Photonic Quantum Information Processing
OPTI 647: Lecture 4

Saikat Guha
September 5, 2019

Associate Professor
College of Optical Sciences
Meinel 523

Plan for today

THE UNIVERSITY OF ARIZONA

- Tracing out a quantum system
- Positive operator valued measure
- Quantizing the field
- Heisenberg Uncertainly Principle (HUP) and Minimum uncertainty product (MUP) states

Recap

$c_n = \langle n|\alpha\rangle$

- Coherent state is *always* single mode $|\alpha\rangle = \sum_{n=0}^{\infty}\left(\frac{e^{-\frac{|\alpha|^2}{2}}\alpha^n}{\sqrt{n!}}\right)|n\rangle$
- "splitting" a coherent state: product of coherent states
- Classical state is a mixture of coherent states
  - Single mode classical state $\rho = \int_{\mathbb{C}} P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$
    - P function
  - n-mode classical state $\rho = \int_{\mathbb{C}^n} P(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| d^{2n}\boldsymbol{\alpha}$

    where $|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle \ldots |\alpha_n\rangle$

- Distinguishing equally-likely states, $\{|\psi_1\rangle, |\psi_2\rangle\}$, $\langle\psi_1|\psi_2\rangle = \sigma$ the minimum average Pr(error), $P_e = \frac{1}{2}\left[1 - \sqrt{1-|\sigma|^2}\right]$

Subsystem: concept of "tracing out"

- Consider a state of two modes, ρ_{AB}
- The (marginal) state of mode A is given by

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B\left(\rho_{AB}\right)\\ &= \sum_{n=0\infty} {}_B\langle n|\rho_{AB}|n\rangle_B\end{aligned}$$

Orthonormal basis for mode B

- Example: $|\psi_{AB}\rangle = \dfrac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}}$ $\qquad \rho_A = \dfrac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$

- Fact: if quantum systems A and B are entangled, the reduced states of A or B are both mixed states

General quantum measurement: POVM

- Positive operator valued measure (POVM) operators $\{\Pi_j\}, j = 1, \ldots, N$
  - Hermitian: $\Pi_j^\dagger = \Pi_j$
  - Positive: $\Pi_j \geq 0$
  - Complete: $\sum_j \Pi_j = I$
- Measurement statistics

Special case: If POVM elements are **orthogonal projectors**, it is a **projective measurement**. Projective measurement described by orthonormal vectors $\{|w_i\rangle\}$), are **von Neumann measurements** (with POVM elements, $\Pi_j = |w_j\rangle\langle w_j|$)
One **can** construct POVM out of non-orthogonal projectors $\{|w_i\rangle\}$

$$\rho \longrightarrow \boxed{\{\Pi_j\}} \longrightarrow j$$

$$p(j) = \mathrm{Tr}\left(\rho \Pi_j\right)$$

Annihilation operator *of a mode*

- Recall field quantization:

$$E(t) = \sum_{k=1}^{K} a_i \phi_i(t) \quad \Rightarrow \quad \hat{E}(t) = \sum_{k=1}^{K} \hat{a}_i \phi_i(t)$$

Coherent state of modes $\phi_i(t)$ with complex field amplitudes, a_i

Field operator for the set of modes $\phi_i(t)$, with modal annihilation operators $\hat{a}_i$, with mode i excited in coh. st. $|a_i\rangle$

- Annihilation operator $\hat{a}$ of a single mode
  - Eigenstate is a coherent state, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
  - "Annihilates" photon number, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

  - Number operator, $\hat{N} = \hat{a}^\dagger \hat{a}$

$$\hat{N}|n\rangle = n|n\rangle$$

  - Show that, $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I}$

Problem 9

The phase operator

The University of Arizona

- Application of a phase, $U_\theta|\alpha\rangle = |\alpha e^{i\theta}\rangle$
- The phase operator,

$$U_\theta = e^{i\theta\hat{N}} = e^{i\theta\hat{a}^\dagger a}$$

- Show that random phase scrambling of any pure state leads to a number diagonal state

$$\rho = \int_0^{2\pi} U_\theta|\psi\rangle\langle\psi|U_\theta^\dagger d\theta \quad = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|$$

– With $P(n) = |\langle n|\psi\rangle|^2$

Problem 10

Coherent states resolve the identity

- Recall: $\hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|$
- Prove that:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d^2\alpha = \hat{I}$$

