Strehl ratio for primary aberrations: some analytical results for circular and annular pupils

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Imaging systems with circular and annular pupils aberrated by primary aberrations are considered. Both classical and balanced (Zernike) aberrations are discussed. Closed-form solutions are derived for the Strehl ratio, except in the case of coma, for which the integral form is used. Numerical results are obtained and compared with Maréchal's formula for small aberrations. It is shown that, as long as the Strehl ratio is greater than 0.6, the Maréchal formula gives its value with an error of less than 10%. A discussion of the Rayleigh quarter-wave rule is given, and it is shown that it provides only a qualitative measure of aberration tolerance. Nonoptimally balanced aberrations are also considered, and it is shown that, unless the Strehl ratio is quite high, an optimally balanced aberration does not necessarily give a maximum Strehl ratio.

1. INTRODUCTION

In a recent paper,[1] we discussed the problem of balancing a classical aberration in imaging systems having annular pupils with one or more aberrations of lower order to minimize its variance. By using the Gram–Schmidt orthogonalization process, polynomials that are orthogonal over an annulus were obtained from the Zernike circle polynomials. These polynomials, appropriately called Zernike annular polynomials, represent balanced classical aberrations in the sense of minimum variance. By using an approximate formula for small aberrations, according to which the decrease in Strehl ratio of the imaging system is given by the variance of its phase aberration, the tolerance conditions were obtained for the balanced primary aberrations for a Strehl ratio of 0.8.

In this paper,[2] we obtain simple analytical expressions for the Strehl ratio of images formed by imaging systems with circular or annular pupils aberrated by primary aberrations. The only exception is coma, in which case the Strehl ratio is obtained in an integral form. Both classical and balanced (orthogonal) primary aberrations are considered. Numerical results are obtained from these expressions for up to five waves of a classical aberration. These results are compared with the approximate results obtained by using the Maréchal formula to determine its range of validity. It is shown that, in general, as long as the Strehl ratio is greater than 0.6, the Maréchal formula gives results with an error of less than 10%.

The Rayleigh quarter-wave rule, i.e., that the aberration tolerance for a Strehl ratio of 0.80 is a quarter-wave, is discussed. The Strehl ratio is calculated for a primary aberration when its aberration coefficient, peak absolute value, and peak-to-peak value, are each a quarter-wave. It is shown that only a quarter-wave of spherical aberration and a quarter-wave peak value of balanced spherical aberration give a Strehl ratio of 0.80. A quarter-wave of coma and astigmatism do not give a Strehl ratio of 0.80. It is thus concluded that the Rayleigh quarter-wave rule does not provide a quantitative, but only a qualitative, measure of aberration tolerance.

Nonoptimally balanced aberrations are also considered. It is shown that, unless the Strehl ratio is quite high, an optimally balanced aberration (in the sense of minimum variance) does not give the maximum possible Strehl ratio. As an example, spherical aberration is discussed in detail. A certain amount of spherical aberration is balanced with an appropriate amount of defocus to minimize its variance across the pupil. However, it is found that minimum aberration variance does not always lead to a maximum of Strehl ratio. For example, in the case of circular pupils aberrated by a spherical aberration of more than 2.3 waves, a maximum Strehl ratio is obtained with an amount of defocus that does not correspond to minimum variance. In the case of coma, which is balanced with tilt to minimize its variance, the Strehl ratio is higher (but not the highest) with no tilt when the aberration is larger than 2.3 waves. Only for very small aberrations ($\lesssim 0.7\lambda$) does optimally balanced coma give the maximum Strehl ratio; otherwise a nonoptimal value of tilt gives the maximum Strehl ratio.

2. STREHL RATIO AND THE MARÉCHAL FORMULA

The incoherent point-spread function of an imaging system is given by[3]

$$I(\mathbf{r}) = \frac{1}{\lambda^2 R^2} \left| \int A(\boldsymbol{\rho}) \exp[i\Phi(\boldsymbol{\rho})] \exp(-2\pi i \mathbf{r} \cdot \boldsymbol{\rho}/\lambda R) \mathrm{d}\boldsymbol{\rho} \right|^2, \quad (1)$$

where $I(\mathbf{r})$ is the irradiance at a point $\mathbf{r}$ in the image plane for a point object, λ is the wavelength of the object radiation, $A(\boldsymbol{\rho})$ is the amplitude, $\Phi(\boldsymbol{\rho})$ is the phase aberration at a point $\boldsymbol{\rho}$ on the system's exit pupil, and R is the radius of curvature of the reference sphere with respect to which the aberrations are measured. The point $\mathbf{r} = 0$ lies at the center of curvature of the reference sphere. The Strehl ratio of the imaging system is given by the ratio of the central irradiance of its aberrated and unaberrated point-spread functions. From Eq. (1), we can write it in the form

