Logistical information

• Office hours 11 am – noon, Wednesdays, OSC 447
• First problem set due next week, Tuesday by noon to Brianna Moreno (OSC 501)
• Class website: https://wp.optics.arizona.edu/sguha/opti-647/
Laser light pulse: coherent state (recap)

- Laser light pulse: in $\sqrt{\text{photons/m}^2\text{-sec}}$ units.
  "Deterministic" (no stochastic) field:
  $$\tilde{E}(\mathbf{r}, t) = E(\mathbf{r}, t)e^{j(\omega_0 t + \phi)}, \; t \in (0, T], \; \mathbf{r} \in \mathcal{A}$$
  - Quasi-mono-chromatic, temporal bandwidth $W \ll \omega_0$

  - Mean photon number,
    $$N = \int_0^T \int_{\mathcal{A}} |\tilde{E}(\mathbf{r}, t)|^2 d\mathbf{r} dt$$
    $$\alpha = \frac{\sqrt{N}}{N} e^{j\phi}, \; \alpha = (\alpha_1, \alpha_2)$$

Phase space picture: once we identify a spatio-temporal-polarization mode, a complex number describes the state of the laser pulse

No detector can accurately measure the field $E(\mathbf{r}, t)$
Example: Hermite Gaussian (HG) modes

- Consider the one dimensional infinite HG basis

\[ \phi_q(x) = \left( \frac{1}{2\pi \sigma^2} \right)^{1/4} \frac{1}{\sqrt{2^q q!}} H_q \left( \frac{x}{\sqrt{2\sigma}} \right) \exp \left( -\frac{x^2}{4\sigma^2} \right) ; x \in \mathbb{R}, q = 0, 1, \ldots, \infty \]

  - \( H_q(x) \): Hermite polynomials
  - The \( q=0 \) function is the Gaussian: \( \phi_0(x) = \left( \frac{1}{2\pi \sigma^2} \right)^{1/4} e^{-x^2/4\sigma^2} \)
  - Consider the coherent state of the \( q=0 \) mode with mean photon number \( N \) and phase 0; \( |\sqrt{N}\rangle|0\rangle|0\rangle \ldots \)

- Consider the coherent state of \( \phi_0(x - a) \), the shifted Gaussian mode (mean photon number \( N \) and phase 0)
  - Express this coherent state in the HG mode basis

Problem 2
Poisson point process (PPP)

• Counting process that satisfies the following is a PPP
  
  – \( N(0) = 0 \)
  
  – For all \( t_1 \leq t_2 \leq s_1 \leq s_2 \), \( N(s_2) - N(s_1) \) S.I. of \( N(t_2) - N(t_1) \)
  
  – \( \exists \lambda > 0 \), s.t. for any \( 0 \leq t_1 < t_2 \), \( \mathbb{E}[N(t_2) - N(t_1)] = \lambda(t_2 - t_1) \)
  
  – If \( P(s) = P\{N(t + s) - N(t) \geq 2\} \), then \( \lim_{s \to 0} [P(s)/s] = 0 \)

• Let us generate a PPP with constant rate, \( \lambda = N/T \)

\[
T = 1e-9; \\
N = 10; \\
\text{lambda} = N/T; \\
dT = 1e-14; \\
t = 0:dT:T-dT; \\
p = \text{lambda} \times dT; \\
n = 0; \\
\text{clicks} = -(\text{sign(rand(1,length(t))-p)-1)/2}; \\
\text{plot}(t,\text{clicks},'\text{LineWidth}',2); \\
\]
Inter-arrival times and number of arrivals

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<th>1</th>
<th>2</th>
<th>m</th>
<th>n</th>
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<td>Δt</td>
<td>2Δt</td>
<td>τ = mΔt</td>
<td>T = nΔt</td>
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Probability of one arrival in a Δt interval, \( p = \lambda \Delta t \)

Let us denote by \( t \), the time of first arrival; \( P[t = \tau] = (1 - p)^{m-1}p; \ m = 1, 2, \ldots \)

c.d.f., \( F_\tau(\tau) = P[t \leq \tau] = \sum_{j=1}^{m} (1 - p)^{j-1}p = 1 - (1 - p)^{m} \rightarrow 1 - e^{-\lambda \tau} \)

p.d.f., \( P_\tau(\tau) = \frac{d}{d\tau} F_\tau(\tau) = \lambda e^{-\lambda \tau}; \ \tau \geq 0 \)

