Photonic Quantum Information Processing
OPTI 647: Lecture 3

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Outline for today

- Coherent states and linear optics
- Quantizing the field
- Distinguishing pure states

General pure state of a single mode

Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K is a Poisson random variable of mean N

Mode $\phi(t)$, a quantum system, is excited in a number state $|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K = n (exactly so; K is not a random variable).

A mode of ideal laser light is in a coherent state.
Number (Fock) state of a given mode is very hard to produce experimentally

There are infinitely many other types of "states" of the mode $\phi(t)$.
Coherent state and Fock state are just two example class of states

This orthogonality (in the Hilbert state) is different from that of modes (in L_2 norm space)

$|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$ Fock states of a mode are special: they form an orthonormal basis that spans any general quantum state $|\psi\rangle$ of that mode

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ , \ \sum_{n=0}^{\infty} |c_n|^2 = 1$$

Coherent state as a quantum state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \underbrace{\left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right)}_{c_n} |n\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \cdots$$

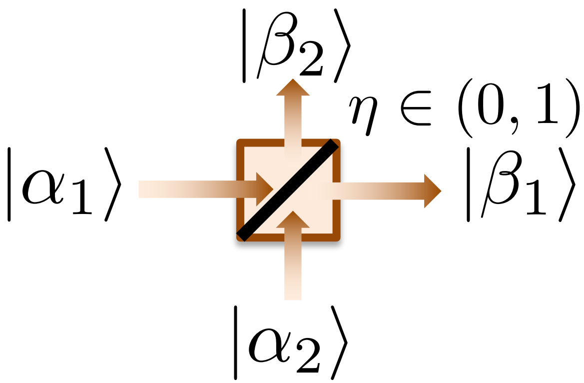
Fock states can be thought of as infinite-length unit-length column vectors corresponding to the orthogonal axes of an infinite-dimensional vector space

Ideal photon detection is a von Neumann quantum measurement described by projectors, $\{|n\rangle\langle n|\}$, $n = 0, 1, \ldots, \infty$

Ideal direct detection on a coherent state $|\alpha\rangle$ produces outcome "n" (i.e., n "clicks") with probability, $p_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = \dfrac{e^{-N} N^n}{n!}$

Poisson detection statistics in a laser pulse is a result of the projection of the quantum state of the laser pulse—a coherent state–on to one of the Fock states

Coherent states and “linear-optical transformations” (beam-splitters)

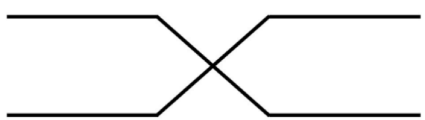
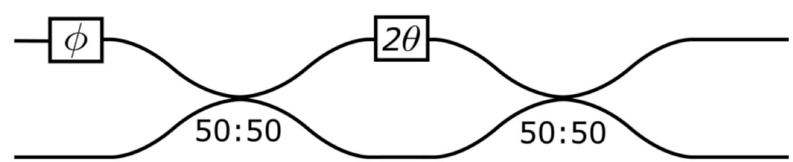


Transmissivity, $\eta = \cos^2\theta$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} e^{i\phi}\cos\theta & -\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{pmatrix}}_{T(\theta,\phi)} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Complex Unitary matrix, $T^{-1} = T^*$

Mach Zehnder Interferometer (MZI):

$T(\theta, \phi) =$  $=$ 

An arbitrary 2-mode linear optical unitary can be realized with a MZI – two 50-50 beam-splitters and two phases

Arbitrary N-mode linear optical unitary

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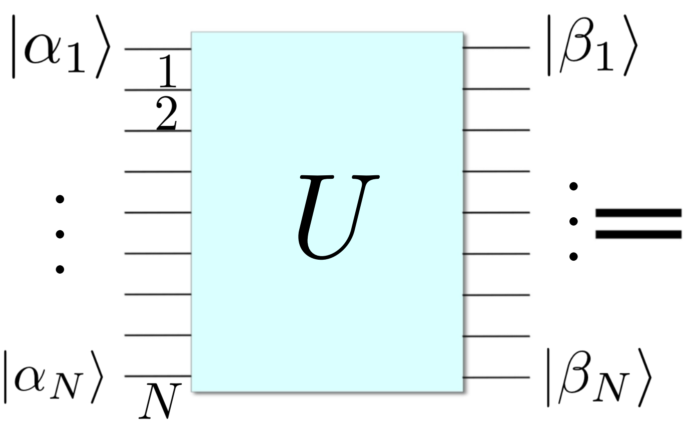
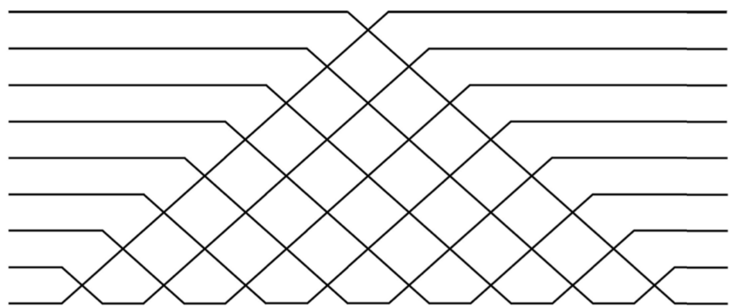
- Any N-by-N unitary U can be realized with M = N(N-1)/2 Mach Zehnder Interferometers, $T_{m,n}(\theta, \phi)$. So, we need N(N-1) 50-50 beam-splitters and N(N-1) tunable phases to realize any N-mode linear optical unitary U



