Photonic Quantum Information Processing
OPTI 647: Lecture 11

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Announcements

- It's nice to be back into class.
- Office hours every Wednesday, 11-12 am, at conf. room 447.
- Office hours are not only for h/w questions: You may want some more details or some literature guidance.
- Sometimes I may write on the whiteboard. Feel free to take notes.
- We will cover multi-mode Gaussian transformations in the Heisenberg picture and phase space.
- We will then examine applications (quantum estimation theory, etc…)

Outline

- Topics to be covered today
  - Gaussian transformations of single mode states (phase, displacement, single-mode squeezing)
  - Gaussian transformations of two-mode states (beam splitter, two-mode squeezing)
  - We’ll stay in the Heisenberg picture (for now).

Renaming quadrature operators

- Notation
  - Saikat: $\hat{a} = \hat{a}_1 + j\hat{a}_2$, $\hat{a}^\dagger = \hat{a}_1 - j\hat{a}_2$
    - Quadrature commutator: $[\hat{a}_1, \hat{a}_2] = j/2$
    - Quadrature variance of vacuum state: $\langle \Delta\hat{a}_1^2 \rangle = \langle \Delta\hat{a}_2^2 \rangle = 1/4$
  - Christos: $\hat{a} = (\hat{q} + j\hat{p})/\sqrt{2}$, $\hat{a}^\dagger = (\hat{q} - j\hat{p})/\sqrt{2}$
    - Quadrature commutator: $[\hat{q}, \hat{p}] = j$
    - Quadrature variance of vacuum state: $\langle \Delta\hat{q}^2 \rangle = \langle \Delta\hat{p}^2 \rangle = 1/2$
- Reason for this switch:
  - We will analyze multi-mode states and would like to reserve the subscript to index modes
  - For a coherent state, $\langle \Delta\hat{a}_1^2 \rangle = \langle \Delta\hat{a}_2^2 \rangle = \dfrac{1}{4}$

$$\langle \Delta\hat{p}^2 \rangle = \langle \Delta\hat{q}^2 \rangle = \frac{1}{2}$$

Gaussian unitaries: Phase shifting

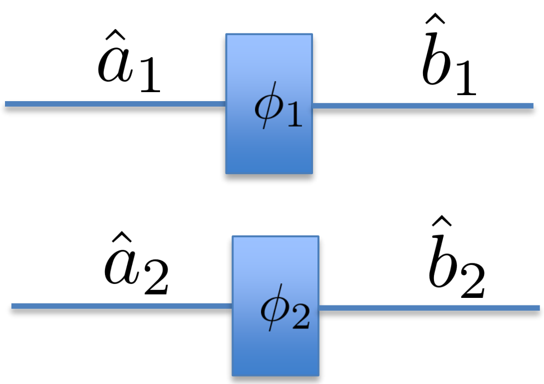
Gaussian unitary operators are generated by quadratric Hamiltonians (quadratic in $\hat{a}$ and $\hat{a}^\dagger$)

<u>Phase shift</u> $H = \hbar\phi\hat{a}^\dagger\hat{a} \rightarrow U(\phi) = \exp\left(-i\phi\hat{a}^\dagger\hat{a}\right)$

Transformed mode: $\hat{b} = U(\phi)\hat{a}U^\dagger(\phi) = \hat{a}e^{-i\phi}$

$$\hat{b} = \hat{a} + (-i\phi)[H,\hat{a}] + \frac{(-i\phi)^2}{2!}[H,[H,\hat{a}]] + \ldots =$$
$$= \hat{a} + (-i\phi)(-\hat{a}) + \frac{(-i\phi)^2}{2!}\hat{a} + \ldots =$$
$$= \hat{a}\sum_{n=0}^{\infty}\frac{(-i\phi)^n}{n!} = \hat{a}e^{-i\phi}$$

we have used the BCH relation

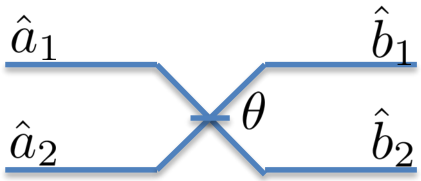


$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix} = \begin{pmatrix}e^{-i\phi_1} & 0\\ 0 & e^{-i\phi_2}\end{pmatrix}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}$$

Gaussian unitaries: 2-modes

Beam splitter

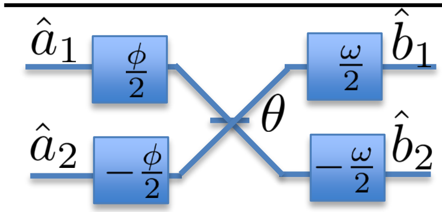
$$H(\theta) = \hbar\theta \hat{a}_1^\dagger \hat{a}_2 + \hbar\theta \hat{a}_1 \hat{a}_2^\dagger$$



