\rangle\langle\psi|$
  - Define $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$
  - Therefore, $\langle\Delta\hat{A}^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$
- Heisenberg uncertainly relation (we will prove it!)

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle \geq \frac{1}{4}\left|\langle\hat{C}\rangle\right|^2$$

- **Mathematical** meaning of non-commuting operators
  - They are not *simultaneously diagonalizable*
- **Physical** meaning of non-commuting observables
  - the observables cannot be simultaneously measured

Heisenberg uncertainty principle, contd.

- For two non-commuting observables, show that

$$[\hat{A}, \hat{B}]^\dagger = -[\hat{A}, \hat{B}]$$

Problem 15

  - Since $[\hat{A}, \hat{B}] = j\hat{C}$, $\hat{C}$ is Hermitian, i.e., $\hat{C}^\dagger = \hat{C}$
- Multiply to verify that: $[\Delta\hat{A}, \Delta\hat{B}] = j\hat{C}$ **Problem 16**
- Cauchy-Schwarz inequality $\langle X^2\rangle\langle Y^2\rangle \geq |\langle XY\rangle|^2$:

$$\langle\psi|\Delta\hat{A}^2|\psi\rangle\langle\psi|\Delta\hat{B}^2|\psi\rangle \geq |\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle|^2$$

  - With equality iff $\Delta\hat{A}|\psi\rangle = j\lambda\Delta\hat{B}|\psi\rangle$ for some $\lambda \in \mathbb{C}$

Heisenberg uncertainty principle, contd.

- The rest is algebra…

$$|\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle|^2 = \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}+[\Delta\hat{A},\Delta\hat{B}]}{2}\right)|\psi\rangle\right|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}}{2}\right)|\psi\rangle + \frac{j}{2}\langle\psi|\hat{C}|\psi\rangle\right|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}}{2}\right)|\psi\rangle\right|^2 + \left|\frac{\langle\psi|\hat{C}|\psi\rangle}{2}\right|^2$$

$$\geq \left|\langle\psi|\hat{C}|\psi\rangle\right|^2/4$$

  - With equality iff $\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle = -\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle$

Heisenberg uncertainty principle, contd.

- So, we have shown: $\langle\psi|\Delta\hat{A}^2|\psi\rangle\langle\psi|\Delta\hat{B}^2|\psi\rangle \geq \left|\langle\psi|\hat{C}|\psi\rangle\right|^2/4$

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle \geq \frac{1}{4}\left|\langle\hat{C}\rangle\right|^2$$

  – With equality iff $\Delta\hat{A}|\psi\rangle = j\lambda\Delta\hat{B}|\psi\rangle$ for a real λ

- States $|\psi\rangle$ that meet the Heisenberg lower bound on the product of variances for a given pair of non-commuting observables are called “minimum uncertainly product” (MUP) states

- Note that the HUP is an “either or” proposition

Field quadrature operators

- Quadrature operators $\hat{a}_1$ and $\hat{a}_2$ defined as:

$$\hat{a} = \hat{a}_1 + j\hat{a}_2 \qquad \hat{a}_1 = (\hat{a} + \hat{a}^\dagger)/2$$
$$\hat{a}^\dagger = \hat{a}_1 - j\hat{a}_2 \qquad \hat{a}_2 = (\hat{a} - \hat{a}^\dagger)/2j$$

  - Show that **Problem 17**

    (1) $\hat{a}_1$ and $\hat{a}_2$ are Hermitian operators

    (2) Their commutator is given by $[\hat{a}_1, \hat{a}_2] = \frac{j}{2}$

  - They are non-commuting observables

- Therefore, the HUP states that:

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \geq \frac{1}{16}$$

Quadrature measurement on number states

- Mean, $\langle n|\hat{a}|n\rangle = 0$
  - Therefore, $\langle \hat{a}_1\rangle = \langle n|\hat{a}_1|n\rangle = 0$ and $\langle \hat{a}_2\rangle = \langle n|\hat{a}_2|n\rangle = 0$
- Second moment,

$$\langle n|\hat{a}_1^2|n\rangle = \langle n|\left(\frac{[\hat{a}+\hat{a}^\dagger]^2}{4}\right)|n\rangle$$

$$= \frac{\langle n|\hat{a}^2|n\rangle + \langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle + \langle n|\hat{a}^{\dagger 2}|n\rangle}{4}$$

$$= \frac{2\langle n|\hat{a}^\dagger\hat{a}|n\rangle + 1}{4} = \frac{2n+1}{4}$$

- Uncertainly product for quadrature variances:

$$\langle\Delta\hat{a}_1^2\rangle\langle\Delta\hat{a}_2^2\rangle = \left(\frac{2n+1}{4}\right)^2 \geq \frac{1}{16}$$