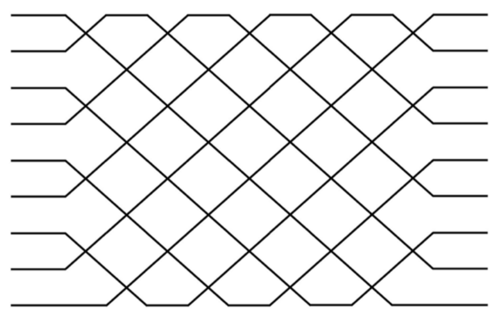
$$\vec{\beta} = U\vec{\alpha}$$

a



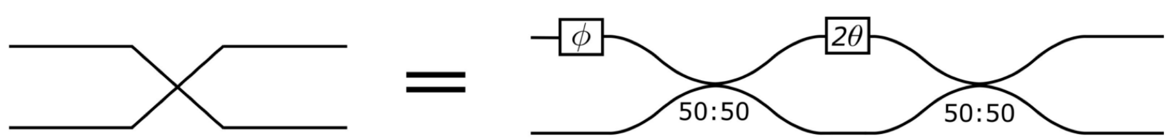
Reck et al., PRL 73, 1 (1994)

b



Clements et al., Optica 3 (12), 1460-1465 (2016)

$T_{m,n}(\theta, \phi)$ =



A coherent state is *always* single mode

- By an appropriate choice of modal basis, any "multi-mode" coherent state can always be expressed as a single mode coherent state

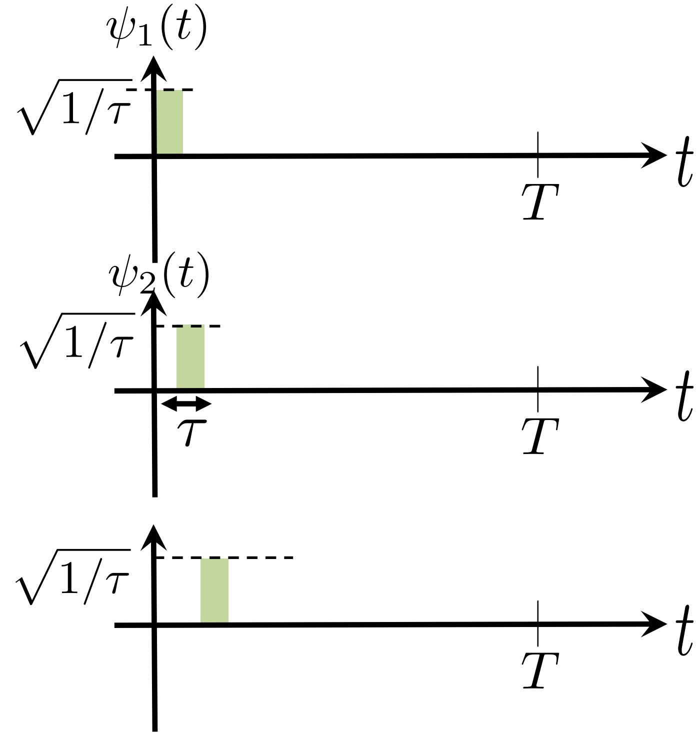
$$|\alpha_1\rangle|\alpha_2\rangle\ldots|\alpha_K\rangle \equiv |\beta\rangle|0\rangle\ldots|0\rangle$$

  – In other words… if we have a deterministic field in any spatio-temporal shape (of any given polarization), we can always represent that as a *single-mode* coherent state of an appropriate normalized mode
  – We will see later, this is not true for other quantum states in general. For example, a multimode thermal state or a multimode squeezed state, etc.

Slicing a coherent state pulse (in time)

(1) Single-mode coherent state of this mode: $\phi(t)$

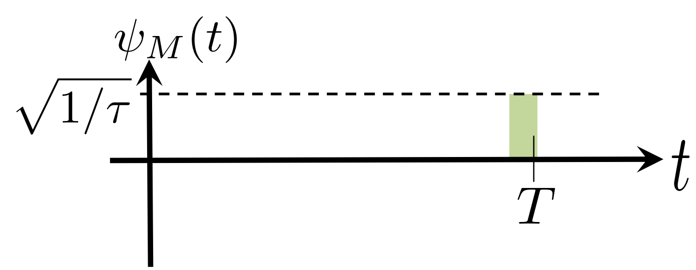
$|\alpha\rangle,\ \alpha = \sqrt{N},\ N = E^2 T$



