Tutorial - Hertz Contact Stress

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ABSTRACT

This paper has been written to discuss the Hertz contact stress and examples of FEA

1. INTRODUCTION

When there are two non-conforming surfaces that come into contact the point or line of contact can be a location of very high stress values. Since the stress is the Force over the Area, any load on a point contact would imply infinite stress at a mathematical point. What happens at the point of contact had been considered many times in an attempt to understand what was going on at this point of "surface of pressure" but the solutions were either approximations or contained some unknown empirical constant.¹ In 1882, Heinrich Hertz developed an analytical method of solving these issues so that the shape of the contact and deformation of the materials allowing the stresses to be resolved. Understanding these stresses is especially important in optomechanics for two major applications, glass to metal interfacing and kinematics. Understanding the stress at the metal-glass interfacing to reduce the risk of fracture and in kinematics it is import to understand the stresses to reduce fatigue so that the kinematic mounts don't become over constrained.

1.1 The Hertzian Contact

The original theory was derived for two non-conforming surfaces that in an no load situation would result in a point or line contact. When the load was applied there is some elastic deformation that takes place to provide a larger area so that the stresses are not infinite.

- Compressive stress that is generated when two non-conforming surfaces are under a forced contact.
- Contact is a point or line when there are no applied forces or moments.
- Area of contact is dependent on the force applied due to elastic deformation.
- Non-adhesive, meaning the two surfaces require no force to separate.
- Hertz Contact theory maintains both surfaces are frictionless. This means only the normal stresses will be translated between the two surfaces.

This is really a geometry problem, that can be

1.2 Improvements on the Original Theory

During the early to mid 1970's two competing theories where developed. Empirical evidence was starting to demonstrate that while the Hertz theory of contact was quite good there cases where the theory did not match to experimentation. This experimentation showed that limitations³ to the Hertz theory at smaller loads(;10g) were

- That the area of contact was larger than predicted.
- The area of contact had a non-zero value even when the load was removed.
- There was strong adhesion if the contacting surfaces were clean and dry.

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Fig. 4.1. Interference fringes at the contact of two equal cylindrical lenses with their axes inclined at 45° : (a) unloaded, (b) loaded.

Figure 1. From *Contact Mechanics*² demonstrating the change in area during conditions of load.

The consequences for these limitations are small for optics and unless there is an argument that the stress must be known exact as possible, for optical engineers using the Hertz contact theory as is can be beneficial when thinking about the lens mount interface. If the loads are small then you will have an upper limit on the actual stress experienced by the optic as the area of deformation will be greater than predicted by the classical theory. These adhesive modifications to the original theory are beyond the scope of this tutorial.

We will cover two classical examples of the Hertz theory of contact plus one practical example in this tutorial.

2. EXAMPLES OF CLASSICAL PROBLEMS

The classical problems fall into two categories, point or line contact. Each of the two contacts have parameters that represent a variety of problems that can be represented by the equations. If you think about what is going on, these are really complicated geometry problems to solve for an area, that depend on the force that is applied and the material on either side of the contact. For these two problems we will define to new terms:

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \tag{1}$$

This is the Contact Modulus, which is a representation of the two materials that are coming into contact. If the materials are the same this reduces to simply

$$\frac{1}{E^*} = \frac{2(1-\nu^2)}{E} \tag{2}$$

The other is a geometrical term that defines the equivalent radius of the two surfaces.

$$\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2} \tag{3}$$

It is useful to remember that when you are dealing with a flat surface that the $R_2 \longrightarrow \infty$ so $R_e = R$

2.1 Point Contact - Two Spheres

2.1.1 Theoretical Example

We will use two different materials and two different radii to work through the problem. So we will use BK7 for the larger sphere $(R_1 = 20 \text{ }mm)$ and 6061 Aluminum $(R_2 = 10 \text{ }mm)$ for the smaller sphere. Using the material properties.

- The Elastic Moduli: $E_1 = 82 \times 10^3 N/mm^2$ and $E_2 = 69 \times 10^3 N/mm^2$
- The Possion Ratios: $\nu_1 = 0.206$ and $\nu_2 = 0.33$

Gives us a Contact Modulus of $E^* = 40663.46 N/mm^2$ and Reduced Radius of $R_e = 6.6667 mm$. Working through the derivation of the geometry solution for the two contact spheres is beyond the scope of this paper. However K. L. Johnson does a very nice job of working through this geometry of contact in Chapter 4 of *Contact Mechanics*² which I recommend the reader to work through on their own. The summary of this is that pressure distribution proposed by Hertz

$$p = p_0 (1 - (r/a)^2)^{1/2} \tag{4}$$

where r is the radius of the measure pressure and p_0 is the maximum pressure of contact between two solids. This relation with the geometry determined for contact of two spheres results in the radius of the area of contact as

$$a = \frac{\pi p_0 R_e}{2E^*} \tag{5}$$

and the mutual approach of points in the solids as

$$\delta = \frac{\pi a p_0}{2E^*} \tag{6}$$

The total load is related to the pressure by

$$F = \int_0^a p(r) 2\pi r dr = \frac{2}{3} p_0 \pi a^2 \tag{7}$$

Since we normally know the load that is being applied then solving for the maximum pressure we get

$$p_0 = \frac{3F}{2\pi a^2} = \left(\frac{6FE^{*2}}{\pi^3 R_e^2}\right)^{1/3} \tag{8}$$

and equations 5 and 6 become

$$a = \left(\frac{3FR_e}{4E^*}\right)^{1/3} \tag{9}$$

and

$$\delta = \frac{a^2}{R_e} = \left(\frac{9F^2}{16R_e E^{*2}}\right)^{1/3} \tag{10}$$