Problem 11

- Coherent states are an "over-complete" basis
- They are not orthogonal (and hence cannot be distinguished perfectly)
- They form a POVM with elements $\{\Pi_\alpha\}\,,\alpha \in \mathbb{C}$

$$\Pi_\alpha = \frac{|\alpha\rangle\langle\alpha|}{\pi}$$

Coherent state overcomplete basis

- Use the fact $\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d^2\alpha = \hat{I}$ to show that

any operator $\hat{G}$ can be expressed in terms of its coherent state matrix elements, $\langle\alpha|\hat{G}|\beta\rangle$, i.e.,

$$\hat{G} = \int\int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\langle\alpha|\hat{G}|\beta\rangle|\alpha\rangle\langle\beta|$$

Problem 12

- Hint: $\hat{G} = \hat{I}\hat{G}\hat{I}$
- Any state can be written in this “basis”:

$$|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \psi(\alpha)|\alpha\rangle d^2\alpha, \; \psi(\alpha) = \langle\alpha|\psi\rangle$$

Representation in the coherent state (overcomplete) basis is NOT unique

THE UNIVERSITY OF ARIZONA

- Coherent state $|\psi\rangle = |\beta\rangle$ can be expressed as:
- (1) $|\psi\rangle = |\beta\rangle$
- (2) $|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \psi(\alpha)|\alpha\rangle d^2\alpha,\ \psi(\alpha) = \langle\alpha|\psi\rangle$

$$\psi(\alpha) = \langle\alpha|\beta\rangle = \exp\left[\alpha^*\beta - \frac{1}{2}(|\alpha|^2 + |\beta|^2)\right]$$

- BUT, the P function of "classical states" is unique. P function of $|\beta\rangle$ is: $P(\alpha) = \delta(\alpha - \beta)$

The Q function

- Trace of an operator

$$\mathrm{Tr}(\hat{G}) = \sum_{n=0}^{\infty} \langle n|\hat{G}|n\rangle = \int \frac{d^2\alpha}{\pi} \langle \alpha|\hat{G}|\alpha\rangle$$

- Since trace of a density operator is 1,

$$\int \frac{d^2\alpha}{\pi} \langle \alpha|\hat{\rho}|\alpha\rangle = 1$$

  - $\langle \alpha|\hat{\rho}|\alpha\rangle \geq 0, \forall \alpha \in \mathbb{C}$
  - $Q(\alpha) = \frac{1}{\pi}\langle \alpha|\hat{\rho}|\alpha\rangle, \alpha \in \mathbb{C}$ is a proper probability distribution
  - We will later see this is the probability distribution of the output of heterodyne detection on $\hat{\rho}$

Examples of Q functions

- Coherent state $\hat{\rho} = |\beta\rangle\langle\beta|$

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{|\langle\alpha|\beta\rangle|^2}{\pi} = \frac{e^{-|\beta-\alpha|^2}}{\pi}$$

- Number state $Q(\alpha) = \dfrac{|\langle\alpha|n\rangle|^2}{\pi} = \dfrac{e^{-|\alpha|^2}|\alpha|^{2n}}{\pi n!}$

Evaluate the Q functions of the following two states:

Problem 13

- Cat state $|\psi\rangle = \mathcal{N}_+(|\alpha\rangle + |-\alpha\rangle)$, $\hat{\rho} = |\psi\rangle\langle\psi|$
  - Use the correct value of $\mathcal{N}_+$ such that $|\psi\rangle$ is normalized
- Thermal state $\hat{\rho} = \int_{\mathbb{C}} P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$, $P(\alpha) = \dfrac{e^{-|\alpha|^2/N}}{\pi N}$

More on the coherent-state basis…

- Show that:

$$\hat{a} = \int \frac{d^2\alpha}{\pi} \alpha |\alpha\rangle\langle\alpha|$$

$$\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi} \alpha^* |\alpha\rangle\langle\alpha|$$

$$\hat{a}\hat{a}^\dagger = \int \frac{d^2\alpha}{\pi} |\alpha|^2 |\alpha\rangle\langle\alpha|$$

$$\hat{a}^\dagger\hat{a} = \int \frac{d^2\alpha}{\pi} (|\alpha|^2 - 1) |\alpha\rangle\langle\alpha|$$