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$$S = I(0)_\Phi/I(0)_{\Phi=0} = |\langle \exp(i\Phi)\rangle|^2, \tag{2}$$

where angle brackets indicate a spatial average over the amplitude-weighted pupil, e.g., the average of a function $f(\rho)$ is given by

$$\langle f\rangle = \int A(\rho)f(\rho)\mathrm{d}\rho/\int A(\rho)\mathrm{d}\rho. \tag{3}$$

Since the average phase aberration $\langle \Phi\rangle$ is independent of ρ, Eq. (2) can also be written as

$$S = |\langle \exp[i(\Phi - \langle \Phi\rangle)]\rangle|^2. \tag{4}$$

By expanding the complex exponential of Eq. (4) in cosine and sine terms, we may write

$$S = \langle\cos(\Phi - \langle \Phi\rangle)\rangle^2 + \langle\sin(\Phi - \langle \Phi\rangle)\rangle^2 \tag{5}$$

so that

$$S \geq \langle\cos(\Phi - \langle \Phi\rangle)\rangle^2, \tag{6}$$

equality holding when $\Phi(\rho) = 0$. By expanding $\cos(\Phi - \langle \Phi\rangle)$ in a power series and retaining the first two terms for small aberrations, we obtain the Maréchal[4] result that

$$S \gtrsim (1 - \tfrac{1}{2}\sigma_\Phi^2)^2, \tag{7}$$

where σ_Φ^2 is the variance of the aberration. Note that, unlike Maréchal, we have obtained this result without assuming that $\langle \Phi\rangle$ be zero. Two approximate expressions for the Strehl ratio for small aberrations used in the literature are

$$S_1 \simeq (1 - \tfrac{1}{2}\sigma_\Phi^2)^2 \tag{8}$$

and

$$S_2 \simeq 1 - \sigma_\Phi^2. \tag{9}$$

Formula (9) is obtained from formula (8) by neglecting the term in σ_Φ^4. It has been used by Nijboer,[5] Born and Wolf,[6] and the author,[1] among others, in the discussion of orthogonal aberrations. We shall refer to formula (8) as the Maréchal formula. Note that S_1 is always positive, as a Strehl ratio has to be. It starts at a value of 1 when $\sigma_\Phi = 0$ and approaches zero as σ_Φ approaches $\sqrt{2}$. It becomes greater than or equal to 1 as σ_Φ becomes greater than or equal to 2, respectively. This is obviously a region where σ_Φ is too large for formula (8) [and formula (7)] to have any validity. An invalid region for S_2 is when $\sigma_\Phi > 1$ since S_2 then becomes negative.

For imaging systems with uniformly illuminated annular pupils having a central obscuration ratio of ϵ, i.e., for

$$A(\rho) = \begin{cases} 1, & \epsilon \leq |\rho| \leq 1 \\ 0, & \text{otherwise,}\end{cases} \tag{10}$$

Eq. (2) becomes

$$S = \frac{1}{\pi^2(1-\epsilon^2)^2}\left|\int_\epsilon^1 \int_0^{2\pi} \exp[i\Phi(\rho,\theta)]\rho\mathrm{d}\rho\mathrm{d}\theta\right|^2. \tag{11}$$

The variance of an aberration $\Phi(\rho, \theta)$ over an annular pupil is given by

$$\sigma_\Phi^2 = \frac{1}{\pi(1-\epsilon^2)}\int_\epsilon^1 \int_0^{2\pi} \Phi^2(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta - \left[\frac{1}{\pi(1-\epsilon^2)}\int_\epsilon^1 \int_0^{2\pi} \Phi(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta\right]^2. \tag{12}$$

3. STREHL RATIO AND STANDARD DEVIATION FOR PRIMARY ABBERRATIONS

Table 1 lists the primary aberrations and their corresponding standard deviations and the Strehl ratios. Both classical and balanced (in the sense of minimum variance) aberrations are considered. The corresponding orthogonal (Zernike) aberrations are given in Ref. 1. The form of an orthogonal aberration is identical with that of a balanced aberration, except in the case of a rotationally symmetric aberration, in which case it differs by a constant (independent of ρ) term. Since the variance or the Strehl ratio for an aberration does not depend on a constant aberration term, a study here of a balanced aberration is equivalent to that of an orthogonal aberration. Some of the Strehl-ratio results for the classical aberrations have been given by Steward.[7] Note that, although in Table 1 the aberration coefficients A_i are in units of radians, we shall specify their values in units of optical wavelength in all of our discussion, as is customary in optics.