In this example we are using 300 Newtons (F = 300 N) as the load, so using the values that we have been carrying through this discussion, $p_0 = 1292.5 N/mm^2$, a = 0.3329 mm, and $\delta = 0.0166 mm = 16.6 \mu m$. Since we are using Hertz's assumption of no friction, we can find the normal stresses.

$$\sigma_3 = \sigma_z = -p_0 (1 + \frac{z^2}{a^2})^{-1} \tag{11}$$

where z is the position from the point of contact along the z-axis. At the point of contact z=0 so the maximum pressure is the maximum stress at contact. The other principal stresses are found by the equation

$$\sigma_1 = \sigma_x = -p_0 \left[(1+\nu) \left(1 - \left| \frac{z}{a} \right| \tan^{-1} \left(\left| \frac{z}{a} \right| \right) - \frac{1}{2(z^2/a^2 + 1)} \right]$$
(12)

which is also σ_2 and σ_y due to symmetry. The principal shear stresses are

$$|\tau_1| = |\tau_2| = \tau_{max} = \left|\frac{\sigma_1 - \sigma_3}{2}\right| \tag{13}$$

2.2 Finite Element Analysis of Two Sphere in Contact



Figure 2. Finite Element Analysis of two spheres in contact. The larger radius is BK7 and the smaller radius is 6061

There are other limitation on these equations as the metals have plastic properties.⁴ Which may account for the difficulty that was had while trying to match the magnitude of the FEA to the theory.

Configuration of this model that the figures included here where done with bonded and no penetration contacts for the two parts. Constraints consisted of holding the spheres in the X and Y directions, to prevent rotation, while applying the force at both ends of the two models. It was also discovered that it was necessary to hold the bottom sphere fixed in all three directions. Since the morphology of the stress fields were known from the various papers cited here. These conditions resulted in the best representation of those fields.

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Figure 3. FEA Representation of the Shear Stress

2.2.1 Point Contact Discussion

With point contact equations we can solve three other types of problems. These problems can be represented with just a change in radii.

- Sphere on a flat surface, $R_2 = \infty$
- Sphere in a spherical groove $R_2 > -R_1$
- Two cylinders with non-parallel axis, again $R_2 = \infty$

The other thing of interest will be the stress profile and shear stress that occurs. This will become more important when we start thinking about the flaws in the glass. Compressive stress on a fracture would be less of a concern than shear which could put the body into tension in places.



Figure 4. The absolute value of the ratio of the σ_3 (Black Astrisk), σ_1 (Blue Triangles), τ_{max} as a function of the contact radius distance from z=0.

Looking at the profiles we see that there are large stresses, which will mostly be compressive, at the surface of the sphere but also the shear stress increases at a lower depths. It has been commented in several places, that the shear stress could be the likely cause of surface fatigue⁵ and has been argued that this surface fatigue will not degrade the strength of glass,⁴ as it does not reach the flaw depth of the glass.

2.3 Line Contact - Two Cylinders, parallel axis

For two cylinders that with parallel axis. The area of contact is no longer circular but a rectangular area that the magnitude depends on the material values and the load applied. Derivation of the area is again out of the scope of this paper but the half-width of the rectangle is

$$b = \left(\frac{4R_e}{\pi L E^*}\right)^{1/3} \tag{14}$$

so then the maximum pressure is

$$p_0 = \frac{2F}{\pi bL} \tag{15}$$

and the principal stresses⁶ are generated along the z-axis and are:

$$\sigma_1 = \sigma_x = -2\nu p_0 \left[\left(\frac{z^2}{b^2} \right)^{1/2} - \left| \frac{z^2}{b^2} \right| \right]$$
(16)

$$\sigma_2 = \sigma_y = -p_0 \left[\left(2 - \left(\frac{z^2}{b^2} + 1 \right)^{-1} \right) \left(\frac{z^2}{b^2} + 1 \right)^{1/2} - 2 \left| \frac{z}{b} \right| \right]$$
(17)

$$\sigma_3 = \sigma_z = -p_0 \left[\left(\frac{z^2}{b^2} + 1 \right) \right]^{-1/2} \tag{18}$$

and the shear stresses become

$$\tau_1 = \left| \frac{\sigma_2 - \sigma_3}{2} \right|, \tau_2 = \left| \frac{\sigma_1 - \sigma_3}{2} \right|, \tau_3 = \left| \frac{\sigma_1 - \sigma_2}{2} \right|$$
(19)

3. CONCLUSIONS

As with anything in optics and precision mechanics, things get complicated very quickly. Far more complicated than can be understood within the scope of this paper. The original contact theory developed by Hertz was and still is a good theory for larger loads. In the regime that one would expect loads that optics would fall under, it is clear that there are other aspect to contact mechanics needed to get the correct amount of stress that a part would undergo. Friction, the plastics of metal, and adhesion would have to be carefully considered before making a definitive solution on the stress that a part would undergo. FEA is again difficult to get right but one would expect a better answer once the boundary conditions are defined well. However, if one was doing a quick calculation to determine the survivability of an optic then Hertz's theory provides a nice cautious upper end to the stress that would be encountered.

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