Photonic Quantum Information Processing
OPTI 647: Lecture 2

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August 29, 2018

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Logistical information

- Office hours 11 am – noon, Wednesdays, OSC 447
- First problem set due next week, Tuesday by noon to Brianna Moreno (OSC 501)
- Class website: https://wp.optics.arizona.edu/sguha/opti-647/

Laser light pulse: coherent state (recap)

- Laser light pulse: in √(photons/m²-sec) units. "Deterministic" (no stochastic) field:

$$\tilde{E}(\boldsymbol{r},t) = E(\boldsymbol{r},t)e^{j(\omega_0 t+\phi)},\ t \in (0,T], \boldsymbol{r} \in \mathcal{A}$$

  – Quasi-mono-chromatic, temporal bandwidth $W \ll \omega_0$

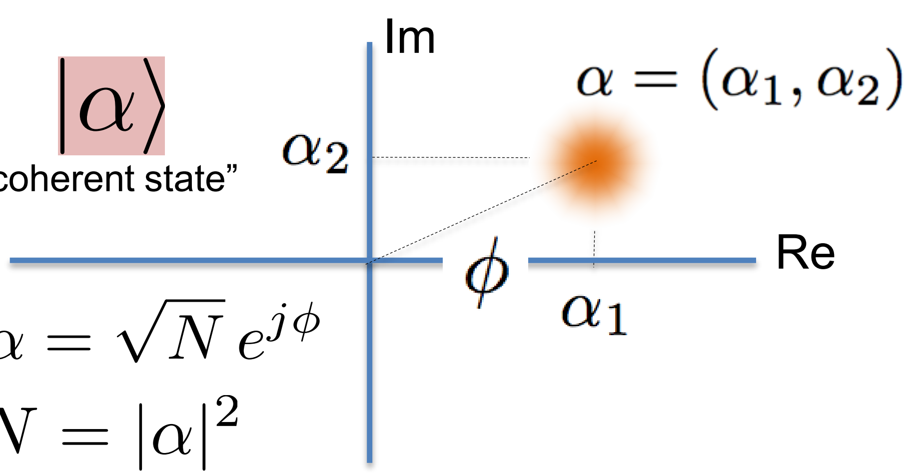
  – Mean photon number,

$$N = \int_0^T \int_{\mathcal{A}} |\tilde{E}(\boldsymbol{r},t)|^2 d\boldsymbol{r} dt$$

$$= \int_0^T \int_{\mathcal{A}} |E(\boldsymbol{r},t)|^2 d\boldsymbol{r} dt$$

$|\alpha\rangle$

"coherent state"



$\alpha = \sqrt{N}\, e^{j\phi}$

$N = |\alpha|^2$

Phase space picture: once we identify a spatio-temporal-polarization mode, a complex number describes the state of the laser pulse

No detector can accurately measure the field $E(\boldsymbol{r},t)$

Example: Hermite Gaussian (HG) modes

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- Consider the one dimensional infinite HG basis

$$\phi_q(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \frac{1}{\sqrt{2^q q!}} H_q\left(\frac{x}{\sqrt{2}\sigma}\right) \exp\left(-\frac{x^2}{4\sigma^2}\right) \; ; x \in \mathbb{R}, q = 0, 1, \ldots, \infty$$

  - $H_q(x)$: Hermite polynomials
  - The q=0 function is the Gaussian: $\phi_0(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-x^2/4\sigma^2}$
  - Consider the coherent state of the q=0 mode with mean photon number N and phase 0; $|\sqrt{N}\rangle|0\rangle|0\rangle\ldots$

- Consider the coherent state of $\phi_0(x-a)$, the shifted Gaussian mode (mean photon number N and phase 0)
  - Express this coherent state in the HG mode basis