$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

Problem 44: Derive the matrix transformation for the beam splitter

General 2-mode passive Gaussian Unitary

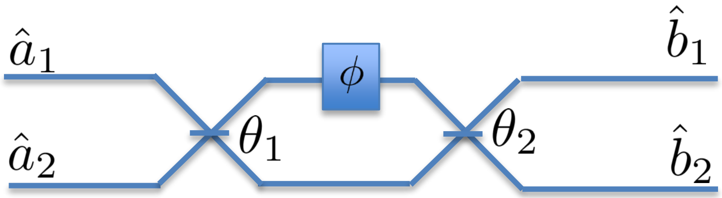


$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = U(\theta, \phi, \omega) \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

$$U(\theta, \phi, \omega) = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\frac{\omega}{2}} & 0 \\ 0 & e^{-i\frac{\omega}{2}} \end{pmatrix}$$

$$U(\theta, \phi, \omega) = \begin{pmatrix} \cos\theta e^{i(\phi+\omega)/2} & -i\sin\theta e^{i(\phi-\omega)/2} \\ -i\sin\theta e^{-i(\phi-\omega)/2} & \cos\theta e^{i(\phi+\omega)/2} \end{pmatrix}$$

Mach-Zehnder interferometer



Mach-Zehnder interferometer

Problem 45: Find the transformed modes and prove that the total photon number of the input and output is conserved. Calculate the energy difference between the two output modes. Assume that both beam splitters are $U\left(\frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{2}\right)$. Verify the relation at the bottom of the page.

The MZ interferometer is a method to estimate some unknown phase by photon counting in each output port and subtracting their values.

$$\langle \hat{N}_{b_1} \rangle - \langle \hat{N}_{b_2} \rangle = \langle \hat{N}_{a_1} + \hat{N}_{a_2} \rangle \cos\phi - i\langle \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \rangle \sin\phi$$

Reck decomposition of U(N)

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N-mode passive transformation U(N).

We need **N(N-1)/2** beam splitters and phase shifters.

[Reck et al. 1994]

$$U(N)T_{N,N-1}\dots T_{2,1} = \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_N} \end{pmatrix}$$

$$U(N) = \begin{pmatrix} e^{i\phi_1} & & \\ & \ddots & \\ & & e^{i\phi_N} \end{pmatrix} T^\dagger_{2,1}\dots T^\dagger_{N,N-2}T^\dagger_{N,N-1}$$

Where the matrix $T_{n,m}$ is a **unit matrix** with the matrix elements $t_{nn}, t_{mm}, t_{nm}, t_{mn}$ are replaced by the four matrix elements of some 2x2 unitary matrix.

Advanced Problem 3: Study Phys. Rev. Lett. **73**, 58 (1994), present it briefly and apply the Reck decomposition to the Hadamard gate $H = H_1 \otimes \dots \otimes H_1$

$$H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$U(N)T_{N,N-1}T_{N,N-2}\dots T_{N-1,1} = \begin{pmatrix} U(N-1) & \\ & e^{i\phi_N} \end{pmatrix}$$

$$U(N-1)T_{N-1,N-2}T_{N-1,N-3}\dots T_{N-2,1} = \begin{pmatrix} U(N-2) & & \\ & e^{i\phi_{N-1}} & \\ & & e^{i\phi_N} \end{pmatrix}$$

$$\vdots$$

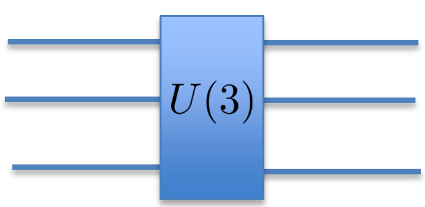
Example on Reck decomposition

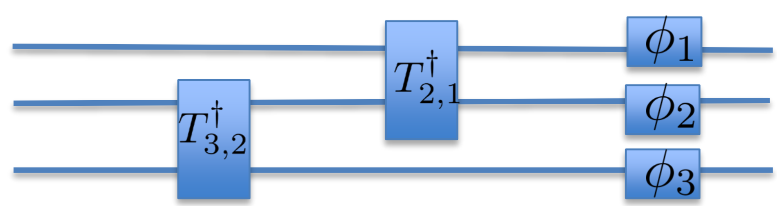
$$U(3) = \begin{pmatrix} e^{i\phi 1}\cos(\theta 2) & e^{i\phi 1}\cos(\theta 1)\sin(\theta 2) & e^{i\phi 1}\sin(\theta 1)\sin(\theta 2) \\ -e^{i\phi 2}\sin(\theta 2) & e^{i\phi 2}\cos(\theta 1)\cos(\theta 2) & e^{i\phi 2}\cos(\theta 2)\sin(\theta 1) \\ 0 & -e^{i\phi 3}\sin(\theta 1) & e^{i\phi 3}\cos(\theta 1) \end{pmatrix}$$