Equality holds only for the vacuum state. Number states are not minimum uncertainly-product states

Quadrature measurement on coherent states

- Mean, $\langle\alpha|\hat{a}|\alpha\rangle = \alpha \equiv \alpha_1 + j\alpha_2$
  - Therefore, $\langle\hat{a}_1\rangle = \langle\alpha|\hat{a}_1|\alpha\rangle = \alpha_1$ and $\langle\hat{a}_2\rangle = \langle\alpha|\hat{a}_2|\alpha\rangle = \alpha_2$
- Second moment,

$$\langle\alpha|\hat{a}_1^2|\alpha\rangle = \langle\alpha|\left(\frac{[\hat{a}+\hat{a}^\dagger]^2}{4}\right)|\alpha\rangle$$

$$= \ldots$$

- Complete this calculation and show that the uncertainly product for quadrature variances is:

$$\langle\Delta\hat{a}_1^2\rangle = \langle\Delta\hat{a}_2^2\rangle = \frac{1}{4}$$

$$\langle\Delta\hat{a}_1^2\rangle\langle\Delta\hat{a}_2^2\rangle = \frac{1}{16}, \ \forall\alpha$$

Problem 18

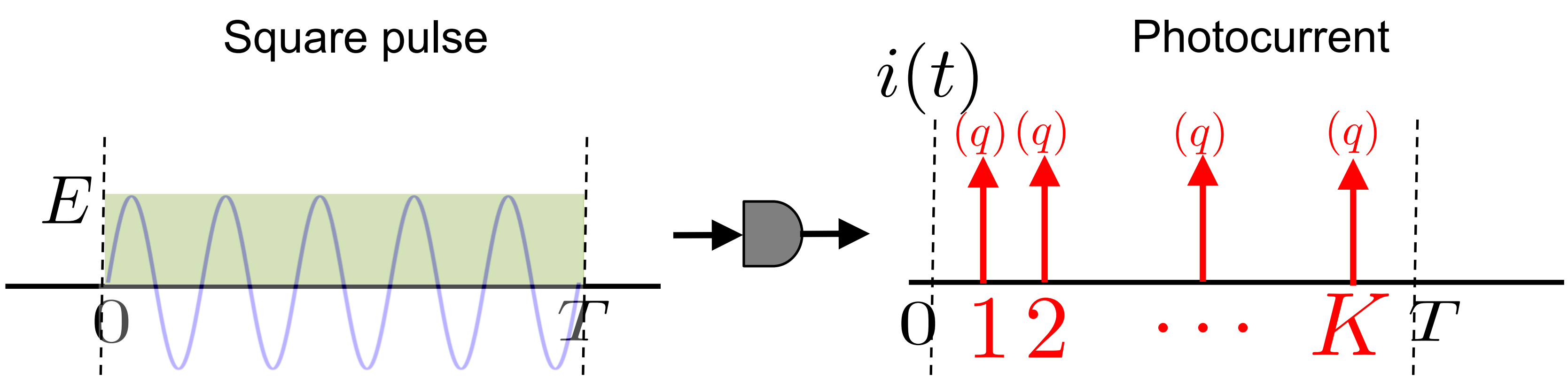
Coherent states are minimum uncertainly product states

Signal-to-noise ratio

THE UNIVERSITY OF ARIZONA

- Quadrature measurement on $|\alpha\rangle, |\alpha|^2 = N$
  - $\mathrm{SNR}_{\mathrm{quadrature}} \equiv \dfrac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} = \dfrac{\mathrm{Re}(\alpha)}{1/4} = 4\,\mathrm{Re}(\alpha)$
  - Measurement of $\hat{a}_1 e^{j\phi}$ yields a Gaussian random variable with mean $\mathrm{Re}(\alpha e^{j\phi})$ and variance 1/4
- Number measurement on coherent state
  - Prove that $\mathrm{SNR}_{\mathrm{number}} \equiv \dfrac{\langle \hat{N} \rangle^2}{\langle \Delta \hat{N}^2 \rangle} = |\alpha|^2 = N$ **Problem 19**
  - This is consistent with the fact that $P(n) = e^{-N} N^n / n!$, the Poisson distribution has mean N and variance N