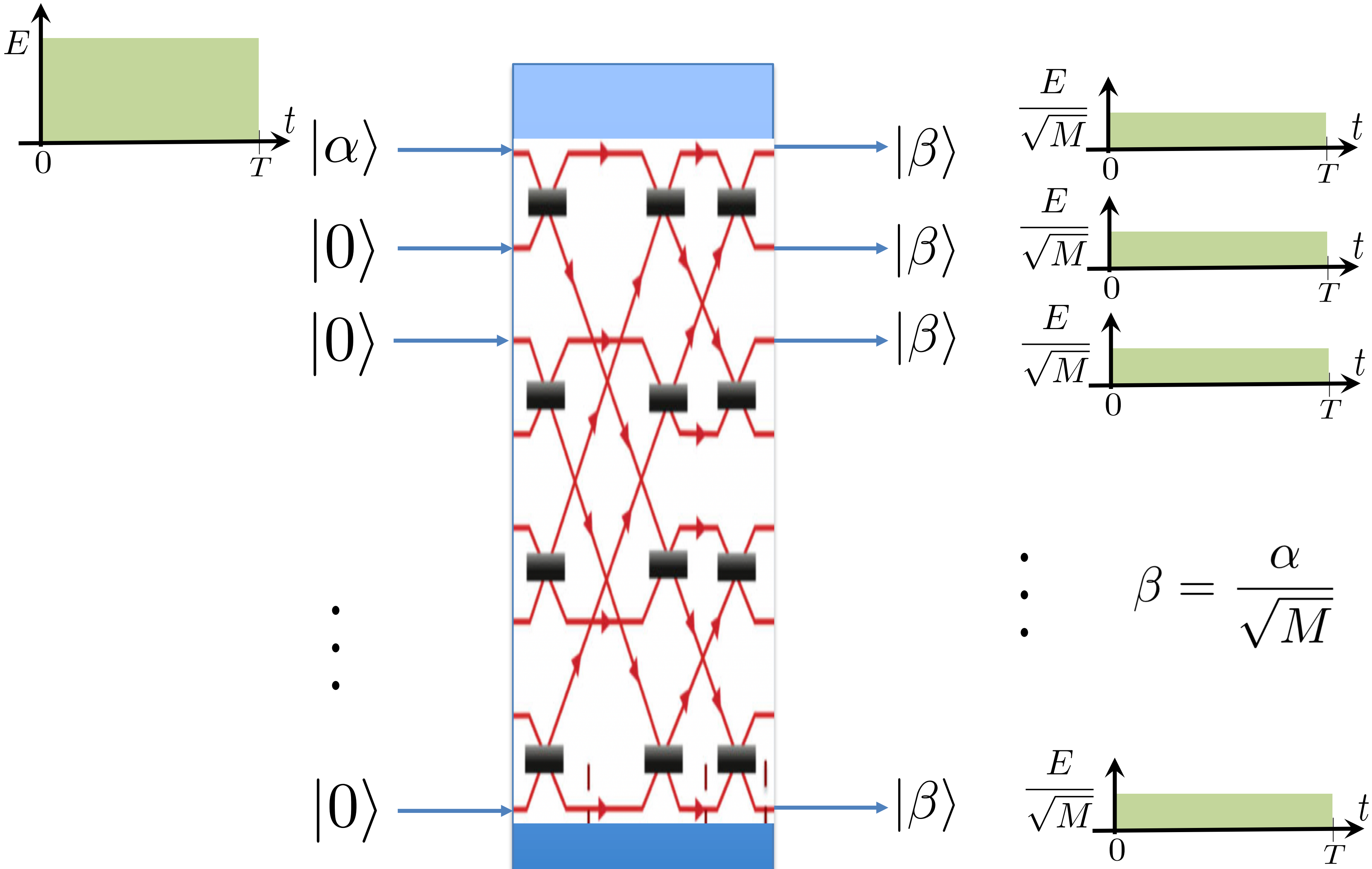
(2) M-mode coherent state of the modes: $\psi_k(t), k = 1, \ldots, M$

$$|\beta\rangle|\beta\rangle\ldots|\beta\rangle \quad , \beta = \sqrt{\frac{N}{M}} = \frac{\alpha}{\sqrt{M}}$$

$M = \dfrac{T}{\tau}$ Orthogonal temporal modes



Slicing a coherent state pulse (in space)



$$\beta = \frac{\alpha}{\sqrt{M}}$$

Examples of optical qubits

- Single-rail qubit $|\mathbf{0}\rangle = |0\rangle, |\mathbf{1}\rangle = |1\rangle$
- Dual-rail qubit $|\mathbf{0}\rangle = |0,1\rangle, |\mathbf{1}\rangle = |1,0\rangle$
- Cat-basis qubit $|\mathbf{0}\rangle = N_+(|\alpha\rangle + |-\alpha\rangle),$

$$|\mathbf{1}\rangle = N_-(|\alpha\rangle - |-\alpha\rangle)$$

Prove that the cat-basis qubit states are mutually orthogonal, and find the normalization constants N_+ and N_- in terms of α