In the case of spherical aberration, the Strehl ratio is given in terms of the Fresnel integrals $C(\cdot)$ and $S(\cdot)$. Note that, for circular pupils ($\epsilon = 0$), the Strehl ratio for a given amount of classical spherical aberration is the same as that for four times that amount of balanced spherical aberration. In other words, an equal and opposite amount of defocus exactly quadruples the spherical aberration tolerance for a given value of Strehl ratio. (For $\epsilon = 0.50, 0.75$, an appropriate amount of defocus increases the tolerance by a factor that is even larger than 4, as may be seen from Figs. 1 and 2.) In the case of coma, the Strehl ratio has to be obtained by a numerical integration. In the case of balanced astigmatism, the Strehl ratio is given in terms of an infinite series of odd-order Bessel functions of the first kind.[8] However, this series converges rapidly, and only a few terms need be considered for adequate precision. The Strehl ratio for classical astigmatism is derived in Appendix A as an example of the results given in Table 1.

The results for curvature of field or defocus and distortion or tilt are well known. If the image (focal) plane is moved along an axis normal to the pupil plane by a small amount z, the amount of defocus aberration produced is approximately given by $A_d = z/8F^2$, where F is the focal ratio of the imaging system. It is evident from the expression for the Strehl ratio that, as a function of z, the peak irradiance occurs when $z = 0$, i.e., when the defocus is zero. This is true of all aberrations; any aberration gives a Strehl ratio of less than unity.[9] However, a refinement of Eq. (1) shows that the principal maximum of axial irradiance occurs for $z < 0$, i.e., it lies at an axial point between the pupil and focal planes.[10–13]

4. NUMERICAL RESULTS

Using the expressions given in Table 1, Figs. 1–6 show how the Strehl ratio varies with the coefficient of a primary aberration. The maximum value of the coefficient is 5λ (or 10π in radian units). Both circular ($\epsilon = 0$) and annular pupils ($\epsilon = 0.50, 0.75$) are considered. It is evident that, in each case, the Strehl ratio decreases monotonically for small values of the aberration coefficient but fluctuates for its large values. For small aberrations, obscured pupils are less sensitive to spherical aberration (and defocus). The opposite is true of astigmatism and coma.

The variation of the percent difference $100(S - S_1)/S$ between the Strehl ratio S and its approximate value S_1 for

Table 1. Standard Deviation and Strehl Ratio for Primary Aberrations[a]

Aberration	$\Phi(\rho, \theta)$	σ_Φ	S
Spherical	$A_s\rho^4$	$(1/3\sqrt{5})(4 - \epsilon^2 - 6\epsilon^4 - \epsilon^6 + 4\epsilon^8)^{1/2}A_s$	$[\pi/2A_s(1-\epsilon^2)^2] \times (\{C[(2A_s/\pi)^{1/2}] - C[(2A_s/\pi)^{1/2}\epsilon^2]\}^2 + \{S[(2A_s/\pi)^{1/2}] - S[(2A_s/\pi)^{1/2}\epsilon^2]\}^2)$
Balanced spherical	$A_s[\rho^4 - (1+\epsilon^2)\rho^2]$	$(1/6\sqrt{5})(1-\epsilon^2)^2A_s$	$[2\pi/A_s(1-\epsilon^2)^2[\{C^2[A_s/2\pi)^{1/2}(1-\epsilon^2)] + S^2[(A_s/2\pi)^{1/2}(1-\epsilon^2)]\}$
Coma	$A_c\rho^3\cos\theta$	$(1/2\sqrt{2})(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}A_c$	$(1-\epsilon^2)^{-2}\left\vert\int_{\epsilon^2}^1 J_0(A_cx^{3/2})\mathrm{d}x\right\vert^2$
Balanced coma	$A_c\left[\rho^3 - \frac{2}{3}\frac{(1+\epsilon^2+\epsilon^4)}{(1+\epsilon^2)}\rho\right]\cos\theta$	$\frac{(1-\epsilon^2)(1+4\epsilon^2+\epsilon^4)^{1/2}}{6\sqrt{2}(1+\epsilon^2)^{1/2}}A_c$	$(1-\epsilon^2)^{-2}\left\vert\int_{\epsilon^2}^1 J_0\left[A_c\left(x^{3/2} - \frac{2}{3}\frac{1+\epsilon^2+\epsilon^4}{1+\epsilon^2}x^{1/2}\right)\right]\mathrm{d}x\right\vert^2$
Astigmatism	$A_a\rho^2\cos^2\theta$	$(1/4)(1+\epsilon^4)^{1/2}A_a$	$(1-\epsilon^2)^{-2}\{H^2(A_a/2) + \epsilon^2H^2(\epsilon^2A_a/2) - 2\epsilon^2H(A_a/2)H(\epsilon^2A_a/2) \times \cos[(1/2)(1-\epsilon^2)A_a - \alpha(A_a/2) + \alpha(A_a\epsilon^2/2)]\}$
Balanced astigmatism	$A_a\rho^2(\cos^2\theta\ 1/2)$	$(1/2\sqrt{6})(1+\epsilon^2+\epsilon^4)^{1/2}A_a$	$\left\{[4/A_a(1-\epsilon^2)]\sum_{k=0}^{\infty} J_{2k+1}(A_a/2) - J_{2k+1}(\epsilon^2A_a/2)\right\}^2$
Curvature of field (defocus)	$A_d\rho^2$	$[(1-\epsilon^2)/2\sqrt{3}]A_d$	$\left[\frac{\sin[A_d(1-\epsilon^2)/2]}{A_d(1-\epsilon^2)/2}\right]^2$
Distortion (tilt)	$A_t\rho\cos\theta$	$[(1+\epsilon^2)/2]A_t$	$(1-\epsilon^2)^{-2}\left[\frac{2J_1(A_t)}{A_t} - \epsilon^2\frac{2J_1(\epsilon A_t)}{\epsilon A_t}\right]^2$