Problem 14

Lets revisit some quantum mechanics

- Observable: Hermitian operator $\hat{A}$, i.e., $\hat{A}^\dagger = \hat{A}$
- $\hat{A}|a_k\rangle = a_k|a_k\rangle$
- Eigenvalues a_k are real valued
- "Measuring observable $\hat{A}$" = von Neumann measurement described by, $\{\Pi_k = |a_k\rangle\langle a_k|\}$
- If $\hat{A}$ measured on the state $\hat{\rho}$,
  - probability of outcome k, $P(k) = \langle a_k|\hat{\rho}|a_k\rangle$
  - post-measurement state, $|a_k\rangle$
  - Average value of measurement outcome

$$\langle \hat{A}\rangle_{\hat{\rho}} = \sum_k a_k P(k) = \mathrm{Tr}(\rho\hat{A})$$

Heisenberg uncertainty principle (HUP)

THE UNIVERSITY OF ARIZONA

- Non-commuting observables, $\hat{A}$ and $\hat{B}$
  - $[\hat{A}, \hat{B}] = j\hat{C}$
  - Define $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\rho) = \langle \psi | \hat{A} | \psi \rangle, \text{if } \rho = |\psi\rangle\langle\psi|$
  - Define $\Delta\hat{A} = \hat{A} - \langle \hat{A} \rangle$
  - Therefore, $\langle \Delta\hat{A}^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$
- Heisenberg uncertainly relation (we will prove it!)

$$\langle \Delta\hat{A}^2 \rangle \langle \Delta\hat{B}^2 \rangle \geq \frac{1}{4} \left| \langle \hat{C} \rangle \right|^2$$

- **Mathematical** meaning of non-commuting operators
  - They are not *simultaneously diagonalizable*
- **Physical** meaning of non-commuting observables
  - the observables cannot be simultaneously measured

Heisenberg uncertainty principle, contd.

- For two non-commuting observables, show that

$$[\hat{A}, \hat{B}]^\dagger = -[\hat{A}, \hat{B}]$$

Problem 15

  - Since $[\hat{A}, \hat{B}] = j\hat{C}$, $\hat{C}$ is Hermitian, i.e., $\hat{C}^\dagger = \hat{C}$
- Multiply to verify that: $[\Delta\hat{A}, \Delta\hat{B}] = j\hat{C}$ **Problem 16**
- Cauchy-Schwarz inequality $\langle X^2\rangle\langle Y^2\rangle \geq |\langle XY\rangle|^2$:

$$\langle\psi|\Delta\hat{A}^2|\psi\rangle\langle\psi|\Delta\hat{B}^2|\psi\rangle \geq |\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle|^2$$

  - With equality iff $\Delta\hat{A}|\psi\rangle = j\lambda\Delta\hat{B}|\psi\rangle$ for some $\lambda \in \mathbb{C}$

Heisenberg uncertainty principle, contd.

- The rest is algebra…

$$|\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle|^2 = \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}+[\Delta\hat{A},\Delta\hat{B}]}{2}\right)|\psi\rangle\right|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}}{2}\right)|\psi\rangle + \frac{j}{2}\langle\psi|\hat{C}|\psi\rangle\right|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}\Delta\hat{B}+\Delta\hat{B}\Delta\hat{A}}{2}\right)|\psi\rangle\right|^2 + \left|\frac{\langle\psi|\hat{C}|\psi\rangle}{2}\right|^2$$

$$\geq \left|\langle\psi|\hat{C}|\psi\rangle\right|^2/4$$

  - With equality iff $\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle = -\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle$

Heisenberg uncertainty principle, contd.

- So, we have shown: $\langle\psi|\Delta\hat{A}^2|\psi\rangle\langle\psi|\Delta\hat{B}^2|\psi\rangle \geq \left|\langle\psi|\hat{C}|\psi\rangle\right|^2/4$

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle \geq \frac{1}{4}\left|\langle\hat{C}\rangle\right|^2$$

  - With equality iff $\Delta\hat{A}|\psi\rangle = j\lambda\Delta\hat{B}|\psi\rangle$ for a real λ

- States $|\psi\rangle$ that meet the Heisenberg lower bound on the product of variances for a given pair of non-commuting observables are called "minimum uncertainly product" (MUP) states