Prove that the Probability distribution of the total number of arrivals \( K \) is given by,

\[
P_K[k] = \frac{e^{-N}N^k}{k!}, \ k \in \mathbb{Z}
\]
Ideal photon detection on a laser pulse

- Poisson point process (PPP) with rate, \( \lambda(t) = |s(t)|^2 \)
  - For constant rate PPP, interarrival time is exponentially distributed
    \[
    p(\tau) = \lambda e^{-\lambda \tau}, \quad \tau \geq 0
    \]

\[
N = \int_0^T \lambda(t) \, dt \quad \text{Mean photon number}
\]

\[
P_K[k] = \frac{e^{-N} N^k}{k!}, \quad k \in \mathbb{Z}
\]
Example: flat-top temporal mode

\[ \phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases} \]

Coherent state of this mode:

\[ E(t) = a \phi(t)e^{j(\omega_0 t + \phi)} \]

Phase-space representation:

\[ \alpha = \sqrt{N}e^{j\theta} \]

Ideal direct detection produces a PPP of rate,

\[ \lambda = |E(t)|^2 = \left| \frac{a}{\sqrt{T}} \right|^2 = \frac{a^2}{T} = \frac{N}{T} \]

Mean photon number (average number of clicks seen during the pulse interval) = N
Review of basic quantum mechanics
State of a quantum system

• Complete knowledge is a pure state $|\psi\rangle$
  – $|\psi\rangle$ is a unit-norm column vector (ket) in a complex vector space
  – $\langle \psi |$ is a unit-norm row vector (bra) – complex conjugate of $|\psi\rangle$
  – Unit-norm condition: $\langle \psi | \psi \rangle = 1$

• Example: one qubit (two level system), $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$
  – $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\langle 0 | 1 \rangle = \langle 1 | 0 \rangle = 0$; $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
  – Unit-norm condition: $\langle \psi | \psi \rangle = |\alpha|^2 + |\beta|^2 = 1$

• Representing qubit’s state in a different orthonormal basis
  – Define a 45° rotated basis, $|\pm\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
  – Express $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ in this basis:
    $= \gamma |+\rangle + \delta |-\rangle$
Product vs. Entangled state

• State of two qubits: qubit A and qubit B
  – Product state: $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$
    
  • Example: $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$, $|\psi\rangle_B = \gamma|0\rangle_B + \delta|1\rangle_B$
    
    $\Rightarrow |\psi\rangle_{AB} = \alpha\gamma|0\rangle_A|0\rangle_B + \alpha\delta|0\rangle_A|1\rangle_B + \beta\gamma|1\rangle_A|0\rangle_B + \beta\delta|1\rangle_A|1\rangle_B$

  • General state: $|\psi\rangle_{AB} = \alpha_{00}|0\rangle_A|0\rangle_B + \alpha_{01}|0\rangle_A|1\rangle_B + \alpha_{10}|1\rangle_A|0\rangle_B + \alpha_{11}|1\rangle_A|1\rangle_B$
    – Unit-norm condition: $\langle\psi|\psi\rangle_{AB} = |\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

• Entangled state
  – State of two (or more) quantum systems that is not a product state, i.e., state of system A and state of system B
    • Bell state; EPR (Einstein-Podolsky-Rosen) state

    $|\psi\rangle_{AB} = \sqrt{\frac{1}{2}}|0\rangle_A|0\rangle_B + \sqrt{\frac{1}{2}}|1\rangle_A|1\rangle_B$

    $\neq \alpha|0\rangle_A + \beta|1\rangle_B \otimes \gamma|0\rangle_B + \delta|1\rangle_B$
Projective measurements

- Measurement described by a set of projectors, \( \{\Pi_1, \Pi_2, \ldots\} \)
  - \( \Pi_k^2 = \Pi_k, \forall k \) : Projector \( \Pi_k \) corresponds to measurement outcome “k”
  - \( \sum_k \Pi_k = \hat{I} \) (identity of the space that the state being measured resides in)