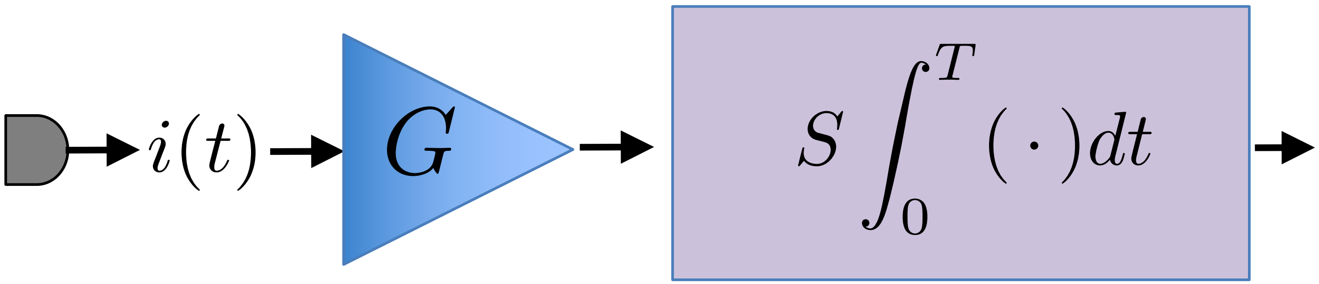
Ideal direct direction for a strong pulse



Mean photon number

$$N = E^2 T$$

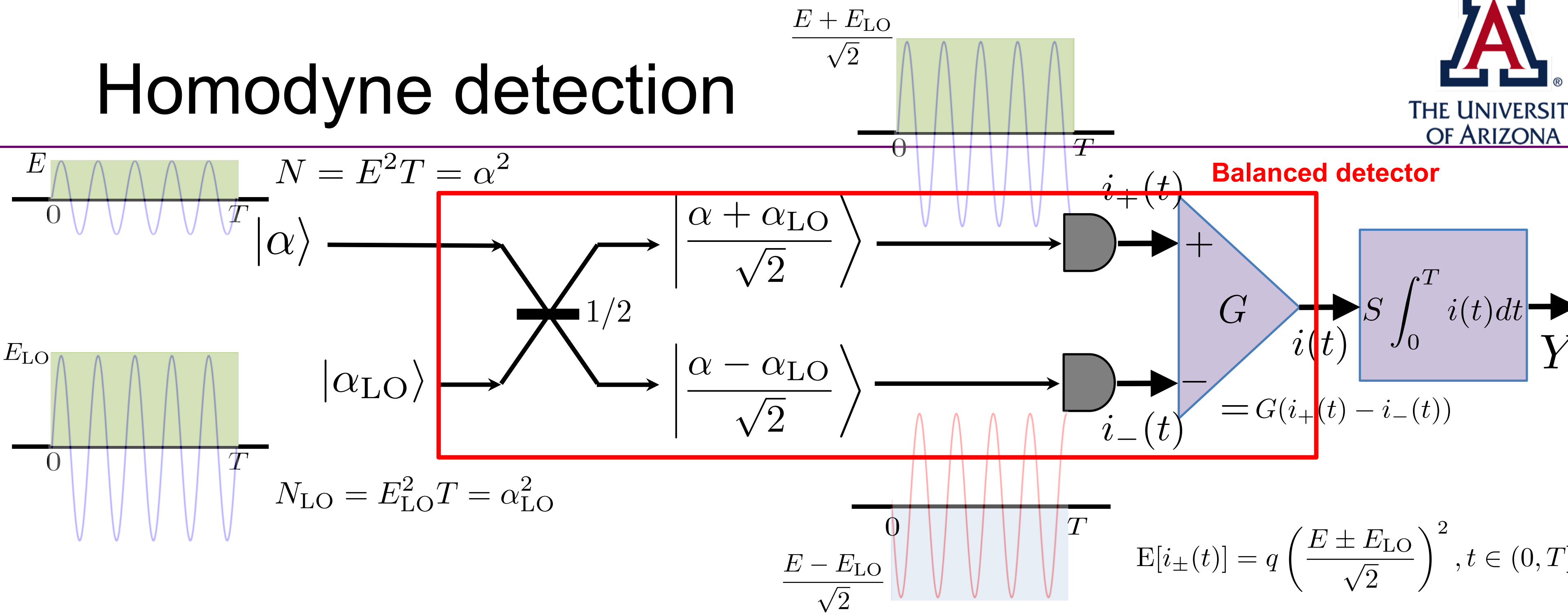
$$K \sim \text{Poisson}(N)$$



$$Y = qGSK$$

When N is large, $Y \sim \text{Gaussian}(\mu, \sigma^2)$, $\mu = qGSN, \sigma^2 = (qGS)^2 N$