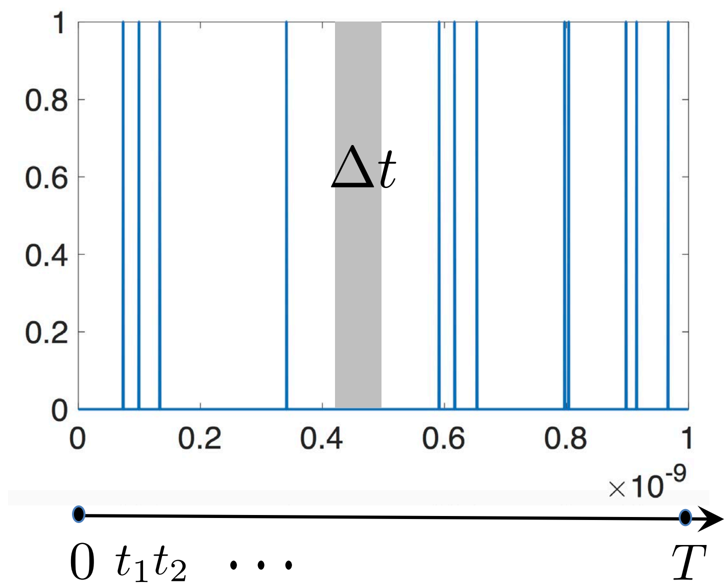
Problem 2

Poison point process (PPP)

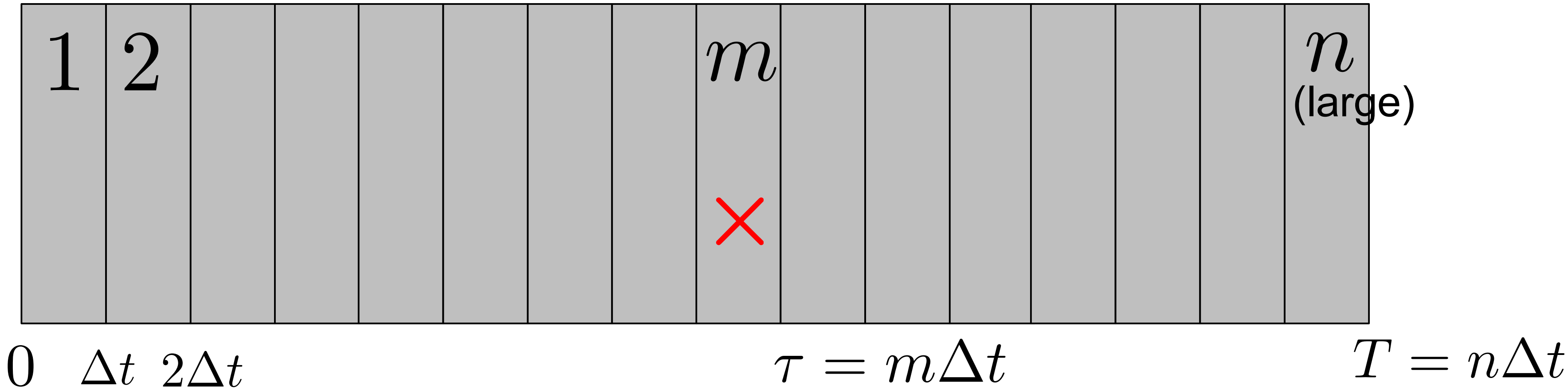
- Counting process that satisfies the following is a PPP
  - $N(0) = 0$
  - For all $t_1 \leq t_2 \leq s_1 \leq s_2$, $N(s_2) - N(s_1)$ S.I. of $N(t_2) - N(t_1)$ — Independent increments (memoryless)
  - $\exists \lambda > 0$, s.t. for any $0 \leq t_1 < t_2$, $\mathrm{E}[N(t_2) - N(t_1)] = \lambda(t_2 - t_1)$ — Stationary increments
  - If $P(s) = P\{N(t+s) - N(t) \geq 2\}$, then $\lim_{s \to 0} [P(s)/s] = 0$ — No more than one arrival in a small increment
- Let us generate a PPP with constant rate, $\lambda = N/T$

```
T = 1e-9;
N = 10;
lambda = N/T;
dT = 1e-14;
t = 0:dT:T-dT;
p = lambda*dT;
n = 0;
clicks = -(sign(rand(1,length(t))-p)-1)/2;
plot(t,clicks,'LineWidth',2);
```