[a] A_i is the coefficient of the ith abberation (measured in radians) and ϵ is the obscuration ratio of an annular pupil. $\epsilon \le \rho \le 1$, $0 < \theta \le 2\pi$, $C(a) = \int_0^a \cos(\pi x^2/2)\mathrm{d}x$, $S(a) = \int_0^a \sin(\pi x^2/2)\mathrm{d}x$, $H(a) = [J_0^2(a) + J_1^2(a)]^{1/2}$, $\alpha(a) = \tan^{-1}[J_1(a)/J_0(a)]$.

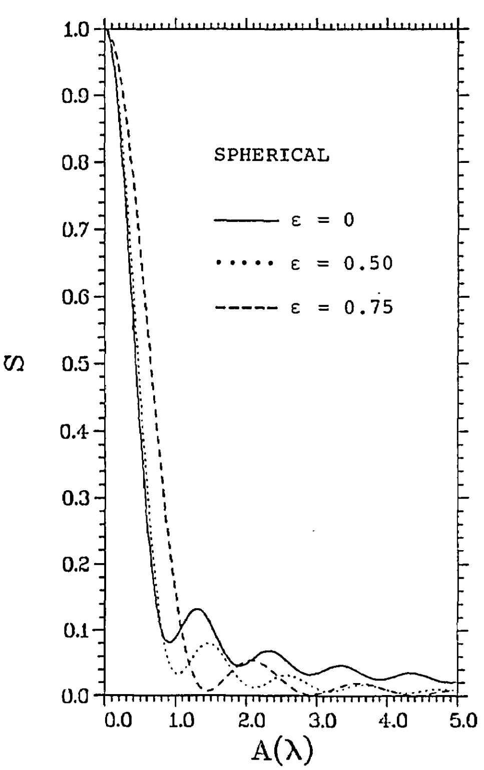

Fig. 1. Strehl ratio for spherical aberration.

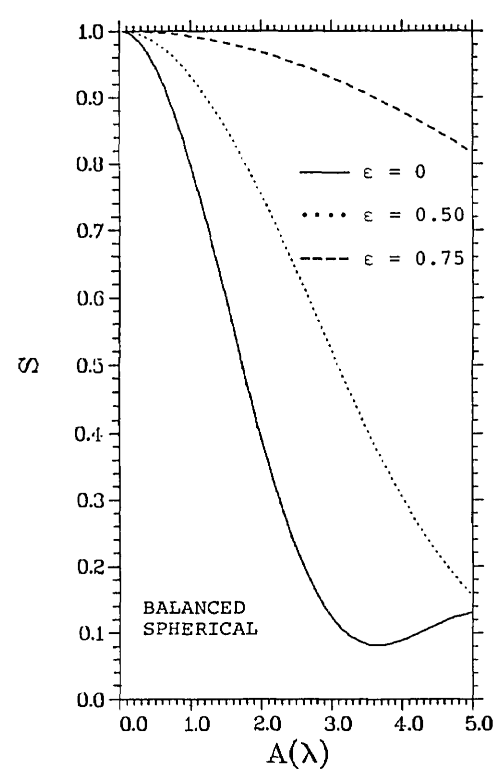

Fig. 2. Strehl ratio for balanced spherical aberration.

primary aberrations is shown in Fig. 7 when $\epsilon = 0, 0.50, 0.75$. When $\epsilon = 0$, the curves for spherical and balanced spherical aberrations are identical because a given value of the Strehl ratio is obtained for the same value of σ_Φ of the two aberrations. When $\epsilon = 0.50$, the percent error curve for balanced spherical aberration is shown in Fig. 7(b) by the curve that approaches only approximately 88% (as opposed to 100% for others). When $\epsilon = 0.75$, the Strehl ratio for balanced spherical aberration for the range of A_s values considered here ($A_s \leq 5\lambda$) is quite large (≥ 0.8157) so that the percent error is negligible (≤ 0.79). Therefore the corresponding curve overlaps the others in the region of its existence.