Problem 4

Binary pure-state discrimination

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$$|\psi_1\rangle \;(\text{hypothesis}\,\mathrm{H}_1) \text{ vs. } |\psi_2\rangle \;(\text{hypothesis}\,\mathrm{H}_2)$$

– Assume equal priors: $p_1 = p_2 = \dfrac{1}{2}$

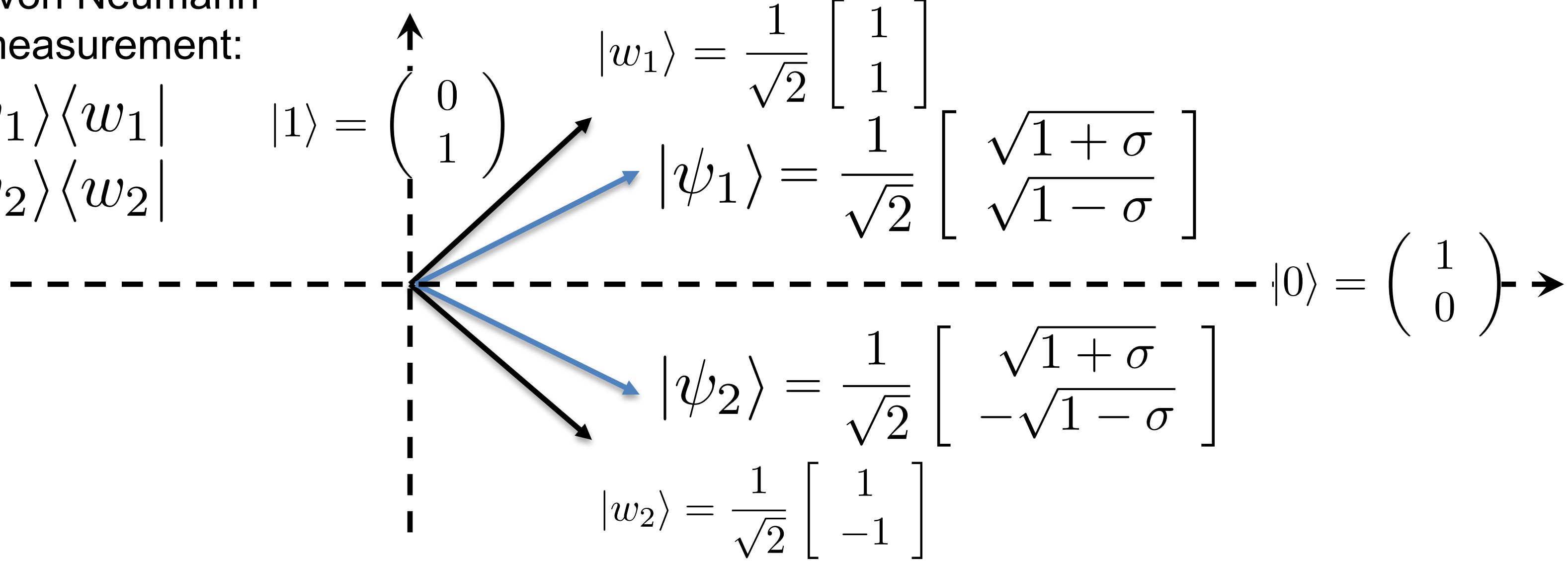
Inner product between the two states

$$\langle\psi_1|\psi_2\rangle = \sigma$$

Consider a von Neumann projective measurement:

$$\Pi_1 = |w_1\rangle\langle w_1|$$

$$\Pi_2 = |w_2\rangle\langle w_2|$$



$$P_e = P(\mathrm{H}_1)P(\mathrm{H}_2|\mathrm{H}_1) + P(\mathrm{H}_2)P(\mathrm{H}_1|\mathrm{H}_2)$$

$$= \frac{1}{2}\left|\langle w_2|\psi_1\rangle\right|^2 + \frac{1}{2}\left|\langle w_1|\psi_2\rangle\right|^2$$

Show that: $P_e = \dfrac{1}{2}\left[1 - \sqrt{1-|\sigma|^2}\right]$ and find the expression for minimum average error probability for $P(\mathrm{H}_1) = p, P(\mathrm{H}_2) = 1-p$