Probability of one arrival in a Δt interval, $p = \lambda \Delta t$



Inter-arrival times and number of arrivals



Probability of one arrival in a Δt interval, $p = \lambda\, \Delta t$

Let us denote by t, the time of first arrival; $P[t = \tau] = (1-p)^{m-1} p; \;\; m = 1, 2, \ldots$

c.d.f., $F_{\mathcal{T}}(\tau) = P[t \leq \tau] = \sum_{j=1}^{m} (1-p)^{j-1} p = 1 - (1-p)^m$

$$\xrightarrow[\Delta t \to 0]{} 1 - e^{-\lambda \tau}$$

p.d.f., $P_{\mathcal{T}}(\tau) = \dfrac{d}{d\tau} F_{\mathcal{T}}(\tau) = \lambda e^{-\lambda \tau}; \; \tau \geq 0$

Prove that the Probability distribution of the total number of arrivals K is given by, **Problem 3** $P_K[k] = \dfrac{e^{-N} N^k}{k!}, k \in \mathbb{Z}$

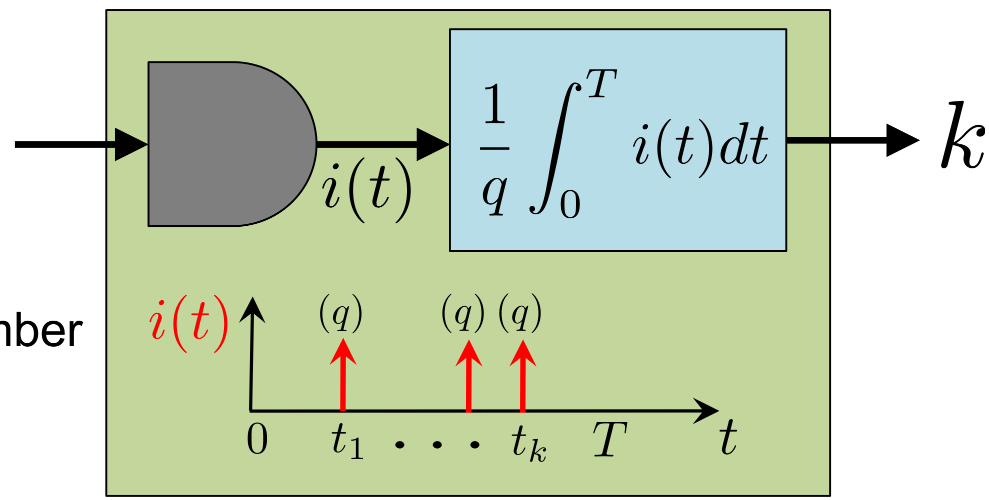
Ideal photon detection on a laser pulse

- Poisson point process (PPP) with rate, $\lambda(t) = |s(t)|^2$
  - For constant rate PPP, interarrival time is exponentially distributed