As the coefficient of an aberration increases in the region where the Strehl ratio decreases monotonically, the percent error increases slowly first and then rapidly, approaching 100% as σ_Φ approaches $\sqrt{2}$; it decreases to zero and finally becomes negative. In the region of fluctuating Strehl ratio (for large aberrations), $S - S_1$ may change sign from negative to posi-

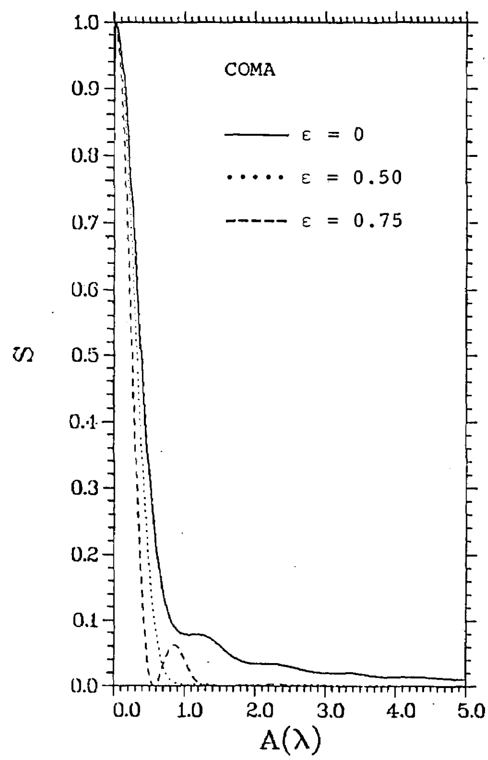

Fig. 3. Strehl ratio for coma.

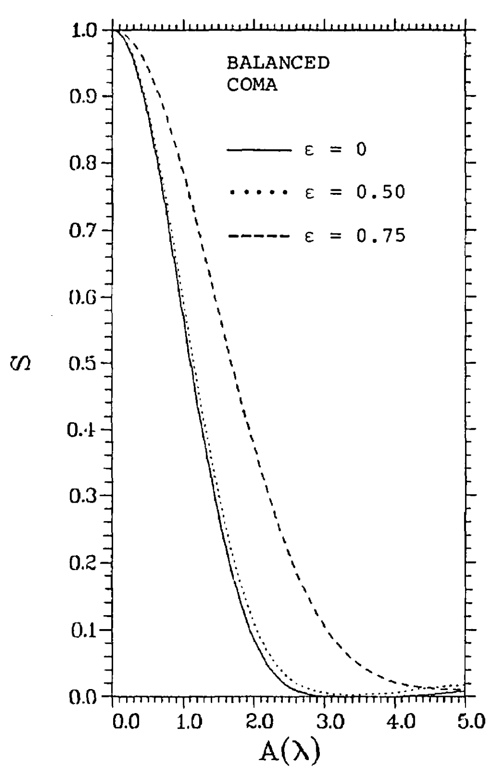

Fig. 4. Strehl ratio for balanced coma.

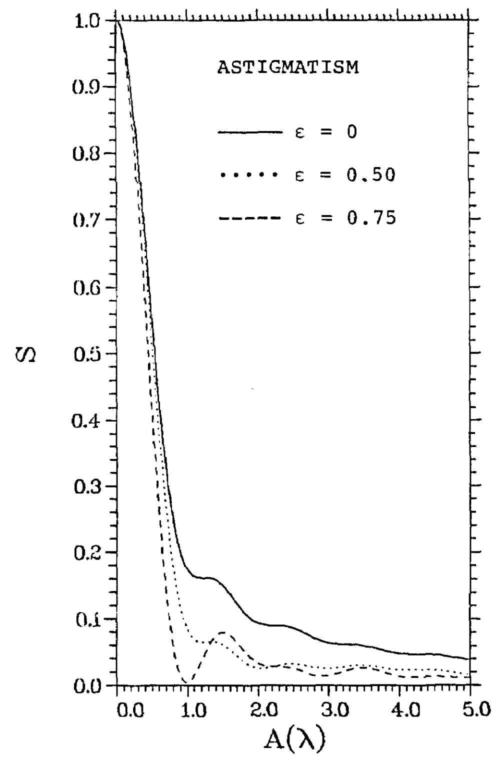

Fig. 5. Strehl ratio for astigmatism.