$$U(3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta 1) & -\sin(\theta 1) \\ 0 & \sin(\theta 1) & \cos(\theta 1) \end{pmatrix} = \begin{pmatrix} e^{i\phi 1}\cos(\theta 2) & e^{i\phi 1}\sin(\theta 2) & 0 \\ -e^{i\phi 2}\sin(\theta 2) & e^{i\phi 2}\cos(\theta 2) & 0 \\ 0 & 0 & e^{i\phi 3} \end{pmatrix}$$

$$U(3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta 1) & -\sin(\theta 1) \\ 0 & \sin(\theta 1) & \cos(\theta 1) \end{pmatrix}\begin{pmatrix} \cos(\theta 2) & -\sin(\theta 2) & 0 \\ \sin(\theta 2) & \cos(\theta 2) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\phi 1} & 0 & 0 \\ 0 & e^{i\phi 2} & 0 \\ 0 & 0 & e^{i\phi 3} \end{pmatrix}$$

$$U(3)T_{3,2}T_{2,1} = \begin{pmatrix} e^{i\phi 1} & 0 & 0 \\ 0 & e^{i\phi 2} & 0 \\ 0 & 0 & e^{i\phi 3} \end{pmatrix} \Rightarrow U(3) = \begin{pmatrix} e^{i\phi 1} & 0 & 0 \\ 0 & e^{i\phi 2} & 0 \\ 0 & 0 & e^{i\phi 3} \end{pmatrix} T^{\dagger}_{2,1}T^{\dagger}_{3,2}$$





Normal modes

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Since U(N) can be decomposed in sequence of beam splitters and phases, the general Hamiltonian of U(N) can be decomposed as:

$$H = \hbar \sum_{n,m=1}^{N} \hat{a}_n^\dagger B_{nm} \hat{a}_m \longrightarrow \hat{b}_n = \sum_{n=1}^{N} U_{nm} \hat{a}_n$$

Where $U_{nm} = V_{nm} V_{mn}^*$ and $\hat{b}$ are the *normal modes*.

Where B is a Hermitian matrix and U_{nm} are the matrix elements of U(N).

For some unitary matrix V:

$$VHV^\dagger = \hbar \sum_{j,n,m=1}^{N} V_{jn} \hat{a}_n^\dagger V_{nj}^* V_{jn} B_{nm} V_{mj}^* V_{jm} \hat{a}_m V_{mj}^* = \hbar \sum_{j=1}^{N} \hat{b}_j^\dagger D_{jj} \hat{b}_j$$

Note that in this way, beam splitters can be transformed away. Therefore beam splitters do not constitute a “real” interaction between modes, it can be viewed as a frame choice.

Active transformations

So far we didn't mix annihilation and creation operators. i.e, we covered passive transformations. If the Hamiltonian mixes annihilation and creation operators, we get active (non energy conserving) transformations.

<u>Single-mode squeezer</u> $H = \hbar r e^{i\theta}\hat{a}_1^2 + \hbar r e^{-i\theta}\hat{a}_1^{\dagger 2}, \ r > 0$

Using the BCH relations:

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_1^\dagger \end{pmatrix} = \begin{pmatrix} \cosh 2r & -ie^{-i\theta}\sinh 2r \\ ie^{i\theta}\sinh 2r & \cosh 2r \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \end{pmatrix}$$

$$\mu = \cosh r$$

$$\nu = e^{i\theta}\sinh r$$

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_1^\dagger \end{pmatrix} = \begin{pmatrix} \mu^2 + |\nu|^2 & -2i\mu\nu^* \\ 2i\mu\nu & \mu^2 + |\nu|^2 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \end{pmatrix}$$

<u>Two-mode squeezer</u> $H = \hbar\xi e^{i\phi}\hat{a}_1\hat{a}_2 + \hbar\xi e^{-i\phi}\hat{a}_1^\dagger\hat{a}_2^\dagger$

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_1^\dagger \\ \hat{b}_2 \\ \hat{b}_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\xi & 0 & 0 & -ie^{i\phi}\sinh\xi \\ 0 & \cosh\xi & ie^{-i\phi}\sinh\xi & 0 \\ 0 & ie^{i\phi}\sinh\xi & \cosh\xi & 0 \\ -ie^{-i\phi}\sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_1^\dagger \\ \hat{a}_2 \\ \hat{a}_2^\dagger \end{pmatrix}$$