$$p(\tau) = \lambda e^{-\lambda\tau}, \tau \geq 0$$

Probability of one arrival in a Δt interval, $p = \lambda \Delta t$

$$s(t)e^{j(\omega_0 t+\phi)},\ t \in (0, T]$$



$$N = \int_0^T \lambda(t)dt$$ Mean photon number

$$P_K[k] = \frac{e^{-N}N^k}{k!}, k \in \mathbb{Z}$$

Direct detection has no information about the phase, ϕ

Example: flat-top temporal mode

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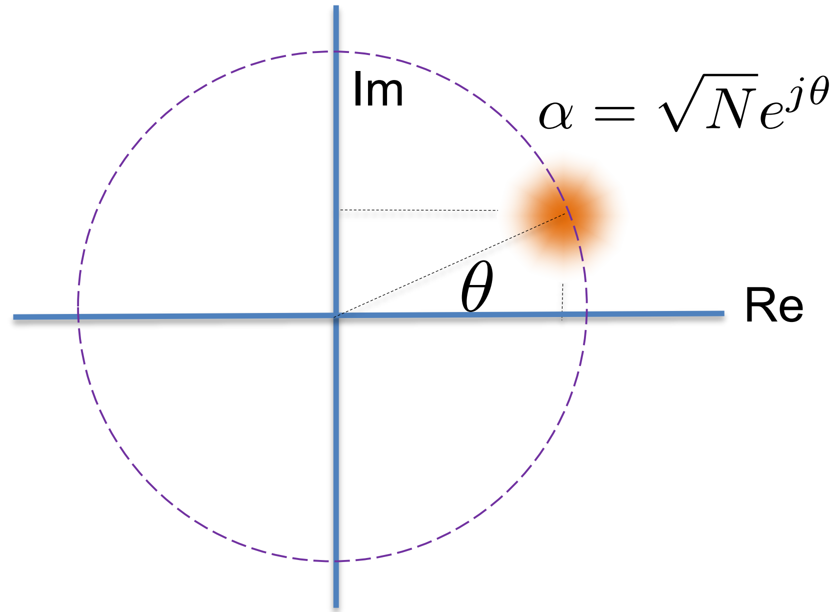
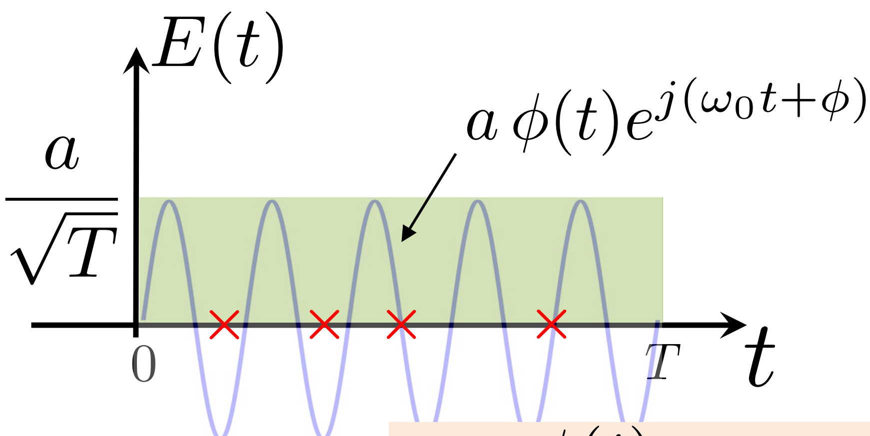
$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

Phase-space representation:



Coherent state of this mode:



Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$

Ideal direct detection produces a PPP of rate, $\lambda = |E(t)|^2 = \left|\frac{a}{\sqrt{T}}\right|^2 = \frac{a^2}{T} = \frac{N}{T}$

Mean photon number (average number of clicks seen during the pulse interval) = N

Review of basic quantum mechanics

State of a quantum system

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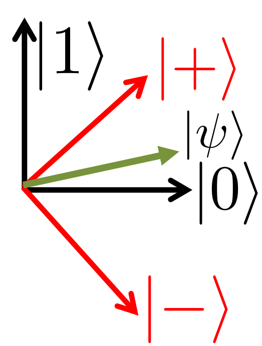
- Complete knowledge is a pure state $|\psi\rangle$
  - $|\psi\rangle$ is a unit-norm column vector (ket) in a complex vector space
  - $\langle\psi|$ is a unit-norm row vector (bra) – complex conjugate of $|\psi\rangle$
  - Unit-norm condition: $\langle\psi|\psi\rangle = 1$

Orthonormal basis

- Example: one qubit (two level system), $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
  - $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\langle 0|1\rangle = \langle 1|0\rangle = 0$; $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
  - Unit-norm condition: $\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$
- Representing qubit's state in a different orthonormal basis
  - Define a 45° rotated basis, $|\pm\rangle = \dfrac{|0\rangle \pm |1\rangle}{\sqrt{2}}$
  - Express $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in this basis:

$$= \gamma|+\rangle + \delta|-\rangle$$



Product vs. Entangled state

- State of two qubits: qubit A and qubit B
  - Product state: $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$
    - Example: $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, $|\psi\rangle_B = \gamma|0\rangle_B + \delta|1\rangle_B$

      $$\Rightarrow \ |\psi\rangle_{AB} = \alpha\gamma|0\rangle_A|0\rangle_B + \alpha\delta|0\rangle_A|1\rangle_B + \beta\gamma|1\rangle_A|0\rangle_B + \beta\delta\overbrace{|1\rangle_A|1\rangle_B}^{|1\rangle_A \otimes |1\rangle_B}$$