Problem 5

Coherent states are not orthogonal

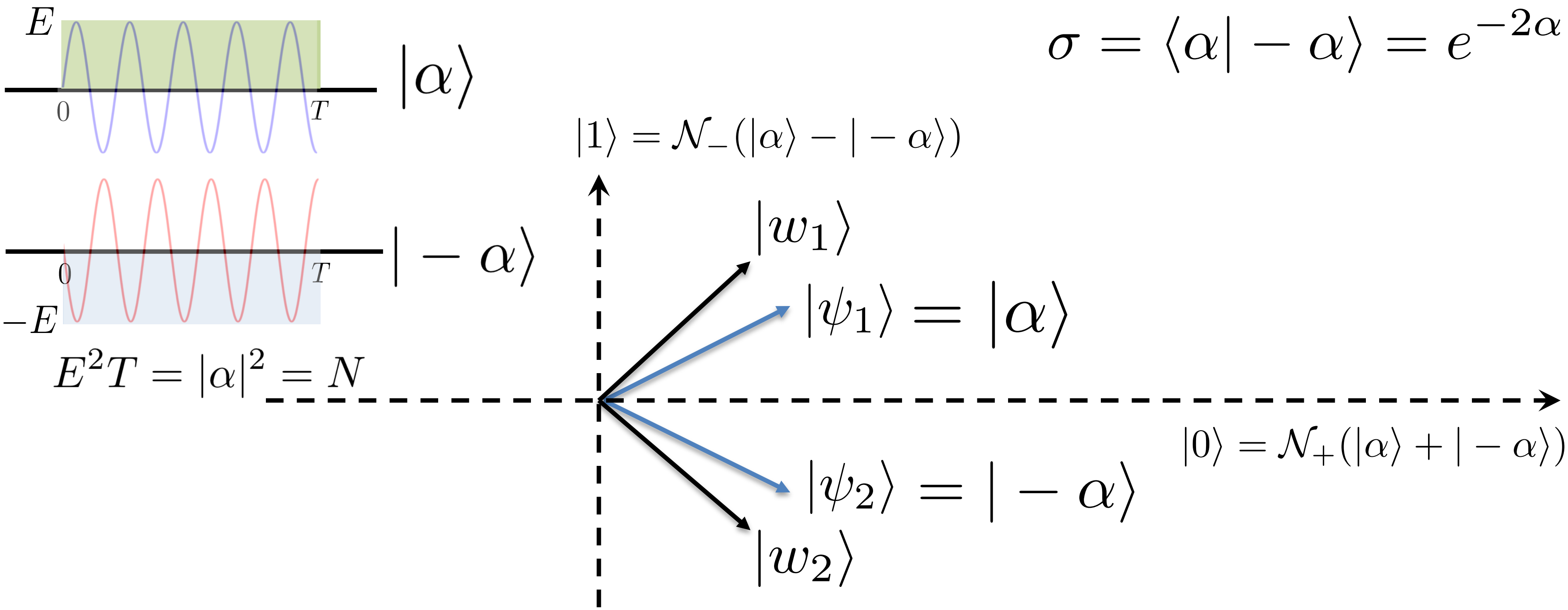
- Distinguishability of two coherent states
  - Recall: $|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right) |n\rangle$
  - Inner product between two coherent states:

$$\langle\alpha|\beta\rangle = \exp\left[\alpha^*\beta - \frac{1}{2}(|\alpha|^2 + |\beta|^2)\right]$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$$

Binary phase shift keying (BPSK) coherent-state modulation

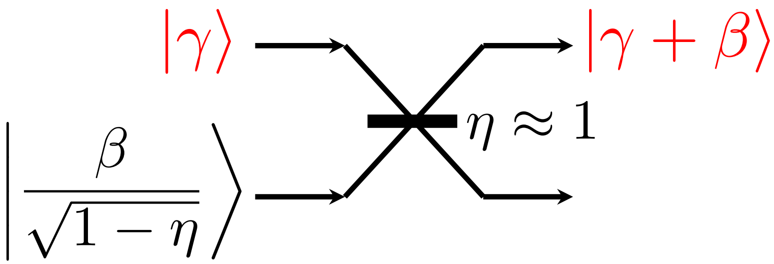
$$\sigma = \langle \alpha | -\alpha \rangle = e^{-2\alpha}$$