Now we have all the essential ingredients to generalize→

General linear transformation which respects commutation relations:

A general Bogoliubov transformation is given by

$$\hat{b}_i = \sum_j A_{ij}\hat{a}_j + B_{ij}\hat{a}_j^\dagger$$

$$\hat{b}_i^\dagger = \sum_j B_{ij}^*\hat{a}_j + A_{ij}^*\hat{a}_j^\dagger$$

$$\boxed{\begin{array}{c}[\hat{b}_i^\dagger, \hat{b}_i] = 1 \\ [\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0\end{array}} \rightarrow \boxed{\begin{array}{c}AB^T = (AB^T)^T \\ AA^\dagger = BB^\dagger + I\end{array}}$$

$$\hat{a}_i = \sum_j A_{ij}^*\hat{b}_j - B_{ij}^*\hat{b}_j^\dagger$$

$$\hat{a}_i^\dagger = -\sum_j B_{ji}\hat{b}_j + A_{ji}\hat{b}_j^\dagger$$

$$\boxed{\begin{array}{c}[\hat{a}_i, \hat{a}_j^\dagger] = 1 \\ [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0\end{array}} \rightarrow \boxed{\begin{array}{c}A^\dagger B = (A^\dagger B)^T \\ A^\dagger A = (B^\dagger B)^T + I\end{array}}$$

Using singular value decomposition: $A = UA_DV^\dagger \quad B = UB_DW^\dagger$

From the "commutation relations" for A and B, we find $V^* = W$

$A = UA_DV^\dagger \quad B = UB_DV^T$

$$\begin{pmatrix}\vec{b} \\ \vec{b}^\dagger\end{pmatrix} = \underbrace{\begin{pmatrix}U & 0 \\ 0 & U^*\end{pmatrix}}_{\text{Beam splitter and phases}} \underbrace{\begin{pmatrix}A_D & B_D \\ B_D^* & A_D^*\end{pmatrix}}_{\text{Squeezers}} \underbrace{\begin{pmatrix}V^\dagger & 0 \\ 0 & V^T\end{pmatrix}}_{\text{Beam splitter and phases}} \begin{pmatrix}\vec{a} \\ \vec{a}^\dagger\end{pmatrix}$$

Displacement operator

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Displacement operator

$$D(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right) \qquad D(\alpha)^\dagger = D(-\alpha) = D^{-1}(\alpha)$$

Using: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \ [A,[A,B]] = [B,[A,B]] = 0$

$$D(\alpha) = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha \hat{a}^\dagger\right)\exp\left(-\alpha^* \hat{a}\right)$$

Problem 46: Prove this expression.

$$D(\alpha)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha \hat{a}^\dagger\right)\exp\left(-\alpha^* \hat{a}\right)|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left(\alpha \hat{a}^\dagger\right)|0\rangle$$

$$= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\hat{a}^{\dagger\,n}|0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

Problem 47: Prove that $D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha$

Displacement operator

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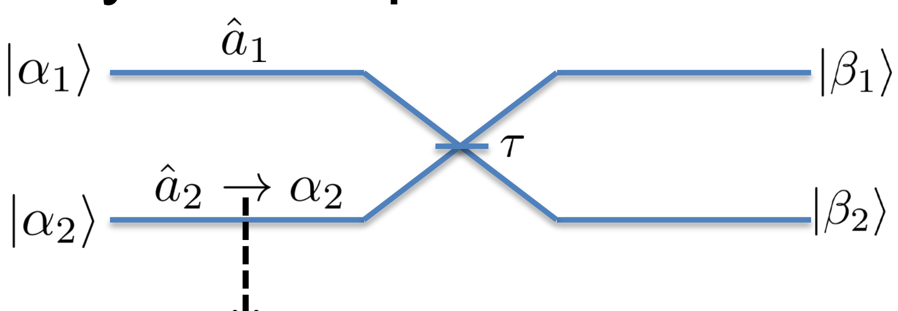
Problem 48: Prove that $D(\alpha)D(\beta) = \exp\left[\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)\right] D(\alpha+\beta)$

Displacement op. on a coherent state

$$D(\beta)|\alpha\rangle = D(\alpha)D(\beta)|0\rangle = \exp\left[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)\right]|\alpha+\beta\rangle$$

Physical implementation



$$\hat{b}_1 = \sqrt{\tau}\hat{a}_1 + \sqrt{1-\tau}\hat{a}_2 \rightarrow$$
$$\rightarrow \sqrt{\tau}\hat{a}_1 + \sqrt{1-\tau}\alpha_2 =$$
$$= \sqrt{\tau}\hat{a}_1 + \alpha$$

where we chose $\alpha_2 = \frac{\alpha}{\sqrt{1-\tau}}$

Treated as classical field, since the coherent state is chosen to be very bright.

$$\hat{b}_1 = \sqrt{\tau}\hat{a}_1 + \alpha \rightarrow |\beta_1\rangle = |\sqrt{\tau}\alpha_1 + \alpha\rangle$$

It implements the displacement operator for $\tau \rightarrow 1$ but in this limit the bright coherent state diverges.