    - General state: $|\psi\rangle_{AB} = \alpha_{00}|0\rangle_A|0\rangle_B + \alpha_{01}|0\rangle_A|1\rangle_B + \alpha_{10}|1\rangle_A|0\rangle_B + \alpha_{11}|1\rangle_A|1\rangle_B$
      - Unit-norm condition: ${}_{AB}\langle\psi|\psi\rangle_{AB} = |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$
- Entangled state
  - State of two (or more) quantum systems that is not a product state, i.e., state of system A *and* state of system B
    - Bell state; EPR (Einstein-Podolsky-Rosen) state

$$|\psi\rangle_{AB} = \sqrt{\frac{1}{2}}|0\rangle_A|0\rangle_B + \sqrt{\frac{1}{2}}|1\rangle_A|1\rangle_B$$

$$\neq \alpha|0\rangle_A + \beta|1\rangle_B \otimes \gamma|0\rangle_B + \delta|1\rangle_B$$

Projective measurements

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- Measurement described by a set of projectors, $\{\Pi_1, \Pi_2, \ldots\}$
  - $\Pi_k^2 = \Pi_k, \forall k$: Projector Π_k corresponds to measurement outcome "k"
  - $\sum_k \Pi_k = \hat{I}$ (identity of the space that the state being measured resides in)
- State $|\psi\rangle$ measured; post-measurement state: $|\phi_k\rangle = \dfrac{\Pi_k|\psi\rangle}{\sqrt{p_k}}$
  - With probability, $p_k = \langle\psi|\Pi_k|\psi\rangle$
- Von Neumann measurement
  - Special case when measurement is described by a set of unit-norm vectors that form a complete orthonormal basis of the vector space that $|\psi\rangle$ is in
  - Orthonormal basis: $\{|w_k\rangle\}$, $\langle w_k|w_j\rangle = \delta_{kj}$; projectors: $\Pi_k = |w_k\rangle\langle w_k|$
  - Probability of outcome k, $p_k = |\langle w_k|\psi\rangle|^2$, post-measurement state, $|\phi_k\rangle = |w_k\rangle$
- Example: single qubit von Neumann measurement $\{\Pi_0 = |0\rangle\langle 0|, \Pi_1 = |1\rangle\langle 1|\}$
  - State measured, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$
  - Result "0" with probability, $p_0 = |\langle 0|\psi\rangle|^2 = |\alpha|^2$, and "1" with $p_1 = |\langle 1|\psi\rangle|^2 = |\beta|^2$
- Example: single qubit measurement $\{\Pi_+ = |+\rangle\langle +|, \Pi_- = |-\rangle\langle -|\}$
  - State measured, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
  - What are the post-measurement states and probabilities of each of two outcomes?

Bell states

- Orthonormal basis for 2 qubits

Computational basis

$$|0\rangle_A|0\rangle_B$$
$$|0\rangle_A|1\rangle_B$$
$$|1\rangle_A|0\rangle_B$$
$$|1\rangle_A|1\rangle_B$$

Bell basis

$$|\Phi^+\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$$
$$|\Phi^-\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B}{\sqrt{2}}$$
$$|\Psi^+\rangle_{AB} = \frac{|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B}{\sqrt{2}}$$
$$|\Psi^-\rangle_{AB} = \frac{|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B}{\sqrt{2}}$$

- Bell states expressed in the $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$ basis

Show that:
$$\begin{cases} |\Phi^+\rangle_{AB} = \dfrac{|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B}{\sqrt{2}} \\ |\Phi^-\rangle_{AB} = \dfrac{|+\rangle_A|-\rangle_B + |-\rangle_A|+\rangle_B}{\sqrt{2}} \end{cases}$$

Quantum key distribution (QKD)

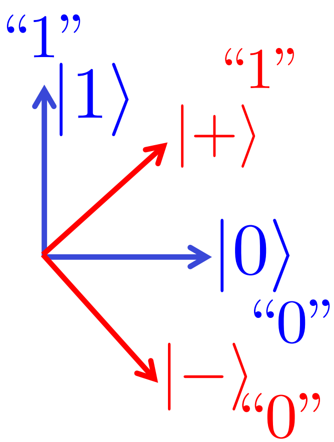
Alice

basis	bit
red	0
blue	1
red	1
red	0
blue	1
blue	1
red	0
blue	1
red	0

Eve

Bob

basis	bit
blue	1
red	1
red	1
blue	0
blue	1
red	0
red	0
blue	1
blue	0



Start with many copies of this EPR state between Alice and Bob

$$\frac{|0,0\rangle + |1,1\rangle}{\sqrt{2}} = \frac{|+,+\rangle + |-,-\rangle}{\sqrt{2}}$$

Number (Fock) state of a mode

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Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K is a Poisson random variable of mean N

Mode $\phi(t)$, a quantum system, is excited in a number state $|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K = n (exactly so; K is not a random variable).