- Note that the HUP is an "either or" proposition

Field quadrature operators

- Quadrature operators $\hat{a}_1$ and $\hat{a}_2$ defined as:

$$\hat{a} = \hat{a}_1 + j\hat{a}_2 \qquad \hat{a}_1 = (\hat{a} + \hat{a}^\dagger)/2$$
$$\hat{a}^\dagger = \hat{a}_1 - j\hat{a}_2 \qquad \hat{a}_2 = (\hat{a} - \hat{a}^\dagger)/2j$$

  - Show that **Problem 17**

    (1) $\hat{a}_1$ and $\hat{a}_2$ are Hermitian operators

    (2) Their commutator is given by $[\hat{a}_1, \hat{a}_2] = \dfrac{j}{2}$
  - They are non-commuting observables
- Therefore, the HUP states that:

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \geq \frac{1}{16}$$

Quadrature measurement on number states

- Mean, $\langle n|\hat{a}|n\rangle = 0$
  - Therefore, $\langle \hat{a}_1\rangle = \langle n|\hat{a}_1|n\rangle = 0$ and $\langle \hat{a}_2\rangle = \langle n|\hat{a}_2|n\rangle = 0$
- Second moment,

$$\langle n|\hat{a}_1^2|n\rangle = \langle n|\left(\frac{[\hat{a}+\hat{a}^\dagger]^2}{4}\right)|n\rangle$$

$$= \frac{\langle n|\hat{a}^2|n\rangle + \langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle + \langle n|\hat{a}^{\dagger 2}|n\rangle}{4}$$

$$= \frac{2\langle n|\hat{a}^\dagger\hat{a}|n\rangle + 1}{4} = \frac{2n+1}{4}$$

- Uncertainly product for quadrature variances:

$$\langle \Delta\hat{a}_1^2\rangle\langle \Delta\hat{a}_2^2\rangle = \left(\frac{2n+1}{4}\right)^2 \geq \frac{1}{16}$$

Equality holds only for the vacuum state. Number states are not minimum uncertainly-product states

Quadrature measurement on coherent states

- Mean, $\langle \alpha | \hat{a} | \alpha \rangle = \alpha \equiv \alpha_1 + j\alpha_2$
  - Therefore, $\langle \hat{a}_1 \rangle = \langle \alpha | \hat{a}_1 | \alpha \rangle = \alpha_1$ and $\langle \hat{a}_2 \rangle = \langle \alpha | \hat{a}_2 | \alpha \rangle = \alpha_2$
- Second moment,

$$\langle \alpha | \hat{a}_1^2 | \alpha \rangle = \langle \alpha | \left(\frac{[\hat{a} + \hat{a}^\dagger]^2}{4} \right) | \alpha \rangle$$

$$= \ldots$$

- Complete this calculation and show that the uncertainly product for quadrature variances is:

$$\langle \Delta \hat{a}_1^2 \rangle = \langle \Delta \hat{a}_2^2 \rangle = \frac{1}{4}$$

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle = \frac{1}{16}, \ \forall \alpha$$

Problem 18

Coherent states are minimum uncertainly product states

Signal-to-noise ratio

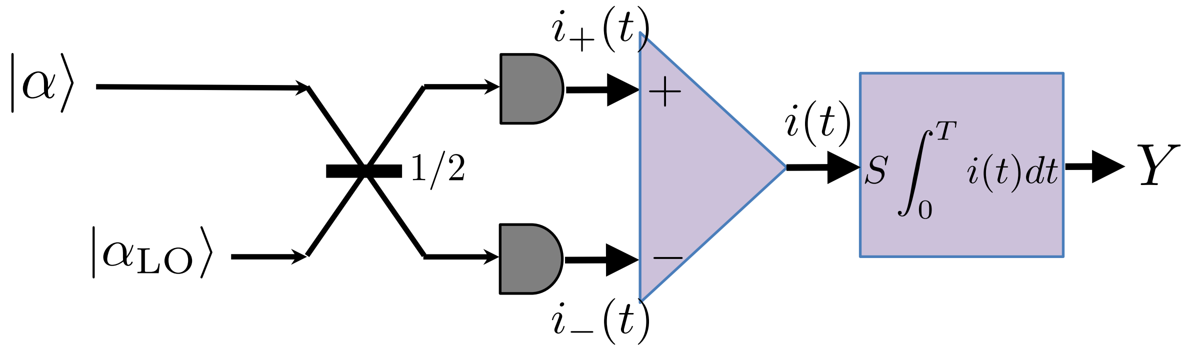
- Quadrature measurement on $|\alpha\rangle, |\alpha|^2 = N$
  - $\mathrm{SNR}_{\mathrm{quadrature}} \equiv \dfrac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle} = \dfrac{\mathrm{Re}(\alpha)}{1/4} = 4\,\mathrm{Re}(\alpha)$
  - Measurement of $\hat{a}_1 e^{j\phi}$ yields a Gaussian random variable with mean $\mathrm{Re}(\alpha e^{j\phi})$ and variance 1/4
- Number measurement on coherent state
  - Prove that $\mathrm{SNR}_{\mathrm{number}} \equiv \dfrac{\langle \hat{N} \rangle^2}{\langle \Delta \hat{N}^2 \rangle} = |\alpha|^2 = N$ **Problem 19**
  - This is consistent with the fact that $P(n) = e^{-N} N^n / n!$, the Poisson distribution has mean N and variance N