Single-mode squeezing

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We show that the single-mode squeezer Hamiltonian is

$H = \hbar r e^{i\theta}\hat{a}_1^2 + \hbar r e^{-i\theta}\hat{a}_1^{\dagger 2}, r > 0$ $\qquad U(s) = \exp\left(-\frac{i}{\hbar}Hs\right) \equiv S(s)$

Setting $s \equiv 2ire^{-i\theta}$, the squeezing unitary is written:

$$S(s) = \exp\left(-s\frac{\hat{a}^{\dagger 2}}{2} + s^*\frac{\hat{a}^2}{2}\right) \qquad S(-s) = S^\dagger(s) = S^{-1}(s)$$

Acting on vacuum state:

Advanced Problem 4: Derive the expression:

$$S(s) = \exp\left(-\tau\frac{\hat{a}^{\dagger 2}}{2}\right)\exp\left[-\nu\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)\right]\exp\left(\tau^*\frac{\hat{a}^2}{2}\right)$$

Where: $\tau = \frac{s}{|s|}\tanh|s|,\ \nu = \ln(\cosh|s|)$

$$S(s)|0\rangle = \exp\left(-\tau\frac{\hat{a}^{\dagger 2}}{2}\right)\exp\left[-\nu\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)\right]\exp\left(\tau^*\frac{\hat{a}^2}{2}\right)|0\rangle =$$

$$= \exp\left(-\tau\frac{\hat{a}^{\dagger 2}}{2}\right)\exp\left[-\nu\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)\right]|0\rangle = \exp\left(-\tau\frac{\hat{a}^{\dagger 2}}{2}\right)\exp\left[-\frac{\nu}{2}\right]|0\rangle =$$

Problem 48: …finish up this calculation…

$$= \frac{1}{\sqrt{\cosh|s|}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n!}\left(-\frac{s}{2|s|}\right)^n \tanh^n s|2n\rangle \equiv |0;s\rangle$$

Single-mode squeezing

$$S^\dagger(s)\hat{a}S(s) = \hat{a}\cosh|s| - \hat{a}^\dagger \frac{s}{|s|}\sinh|s| \qquad S^\dagger(s)\hat{a}^\dagger S(s) = \hat{a}^\dagger\cosh|s| - \hat{a}\frac{s^*}{|s|}\sinh|s|$$

By substituting $s \to -s$:

$$S(s)\hat{a}S^\dagger(s) = \hat{a}\cosh|s| + \hat{a}^\dagger \frac{s}{|s|}\sinh|s|$$

$$S(s)\hat{a}^\dagger S^\dagger(s) = \hat{a}^\dagger\cosh|s| + \hat{a}\frac{s^*}{|s|}\sinh|s|$$

Standard definition: $|\alpha; s\rangle = D(\alpha)S(s)|0\rangle$

$$S(s)D(\alpha)|0\rangle = ?$$

$$S(s)D(\alpha)|0\rangle = S(s)D(\alpha)S^\dagger(s)S(s)|0\rangle = S(s)D(\alpha)S^\dagger(s)|0; s\rangle$$

$$S(s)D(\alpha)S^\dagger(s) = \exp\left(-\frac{|\alpha|^2}{2}\right)\boxed{S(s)\exp\left(\alpha\hat{a}^\dagger\right)S^\dagger(s)}\boxed{S(s)\exp\left(-\alpha^*\hat{a}\right)S^\dagger(s)}$$

Problem 49: … finish up this calculation… Tip: use the formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \; [A,[A,B]] = [B,[A,B]] = 0$$

$$S(s)D(\alpha)|0\rangle = D(\alpha\cosh|s| - \alpha^*\frac{s}{|s|}\sinh|s|)|0; s\rangle = D(\beta)S(s)|0\rangle$$

Upcoming topics

- Two-mode squeezing and entanglement
- Introduction to *symplectic transformations* and phase space approach (covariance matrices, first moments).
- Number basis and coherent basis
- Application to continuous variable teleportation