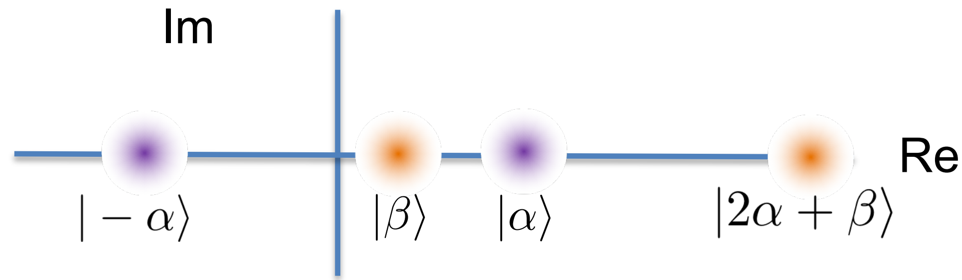


- Optimal measurement operators are cat states
- Minimum probability of error

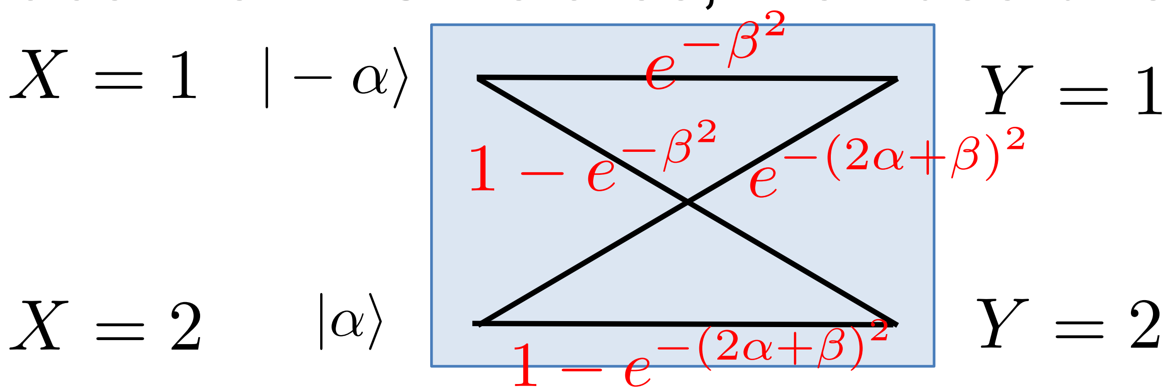
$$P_e = \frac{1}{2}\left[1 - \sqrt{1 - e^{-4|\alpha|^2}}\right]$$

Kennedy receiver: a suboptimal receiver





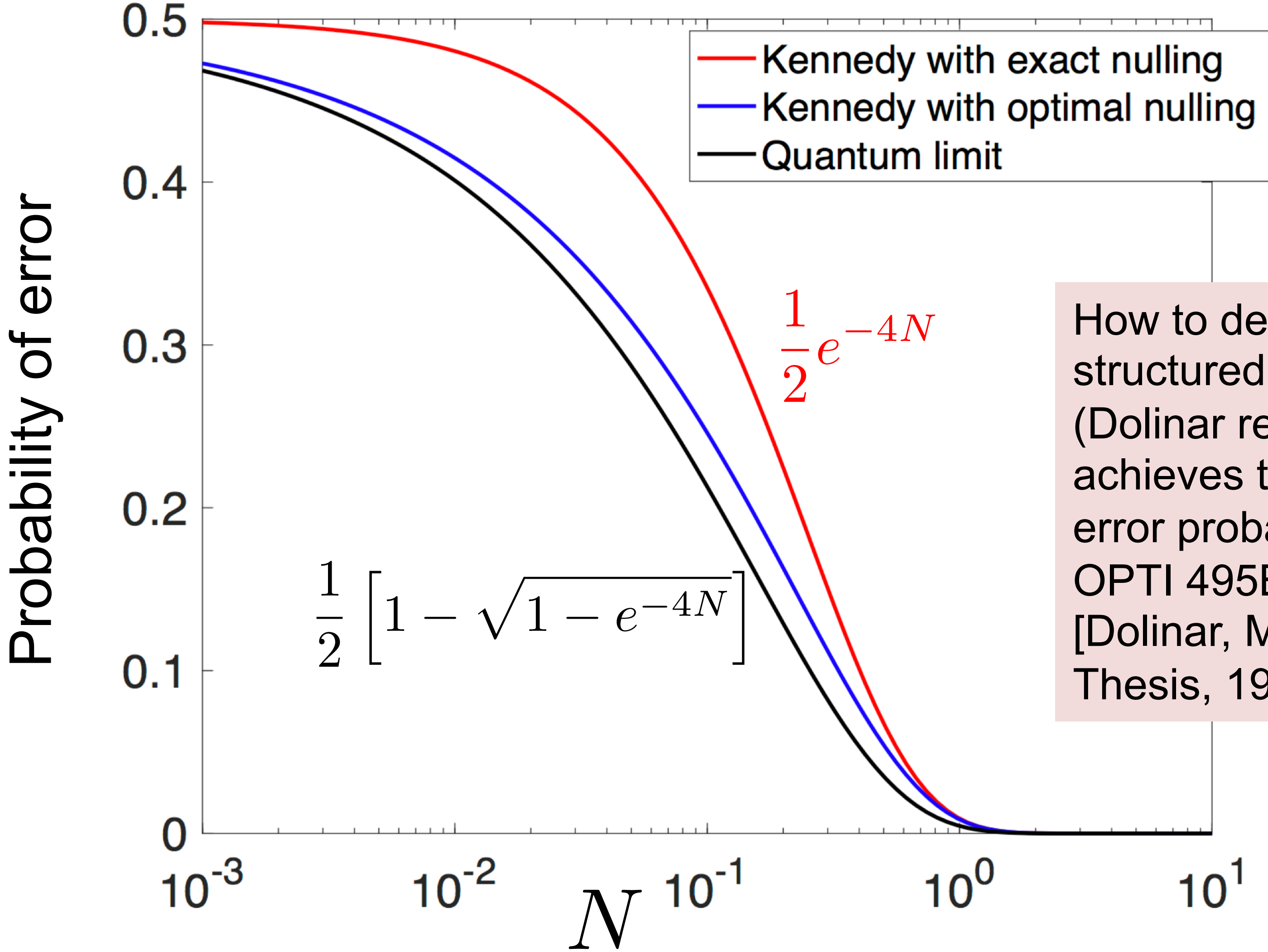
- Displace the BPSK states, then use direct detection



$$P_e(N) = \min_{\beta}\left[\frac{1}{2}e^{-(2\alpha+\beta)^2} + \frac{1}{2}\left(1 - e^{-\beta^2}\right)\right]$$

$$= \frac{1}{2}e^{-4N},\ \beta = 0 \qquad \text{(exact nulling case)}$$

BPSK error probability



How to design a structured receiver (Dolinar receiver) that achieves the minimum error probability. See OPTI 495B/595B [Dolinar, MIT Ph.D. Thesis, 1976]