- State \( |\psi\rangle \) measured; post-measurement state: \( |\phi_k\rangle = \frac{\Pi_k|\psi\rangle}{\sqrt{p_k}} \)
  - With probability, \( p_k = \langle \psi | \Pi_k | \psi \rangle \)

- Von Neumann measurement
  - Special case when measurement is described by a set of unit-norm vectors that form a complete orthonormal basis of the vector space that \( |\psi\rangle \) is in
  - Orthonormal basis: \( \{|w_k\rangle\} \), \( \langle w_k|w_j \rangle = \delta_{k,j} \); projectors: \( \Pi_k = |w_k\rangle\langle w_k| \)
  - Probability of outcome k, \( p_k = |\langle w_k|\psi \rangle|^2 \), post-measurement state, \( |\phi_k\rangle = |w_k\rangle \)

- Example: single qubit von Neumann measurement \( \{\Pi_0 = |0\rangle\langle 0|, \Pi_1 = |1\rangle\langle 1|\} \)
  - State measured, \( |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \), \( |\alpha|^2 + |\beta|^2 = 1 \)
  - Result “0” with probability, \( p_0 = |\langle 0|\psi \rangle|^2 = |\alpha|^2 \), and “1” with \( p_1 = |\langle 1|\psi \rangle|^2 = |\beta|^2 \)

- Example: single qubit measurement \( \{\Pi_+ = |+\rangle\langle +|, \Pi_- = |-\rangle\langle -|\} \)
  - State measured, \( |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \)
  - What are the post-measurement states and probabilities of each of two outcomes?
Bell states

- Orthonormal basis for 2 qubits
  - Computational basis
    - $|0\rangle_A|0\rangle_B$
    - $|0\rangle_A|1\rangle_B$
    - $|1\rangle_A|0\rangle_B$
    - $|1\rangle_A|1\rangle_B$
  - Bell basis
    - $|\Phi^+\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$
    - $|\Phi^-\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B}{\sqrt{2}}$
    - $|\Psi^+\rangle_{AB} = \frac{|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B}{\sqrt{2}}$
    - $|\Psi^-\rangle_{AB} = \frac{|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B}{\sqrt{2}}$

- Bell states expressed in the $|\pm\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ basis
  - Show that:
    - $|\Phi^+\rangle_{AB} = \frac{+\rangle_A+\rangle_B + -\rangle_A-\rangle_B}{\sqrt{2}}$
    - $|\Phi^-\rangle_{AB} = \frac{+\rangle_A-\rangle_B + -\rangle_A+\rangle_B}{\sqrt{2}}$
Quantum key distribution (QKD)

Start with many copies of this EPR state between Alice and Bob

\[
\frac{|0, 0\rangle + |1, 1\rangle}{\sqrt{2}} = \frac{|+, +\rangle + |-, -\rangle}{\sqrt{2}}
\]

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<th>Bob</th>
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Alice and Bob communicate their results, and Eve attempts to intercept the information.
Number (Fock) state of a mode

Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons $K$ is a Poisson random variable of mean $N$.

Mode $\phi(t)$, a quantum system, is excited in a number state $|n\rangle$, $n \in \{0, 1, \ldots, \infty\}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons $K = n$ (exactly so; $K$ is not a random variable).

A mode of ideal laser light is in a coherent state.

Number (Fock) state of a given mode is very hard to produce experimentally.

There are infinitely many other types of “states” of the mode $\phi(t)$.

Coherent state and Fock state are just two example class of states.

This orthogonality (in the Hilbert state) is different from that of modes (in $L_2$ norm space).

Fock states of a mode are special: they form an orthonormal basis that spans any general quantum state $|\psi\rangle$ of that mode.