Homodyne detection; quantum measurement of $\hat{a}_1 \, e^{j\phi}$

- Local Oscillator (LO) is strong w.r.t. signal, $N_{\mathrm{LO}} \gg N$

$$\alpha = \sqrt{N} e^{j\theta}$$

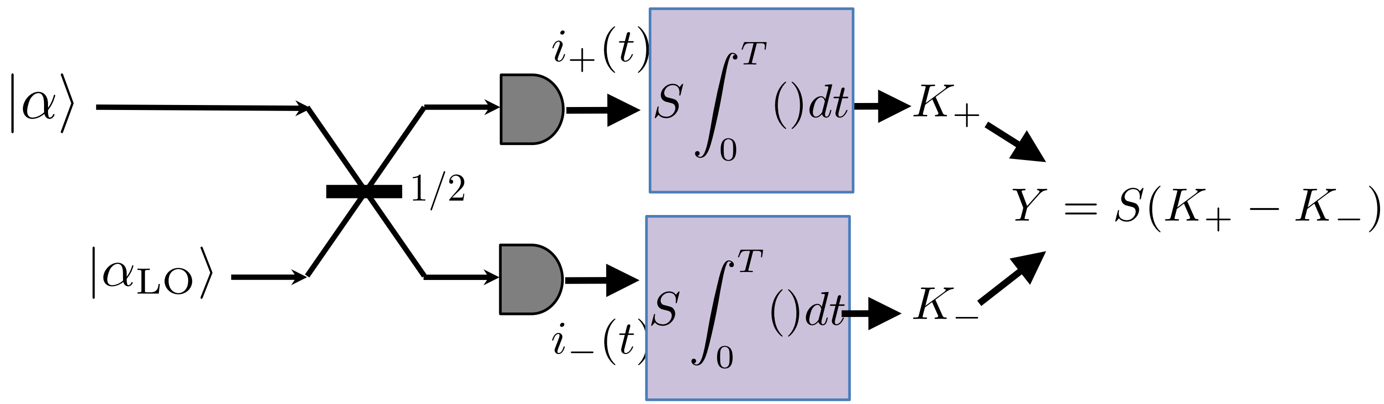
$$\alpha_{\mathrm{LO}} = \sqrt{N_{\mathrm{LO}}}\, e^{j\phi}$$



$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = S(N_+ - N_-)$$

$$\sigma^2 = S^2(N_+ + N_-)$$



Substitute these, take $N_{\mathrm{LO}} \gg N$ limit and pick an appropriate scaling S, to show that:

$$K_+ \sim \mathrm{Poisson}(N_+) \sim \mathcal{N}(N_+, N_+);\; N_+ = \left|\frac{\alpha + \alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$K_- \sim \mathrm{Poisson}(N_-) \sim \mathcal{N}(N_-, N_-);\; N_- = \left|\frac{\alpha - \alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$Y \sim \mathcal{N}\left(\mathrm{Re}(\alpha e^{j\theta_{\mathrm{LO}}}), \frac{1}{4}\right)$$

Problem 20

Upcoming topics

- Quadrature eigenkets
- Squeezed states of light
- Phase space picture of quantum optical states
  - Characteristic functions, and Wigner functions