Optimize (minimize) the probability of error of the optimal-nulling Kennedy receiver (find optimal β) and plot the probabilities of error as function of N, as above

Problem 6

Density operator – pure and mixed states

- State of a quantum system
  - Complete knowledge is a pure state $|\psi\rangle$
  - Incomplete knowledge is a statistical (classically-random) mixture of pure states
    - density operator: positive and unit trace, $\rho = |\psi\rangle\langle\psi|$

$$\rho_X = \sum_x p_X(x)|\psi_x\rangle\langle\psi_x| = \sum_i \lambda_i|\lambda_i\rangle\langle\lambda_i|$$

Mixture of pure states, $\sum_x p_X(x) = 1$

the states don't have to be orthogonal

Spectral decomposition

$$\sum_i \lambda_i = 1$$

$$\langle\lambda_i|\lambda_j\rangle = \delta_{ij}$$

Take the statistical mixture intuition with a pinch of salt -- same density operator can be expressed as different mixtures

Projective measurement on mixed state

- Measure state, ρ with projective measurement

$$\rho \longrightarrow \boxed{\{\Pi_j\}_j \text{ where } \sum_j \Pi_j = I} \longrightarrow j$$

Probability of outcome j

$$p(j) = \mathrm{Tr}(\rho \Pi_j)$$

Conditional post-measurement state

$$\rho_j = \frac{\Pi_j \rho \Pi_j}{p(j)}$$

Projective measurement on mixed state

- Consider ensemble of pure states, $\mathcal{E} = \{p_X(x), |\psi_x\rangle\}$
  - Density operator, $\rho = \sum_x p_X(x)|\psi_x\rangle\langle\psi_x|$
- Measurement projectors, $\{\Pi_j\}_j$ where $\sum_j \Pi_j = I$
- Assume the state in the ensemble was $|\psi_x\rangle$
  - Post-measurement states: $\dfrac{\Pi_j|\psi_x\rangle}{\sqrt{p_{J|X}(j|x)}}$
  - $p_{J|X}(j|x) = \langle\psi_x|\Pi_j|\psi_x\rangle$
- If we get outcome j, we have conditional ensemble

$$\mathcal{E}_j \equiv \left\{p_{X|J}(x|j), \Pi_j|\psi_x\rangle/\sqrt{p_{J|X}(j|x)}\right\}_{x\in\mathcal{X}}$$

  - with $p_{X|J}(x|j) = p_{J|X}(j|x)p_X(x)/p_J(j)$

$$p_J(j) = \sum_{x\in\mathcal{X}} p_{J|X}(j|x)p_X(x)$$

Projective measurement on mixed state

- Density operator of this post-measurement ensemble of states $\mathcal{E}_j$

$$\sum_{x\in\mathcal{X}} p_{X|J}(x|j)\frac{\Pi_j|\psi_x\rangle\langle\psi_x|\Pi_j}{p_{J|X}(j|x)}$$

$$=\Pi_j\left(\sum_{x\in\mathcal{X}}\frac{p_{X|J}(x|j)}{p_{J|X}(j|x)}|\psi_x\rangle\langle\psi_x|\right)\Pi_j$$

$$=\Pi_j\left(\sum_{x\in\mathcal{X}}\frac{p_{J|X}(j|x)p_X(x)}{p_{J|X}(j|x)p_J(j)}|\psi_x\rangle\langle\psi_x|\right)\Pi_j$$

$$=\frac{\Pi_j\left(\sum_{x\in\mathcal{X}}p_X(x)|\psi_x\rangle\langle\psi_x|\right)\Pi_j}{p_J(j)}$$

$$=\frac{\Pi_j\rho\Pi_j}{p_J(j)}.$$

Projective measurement on mixed state

- Probability of outcome j

$$
\begin{aligned}
p_J(j) &= \sum_{x \in \mathcal{X}} p_{J|X}(j|x) p_X(x) \\
&= \sum_{x \in \mathcal{X}} p_X(x) \langle \psi_x | \Pi_j | \psi_x \rangle \\
&= \sum_{x \in \mathcal{X}} p_X(x) \mathrm{Tr}\{ |\psi_x\rangle\langle\psi_x| \Pi_j \} \\
&= \mathrm{Tr}\left\{ \sum_{x \in \mathcal{X}} p_X(x) |\psi_x\rangle\langle\psi_x| \Pi_j \right\} \\
&= \mathrm{Tr}\{ \rho \Pi_j \}.
\end{aligned}
$$

Statistical mixture of coherent states

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- **Classical state**: P function representation

$$\rho = \int_{\mathbb{C}} P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

What is the P function of a coherent state $|\beta\rangle$?