A mode of ideal laser light is in a coherent state.
Number (Fock) state of a given mode is very hard to produce experimentally

This orthogonality (in the Hilbert state) is different from that of modes (in L_2 norm space)

There are infinitely many other types of "states" of the mode $\phi(t)$.
Coherent state and Fock state are just two example class of states

$|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$ Fock states of a mode are special: they form an orthonormal basis that spans any general quantum state $|\psi\rangle$ of that mode

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \,, \; \sum_{n=0}^{\infty} |c_n|^2 = 1$$

Coherent state as a quantum state

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$$|\alpha\rangle = \sum_{n=0}^{\infty} \underbrace{\left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right)}_{c_n} |n\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \cdots$$

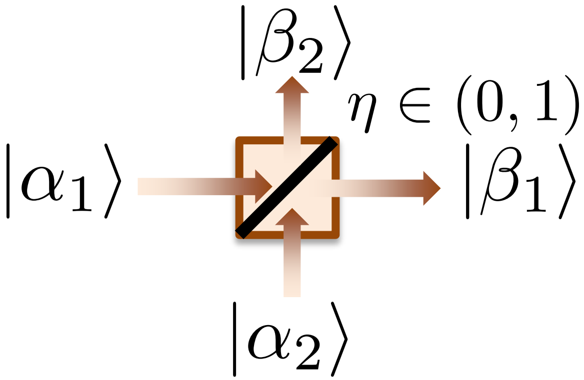
Fock states can be thought of as infinite-length unit-length column vectors corresponding to the orthogonal axes of an infinite-dimensional vector space

Ideal photon detection is a von Neumann quantum measurement described by projectors, $\{|n\rangle\langle n|\}$, $n = 0, 1, \ldots, \infty$

Ideal direct detection on a coherent state $|\alpha\rangle$ produces outcome "n" (i.e., n "clicks") with probability, $p_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = \dfrac{e^{-N} N^n}{n!}$

Poisson detection statistics in a laser pulse is a result of the projection of the quantum state of the laser pulse—a coherent state–on to one of the Fock states

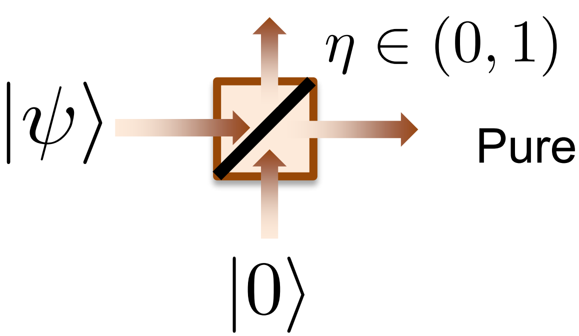
Coherent states and beamsplitters



$$\beta_1 = \sqrt{\eta}\alpha_1 + e^{i\phi}\sqrt{1-\eta}\alpha_2$$

$$\beta_2 = \sqrt{1-\eta}\alpha_1 - e^{i\phi}\sqrt{\eta}\alpha_2$$

The only pure state that remains pure through a pure-loss beam-splitter is a coherent state



Pure state output, only if $|\psi\rangle$ is a coherent state

A coherent state is *always* single mode

- By an appropriate choice of modal basis, any "multi-mode" coherent state can always be expressed as a single mode coherent state