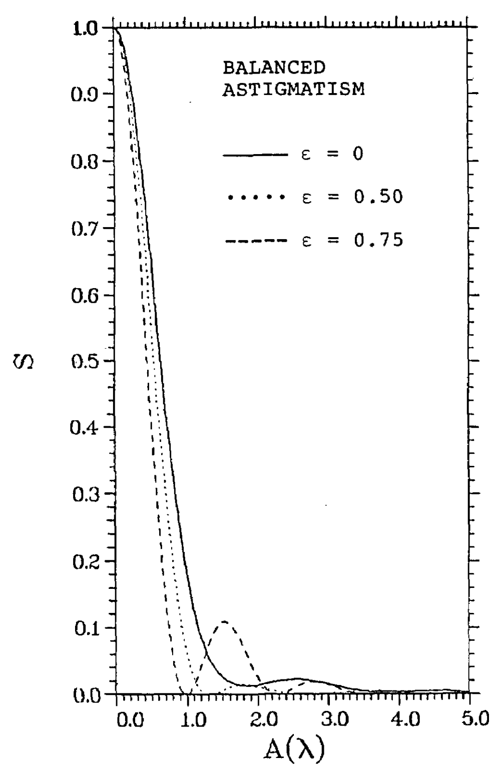

Fig. 6. Strehl ratio for balanced astigmatism.

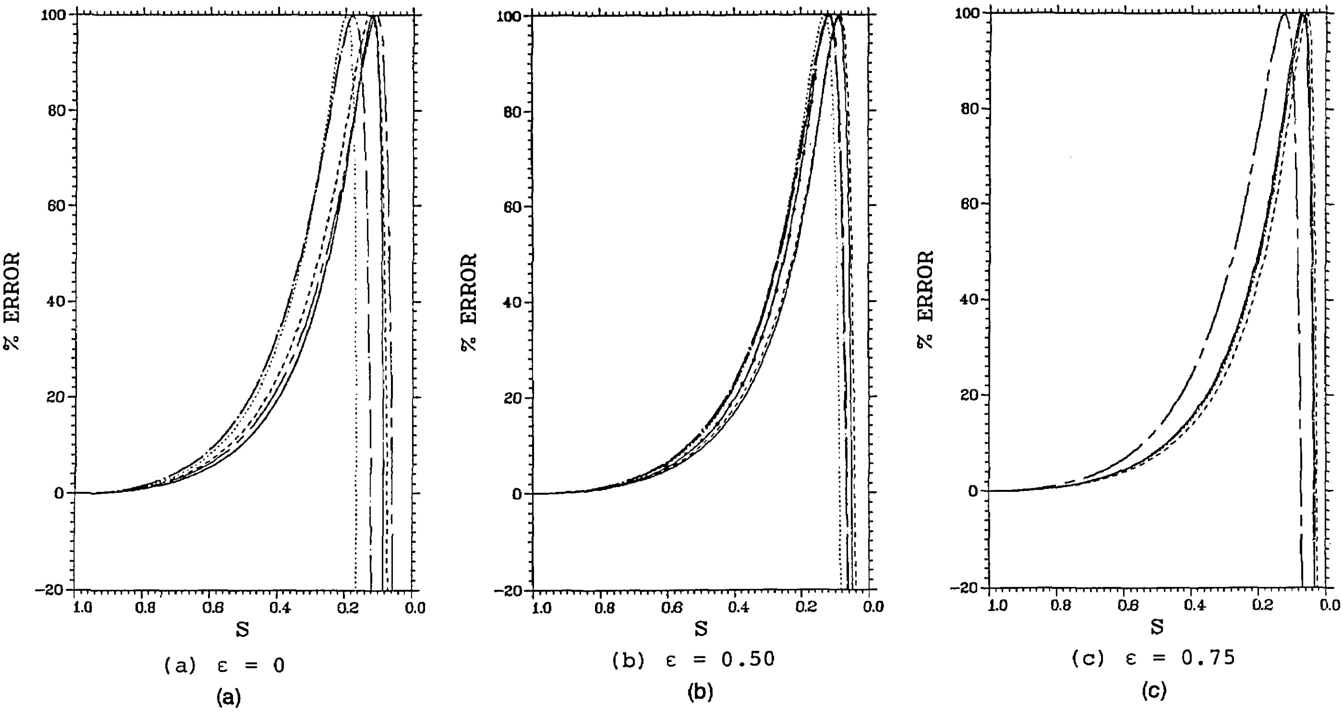

Fig. 7. Percent error $100(S - S_1)/S$ as a function of S. (a) $\epsilon = 0$, (b) $\epsilon = 0.50$, (c) $\epsilon = 0.75$. Spherical, —; coma, - · -; balanced coma, — - —; astigmatism, · · · ·; balanced astigmatism, - - - - -. In (a), the curves for spherical and balanced spherical aberrations are identical. In (b), the curve for balanced spherical aberration is shown by —·—. In (c), this curve exists for $S \gtrsim 0.81$ and overlaps the others in this region.