Homodyne detection



Assume for now that both $\alpha, \alpha_{\mathrm{LO}}$ are real, and $N_{\mathrm{LO}} \gg N$

$$K_+ \sim \mathrm{Poisson}(N_+) \sim \mathcal{N}(N_+, N_+);\ N_+ = \left|\frac{\alpha+\alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$Y = qGS(K_+ - K_-)$$

$$K_- \sim \mathrm{Poisson}(N_-) \sim \mathcal{N}(N_-, N_-);\ N_- = \left|\frac{\alpha-\alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = qGS(N_+ - N_-) = 2qGS\alpha\alpha_{\mathrm{LO}}$$

$$\sigma^2 = (qGS)^2(N_+ + N_-) = (qGS)^2(\alpha^2 + \alpha_{\mathrm{LO}}^2)$$

By picking the constant S appropriately, we get: $Y \sim \mathcal{N}\left(\alpha, \frac{1}{4}\right)$

Shot noise limit

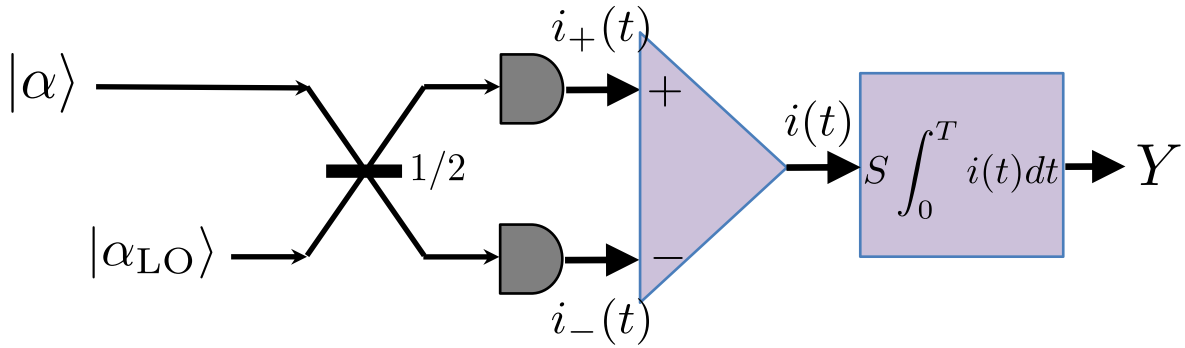
Homodyne detection; quantum measurement of $\hat{a}_1\, e^{j\phi}$