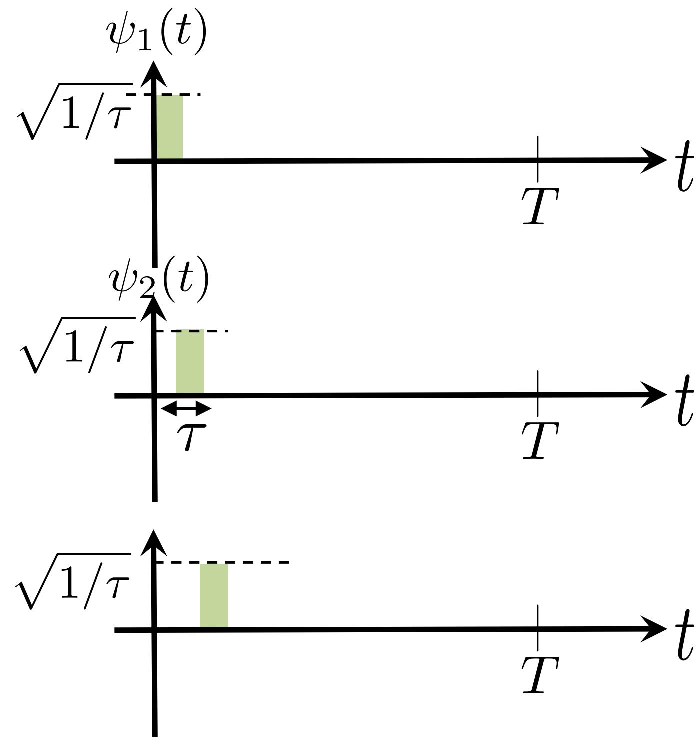
$$|\alpha_1\rangle|\alpha_2\rangle\ldots|\alpha_K\rangle \equiv |\beta\rangle|0\rangle\ldots|0\rangle$$

  - In other words… if we have a deterministic field in any spatio-temporal shape (of any given polarization), we can always represent that as a *single-mode* coherent state of an appropriate normalized mode
  - We will see later, this is not true for other quantum states in general. For example, a multimode thermal state or a multimode squeezed state, etc.

Slicing a coherent state pulse (in time)

(1) Single-mode coherent state of this mode: $\phi(t)$

$|\alpha\rangle$, $\alpha = \sqrt{N}$, $N = E^2 T$

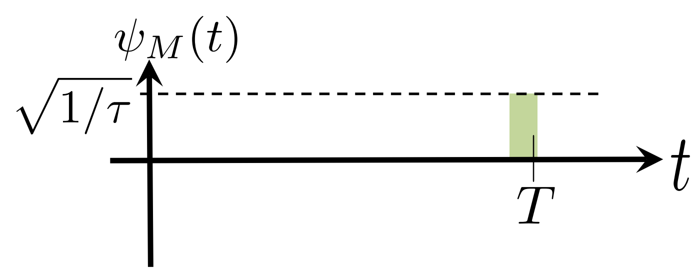


(2) M-mode coherent state of the modes: $\psi_k(t), k = 1, \ldots, M$

$$|\beta\rangle|\beta\rangle \ldots |\beta\rangle \quad , \beta = \sqrt{\frac{N}{M}}$$

$$M = \frac{T}{\tau}$$

Orthogonal temporal modes



Slicing a coherent state pulse (in space)