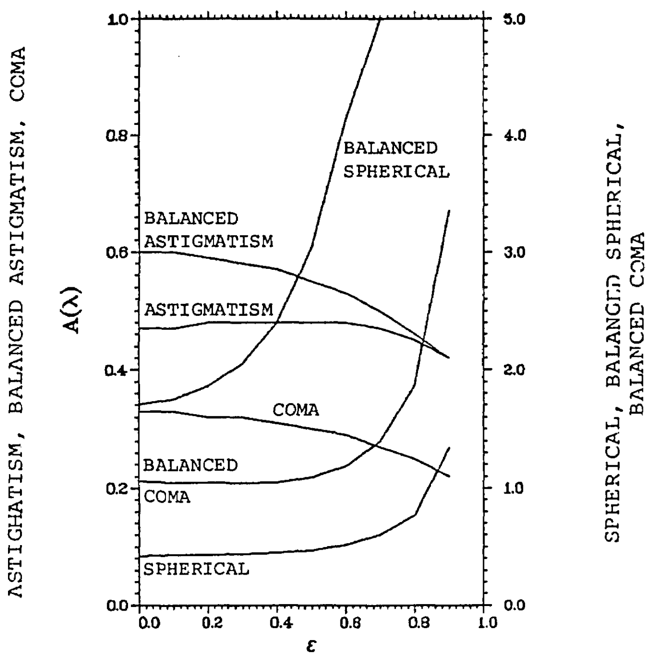

Fig. 8. Variation of a primary aberration coefficient for 10% error with obscuration ratio.

tive, but the relative error is then very large (>100%). When $\epsilon = 0$, $S - S_1$ is positive for $S \gtrsim 0.086$ for spherical or balanced spherical aberration (excluding the region of secondary maxima of the Strehl ratio), for $S \gtrsim 0.123$ for coma, $S \gtrsim 0.063$ for balanced coma, $S \gtrsim 0.166$ for astigmatism, and $S \gtrsim 0.075$ for balanced astigmatism. Thus, unless the Strehl ratio is extremely small (or it belongs to its fluctuating region), S_1 underestimates the value of the Strehl ratio, sometimes by as much as 100%. Since S_2 is smaller than S_1 by $\sigma_\Phi^4/4$, it underestimates the Strehl ratio even more.

It is interesting to note that, for $S \gtrsim 0.8$, the error is negligible and practically insensitive to the type of aberration being considered. The error is less than 10% if $S \gtrsim 0.6$. Thus, as long as $\sigma_\Phi \lesssim 0.67$, or the standard deviation of the wave aberration $\sigma_w = (\lambda/2\pi)\sigma_\Phi \lesssim \lambda/9.4$, S_1 gives Strehl-ratio results with less than 10% error. Figure 8 shows how the aberration coefficient of a primary aberration for 10% error varies with the obscuration ratio. It is evident that this coefficient increases with obscuration in the case of spherical (classical and balanced) aberration and balanced coma but decreases in the case of astigmatism (classical and balanced) and coma.

Since S_1 underestimates the Strehl ratio, $\exp(-\sigma_\Phi^2)$, which, for small σ_Φ, is greater than S_1 by approximately $\sigma_\Phi^4/4$, should approximate the Strehl ratio better. We find that this is indeed true; $\exp(-\sigma_\Phi^2)$ gives the Strehl ratio with less than 10% error as long as $S \gtrsim 0.3$. The error in this region is negative, implying an overestimation. As with S_1, the value of S or σ_Φ for which $\exp(-\sigma_\Phi^2)$ gives a 10% error varies with the type of aberration. The use of a Gaussian model for aberrated point-spread functions to evaluate not only the Strehl ratio but also the encircled energy from a knowledge of the aberration variance is under investigation and will be reported later.

5. RAYLEIGH'S QUARTER-WAVE RULE

Rayleigh[14] showed that a quarter-wave of primary spherical aberration reduces the irradiance at the Gaussian focus by 20%, i.e., the Strehl ratio for this aberration is 0.80. This result has brought forth Rayleigh's quarter-wave rule, namely, that the quality of an aberrated image will be good if the absolute value of the aberration at any point on the pupil is less than a quarter-wave.[15] A variant of this definition is that if the aberrated wave front lies between two concentric spheres that are spaced a quarter-wave apart, the aberrated image will be of good quality.[16] Thus, instead of the maximum or peak absolute value of the aberration ($|W_p|$) being less than a quarter-wave, it is the peak-to-peak aberration (W_{p-p}) that should be less than a quarter-wave. The general implication of a good-quality image is that the Strehl ratio is 0.8.