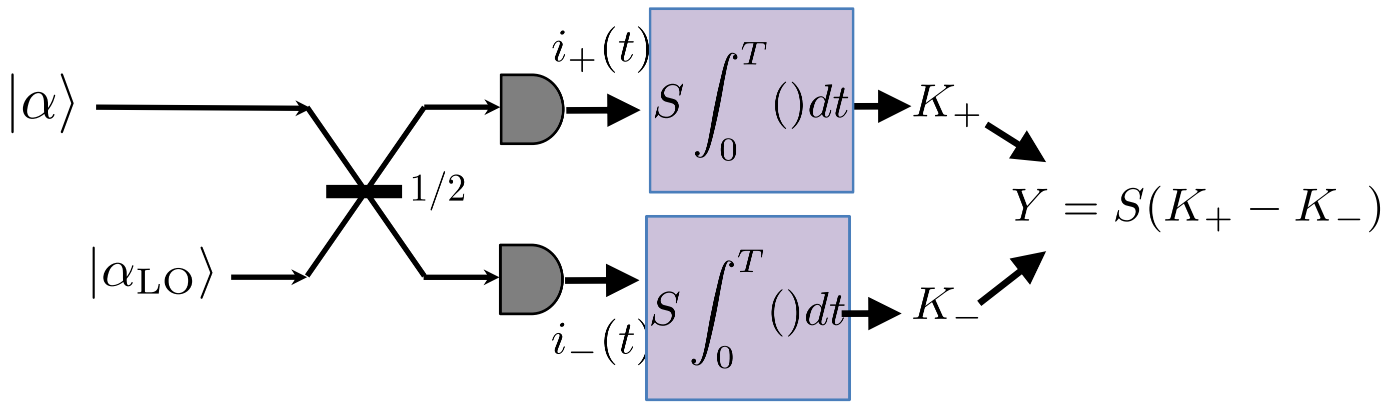
THE UNIVERSITY OF ARIZONA

- Local Oscillator (LO) is strong w.r.t. signal, $N_{\rm LO} \gg N$

$$\alpha = \sqrt{N}e^{j\theta}$$

$$\alpha_{\rm LO} = \sqrt{N_{\rm LO}}\, e^{j\phi}$$





$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = S(N_+ - N_-)$$

$$\sigma^2 = S^2(N_+ + N_-)$$

Substitute these, take $N_{\rm LO} \gg N$ limit and pick an appropriate scaling S, to show that:

$$Y \sim \mathcal{N}\left(\mathrm{Re}(\alpha e^{j\phi}), \frac{1}{4}\right)$$

$$K_+ \sim \mathrm{Poisson}(N_+) \sim \mathcal{N}(N_+, N_+);\ N_+ = \left|\frac{\alpha + \alpha_{\rm LO}}{\sqrt{2}}\right|^2$$

$$K_- \sim \mathrm{Poisson}(N_-) \sim \mathcal{N}(N_-, N_-);\ N_- = \left|\frac{\alpha - \alpha_{\rm LO}}{\sqrt{2}}\right|^2$$

Problem 20

Discriminating BPSK with Homodyne

THE UNIVERSITY OF ARIZONA

$$|\alpha\rangle \text{ vs. } |-\alpha\rangle$$

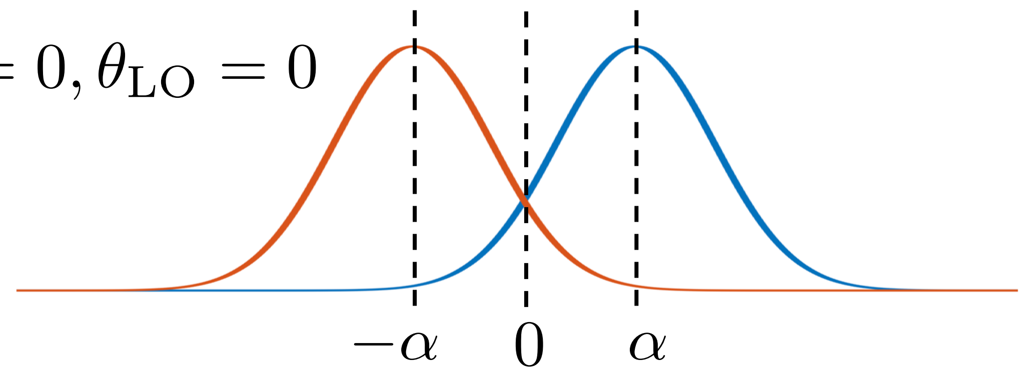
$$|\alpha\rangle \rightarrow \boxed{\theta_{\mathrm{LO}}} \rightarrow Y \sim \mathcal{N}\left(\mathrm{Re}(\alpha e^{j\theta_{\mathrm{LO}}}), \frac{1}{4}\right)$$

$$\alpha = \sqrt{N}e^{j\theta}$$

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \Rightarrow p_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma}e^{-(y-\mu)^2/2\sigma^2}$$

Gaussian probability distribution

$$\theta = 0, \theta_{\mathrm{LO}} = 0$$



$$P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{2N})$$

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt$$

Compare the probability or error achieved by Homodyne detection with that of the Kennedy receiver

Problem 21

$$S := p_{Y|X}(y|x)$$

$$\begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & & \\ 2 & & \end{array}$$

$$S = \begin{pmatrix} 1-\frac{1}{2}\mathrm{erfc}(\sqrt{2N}) & \frac{1}{2}\mathrm{erfc}(\sqrt{2N}) \\ \frac{1}{2}\mathrm{erfc}(\sqrt{2N}) & 1-\frac{1}{2}\mathrm{erfc}(\sqrt{2N}) \end{pmatrix}$$

Upcoming topics

- Quadrature eigenkets
- Squeezed states of light
- Phase space picture of quantum optical states
  - Characteristic functions, and Wigner functions