- Statistical field in classical EM theory

- Multimode classical state $\rho = \int_{\mathbb{C}^n} P(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| d^{2n}\boldsymbol{\alpha}$

  where $|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle \ldots |\alpha_n\rangle$

  – cannot in general be written as a single mode state unlike a multimode coherent state

Single-mode thermal state

- Gaussian mixture of coherent states

$$\rho = \int_{\mathbb{C}} P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha \qquad P(\alpha) = \frac{e^{-|\alpha|^2/N}}{\pi N}$$

  - Probability distribution for photon counting, $\{\Pi_n = |n\rangle\langle n|\}$
    $P(n) = \mathrm{Tr}(|n\rangle\langle n|\rho) = \langle n|\rho|n\rangle$
  - Show that $P(n) = \dfrac{N^n}{(1+N)^{1+n}}; n = 0, 1, \ldots$
  - Show that, $\langle n|\rho|m\rangle = 0, \mathrm{if}\, m \neq n$
  - Hence, $\rho = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|$

Problem 7

Phase scrambled coherent state

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- Application of a phase, $U_\theta|\alpha\rangle = |\alpha e^{i\theta}\rangle$
- Consider the state after application of a random phase to a coherent state:

$$\rho = \int_0^{2\pi} U_\theta|\alpha\rangle\langle\alpha|U_\theta^\dagger d\theta$$

  – Show that: $P(n) = \langle n|\rho|n\rangle = \dfrac{e^{-N}N^n}{n!}; n = 0, 1, \ldots$
  – and that $\langle n|\rho|m\rangle = 0, \text{if } m \neq n$
  – So, ρ is diagonal in the number basis, $\rho = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|$

Problem 8

- Circularly-symmetric states are diagonal in the number basis (we will revisit this later)

Annihilation operator *of a mode*

- Recall field quantization:

$$E(t) = \sum_{k=1}^{K} a_i \phi_i(t) \quad \Longrightarrow \quad \hat{E}(t) = \sum_{k=1}^{K} \hat{a}_i \phi_i(t)$$

Coherent state of modes $\phi_i(t)$ with complex field amplitudes, a_i

Field operator for the set of modes $\phi_i(t)$, with modal annihilation operators $\hat{a}_i$

- Annihilation operator $\hat{a}$ of a single mode
  - Eigenstate is a coherent state $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
  - "Annihilates" photon number, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

    $$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
  - Number operator, $\hat{N} = \hat{a}^\dagger\hat{a}$

    $\hat{N}|n\rangle = n|n\rangle$
  - Show that, $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I}$

Problem 9

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The phase operator

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- Application of a phase, $U_\theta|\alpha\rangle = |\alpha e^{i\theta}\rangle$
- The phase operator,

$$U_\theta = e^{i\theta\hat{N}} = e^{i\theta\hat{a}^\dagger a}$$

- Show that random phase scrambling of any pure state leads to a number diagonal state

$$\rho = \int_0^{2\pi} U_\theta|\psi\rangle\langle\psi|U_\theta^\dagger d\theta \quad = \sum_{n=0}^{\infty} P(n)|n\rangle\langle n|$$

  – With $P(n) = |\langle n|\psi\rangle|^2$

Problem 10

Coherent states resolve the identity

- Recall: $\hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|$
- Prove that:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d^2\alpha = \hat{I}$$

Problem 11

- Coherent states are an “over-complete” basis
- They are not orthogonal (and hence cannot be distinguished perfectly)
- They form a positive operator valued measure (POVM) – the most general description of a quantum measurement

Quantization of the field: summary

- Classical (deterministic) field (coherent state)

$$E(t) = \sum_{k=1}^{K} a_i \phi_i(t)$$

- Quantum description of the field: $\hat{E}(t) = \sum_{k=1}^{K} \hat{a}_i \phi_i(t)$
  - Field becomes an operator
  - Field described by a quantum state of constituent modes
  - Modal annihilation operator: $\hat{a}_i$
  - Classical field is a special case: each mode i is excited in a coherent state $|\alpha_i\rangle$, $\alpha_i = a_i$
  - Classical statistical field is a mixture of coherent states, density operator $\rho = \int P(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|d\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle|\alpha_2\rangle \ldots |\alpha_K\rangle$

Recap of what we learnt today

$c_n = \langle n|\alpha\rangle$

- Coherent state is *always* single mode $|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right) |n\rangle$
- "splitting" a coherent state: product of coherent states
- Classical state is a mixture of coherent states
- Coherent states are not orthogonal $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$ yet they resolve the identity, $\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| d^2\alpha = \hat{I}$
- Distinguishing equally-likely states, $\{|\psi_1\rangle, |\psi_2\rangle\}$, $\langle\psi_1|\psi_2\rangle = \sigma$ the minimum average Pr(error), $P_e = \frac{1}{2}\left[1 - \sqrt{1 - |\sigma|^2}\right]$
- The coherent state is an eigenstate of the "field" operator, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$; $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$
- Canonical commutation relation, $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I}$
- Applying random phase to a pure state gives us a "circularly symmetric" state, which is number diagonal

Upcoming topics

- Single mode quantum optics
  - Phase space, Characteristic functions, Wigner functions, Entanglement
  - Squeezed states