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad \text{with} \quad \sum_{n=0}^{\infty} |c_n|^2 = 1$$
Coherent state as a quantum state

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \left( e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle \]

Fock states can be thought of as infinite-length unit-length column vectors corresponding to the orthogonal axes of an infinite-dimensional vector space.

\[ |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \ldots \]

Ideal photon detection is a von Neumann quantum measurement described by projectors, \( \{ |n\rangle \langle n| \} \), \( n = 0, 1, \ldots, \infty \)

Ideal direct detection on a coherent state \( |\alpha\rangle \) produces outcome “n” (i.e., n “clicks”) with probability,

\[ p_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = \frac{e^{-N} N^n}{n!} \]

Poisson detection statistics in a laser pulse is a result of the projection of the quantum state of the laser pulse—a coherent state—on to one of the Fock states.
Coherent states and beamsplitters

The only pure state that remains pure through a pure-loss beam-splitter is a coherent state

\[ |\psi\rangle \]  
\[ \eta \in (0, 1) \]  
\[ |\beta_2\rangle \]  
\[ |\alpha_1\rangle \]  
\[ |\beta_1\rangle \]  
\[ |\alpha_2\rangle \]

\[ \beta_1 = \sqrt{\eta} \alpha_1 + e^{i\phi} \sqrt{1 - \eta} \alpha_2 \]
\[ \beta_2 = \sqrt{1 - \eta} \alpha_1 - e^{i\phi} \sqrt{\eta} \alpha_2 \]

The only pure state that remains pure through a pure-loss beam-splitter is a coherent state

Pure state output, only if \( |\psi\rangle \) is a coherent state
A coherent state is *always* single mode

- By an appropriate choice of modal basis, any “multi-mode” coherent state can always be expressed as a single mode coherent state

\[ |\alpha_1\rangle|\alpha_2\rangle \ldots |\alpha_K\rangle \equiv |\beta\rangle|0\rangle \ldots |0\rangle \]

- In other words… if we have a deterministic field in any spatio-temporal shape (of any given polarization), we can always represent that as a *single-mode* coherent state of an appropriate normalized mode
- We will see later, this is not true for other quantum states in general. For example, a multimode thermal state or a multimode squeezed state, etc.
Slicing a coherent state pulse (in time)

(1) Single-mode coherent state of this mode: $\phi(t)$

$|\alpha\rangle, \alpha = \sqrt{N}, N = E^2T$

(2) M-mode coherent state of the modes: $\psi_k(t), k = 1, \ldots, M$

$|\beta\rangle|\beta\rangle \ldots |\beta\rangle, \beta = \sqrt{\frac{N}{M}}$

Orthogonal temporal modes
Slicing a coherent state pulse (in space)
Examples of optical qubits

- Single-rail qubit: \( |0\rangle = |0\rangle, |1\rangle = |1\rangle \)
- Dual-rail qubit: \( |0\rangle = |0, 1\rangle, |1\rangle = |1, 0\rangle \)
- Cat-basis qubit: \( |0\rangle = N_+ (|\alpha\rangle + | - \alpha\rangle) \), 
  \( |1\rangle = N_- (|\alpha\rangle - | - \alpha\rangle) \)

Prove that the cat-basis qubit states are mutually orthogonal, and find the normalization constants \(N_+\) and \(N_-\) in terms of \(\alpha\)

Problem 4
Quantization of the field

- Classical (deterministic) field (coherent state)
  \[ E(t) = \sum_{k=1}^{K} a_i \phi_i(t) \]

- Quantum description of the field: \( \hat{E}(t) = \sum_{k=1}^{K} \hat{a}_i \phi_i(t) \)
  - Field becomes an operator
  - Field described by a quantum state of constituent modes
  - Modal annihilation operator: \( \hat{a}_i \)
  - Classical field is a special case: each mode \( i \) is excited in a coherent state \( |\alpha_i\rangle \), \( \alpha_i = a_i \)
  - Classical statistical field is a mixture of coherent states, density operator
  \[
  \rho = \int P(\alpha) |\alpha\rangle \langle \alpha| d\alpha, \quad |\alpha\rangle = |\alpha_1\rangle |\alpha_2\rangle \ldots |\alpha_K\rangle
  \]
Upcoming topics…

• Single mode quantum optics
  – Annihilation and creation operations, Density operators, phase space, Characteristic functions, Wigner functions, Entanglement
  – Gaussian (e.g., squeezed) vs. non-Gaussian (e.g., cat) states
  – Classical (e.g., coherent) vs. quantum (e.g., number) states
  – Photodetection: semiclassical vs. quantum theories
  – Classification of optical quantum transformations