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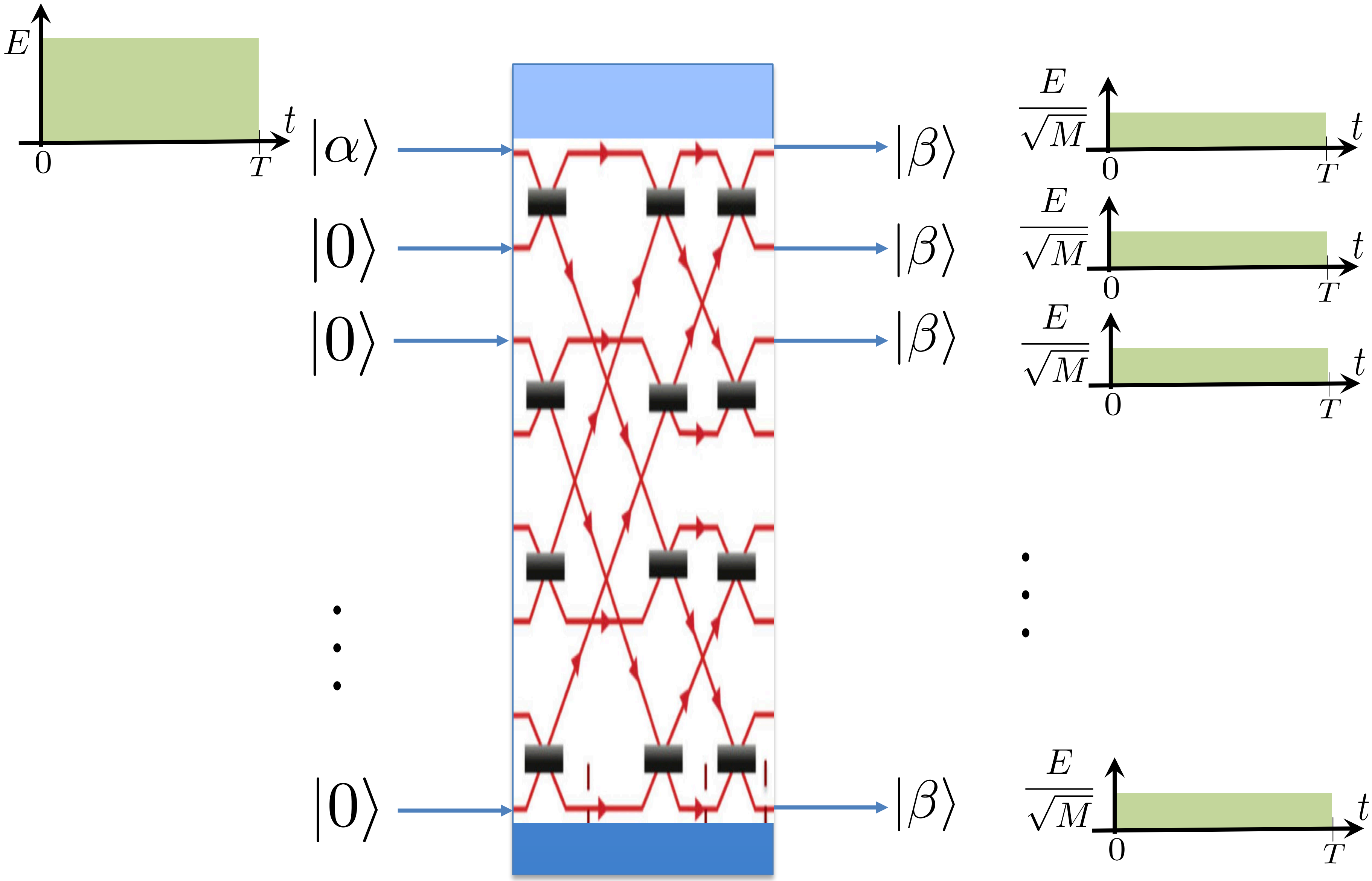

Examples of optical qubits

- Single-rail qubit $|\mathbf{0}\rangle = |0\rangle, |\mathbf{1}\rangle = |1\rangle$
- Dual-rail qubit $|\mathbf{0}\rangle = |0,1\rangle, |\mathbf{1}\rangle = |1,0\rangle$
- Cat-basis qubit $|\mathbf{0}\rangle = N_+(|\alpha\rangle + |-\alpha\rangle),$

$$|\mathbf{1}\rangle = N_-(|\alpha\rangle - |-\alpha\rangle)$$

Prove that the cat-basis qubit states are mutually orthogonal, and find the normalization constants N_+ and N_- in terms of α

Problem 4

Quantization of the field

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- Classical (deterministic) field (coherent state)

$$E(t) = \sum_{k=1}^{K} a_i \phi_i(t)$$

- Quantum description of the field: $\hat{E}(t) = \sum_{k=1}^{K} \hat{a}_i \phi_i(t)$
  - Field becomes an operator
  - Field described by a quantum state of constituent modes
  - Modal annihilation operator: $\hat{a}_i$
  - Classical field is a special case: each mode i is excited in a coherent state $|\alpha_i\rangle$, $\alpha_i = a_i$
  - Classical statistical field is a mixture of coherent states, density operator $\rho = \int P(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|d\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}\rangle = |\alpha_1\rangle|\alpha_2\rangle \ldots |\alpha_K\rangle$

Upcoming topics…

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- Single mode quantum optics
  - Annihilation and creation operations, Density operators, phase space, Characteristic functions, Wigner functions, Entanglement
  - Gaussian (e.g., squeezed) vs. non-Gaussian (e.g., cat) states
  - Classical (e.g., coherent) vs. quantum (e.g., number) states
  - Photodetection: semiclassical vs. quantum theories
  - Classification of optical quantum transformations