It is well known that a quarter-wave of different aberrations does not necessarily give a Strehl ratio of 0.8. Barakat,[17] for example, showed that a slit pupil aberrated by a quarter-wave of Legendre polynomial of order 20 (an orthogonal aberration for the one-dimensional problem[18]), corresponding to a peak-to-peak aberration of a half-wave, gives a Strehl ratio of 0.637. For a quarter-wave peak-to-peak aberration, the Strehl ratio is 0.701.

When the aberration coefficient A_i of a primary aberration, as defined in Table 1, is equal to a quarter-wave, the variation of the Strehl ratio with ϵ is as shown in Fig. 9. When $\epsilon = 0$, i.e., for circular pupils, $|A_i|$ represents the maximum absolute value of the aberration in the case of spherical aberration, coma, and astigmatism. The maximum absolute value is $|A_s|/4$ in the case of balanced spherical aberration, $|A_c|/3$ in the case of coma, and $|A_a|/2$ in the case of balanced astigmatism. The corresponding peak-to-peak values are A_s for spherical aberration, $A_s/4$ for balanced spherical aberration, $2A_c$ for coma, $2A_c/3$ for balanced coma, and A_a for both astigmatism and balanced astigmatism. The maximum absolute and peak-to-peak values of the primary aberrations are tabulated in Table 2.

Table 3 shows the values of the Strehl ratio for primary aberrations of absolute peak value of a quarter-wave. Table 4 shows the Strehl-ratio values when the peak-to-peak aberration is a quarter-wave. Table 5 shows the aberration coefficients (A_i), the absolute peak values $|W_p|$, and the peak-

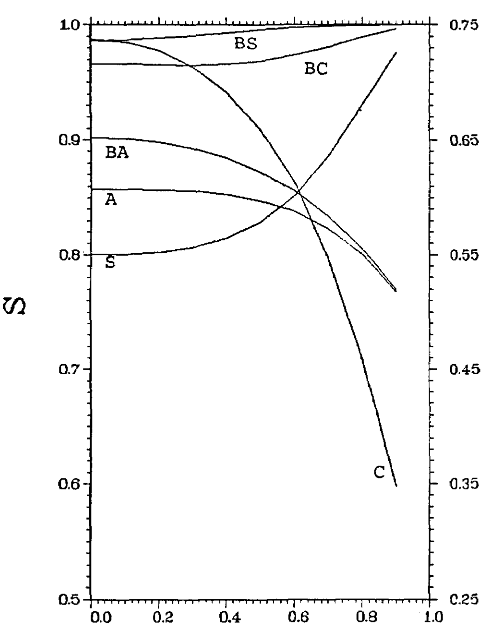

Fig. 9. Strehl ratio for a quarter-wave aberration as a function of obscuration ratio. $A_i = \lambda/4$. S, spherical; BS, balanced spherical; C, coma; BC, balanced coma; A, astigmatism; BA, balanced astigmatism. The right-hand-side vertical scale is only for coma.

Table 2. Aberration Coefficient, Absolute Peak Value, and Peak-to-Peak Value for Primary Aberrations ($\epsilon = 0$)

Aberration	Aberration Coefficient	Absolute Peak Value $|W_p|$	Peak-to-Peak Value W_{p-p}
Spherical	A_s	$|A_s|$	A_s
Balanced spherical	A_s	$|A_s|/4$	$A_s/4$
Coma	A_c	$|A_c|$	$2A_c$
Balanced coma	A_c	$|A_c|/3$	$2A_c/3$
Astigmatism	A_a	$|A_a|$	A_a
Balanced astigmatism	A_a	$|A_a|/2$	A_a

Table 3. Strehl Ratio for a Quarter-Wave Absolute Peak Value of a Primary Aberration ($|W_p| = \lambda/4$)[a]

Aberration	A_i (λ)	S
Spherical	0.25	0.8003
Balanced spherical	1	0.8003
Coma	0.25	0.7374
Balanced coma	0.75	0.7317
Astigmatism	0.25	0.8572
Balanced astigmatism	0.5	0.6602

[a] The corresponding aberration coefficient A_i is given in units of wavelength ($\epsilon = 0$).

Table 4. Strehl Ratio for a Quarter-Wave Peak-to-Peak Value of a Primary Aberration ($W_{p-p} = \lambda/4$)[a]

Aberration	A_i (λ)	S
Spherical	0.25	0.8003
Balanced spherical	1	0.8003
Coma	0.125	0.92
Balanced coma	0.375	0.92
Astigmatism	0.25	0.8572
Balanced astigmatism	0.25	0.9021

[a] The corresponding aberration coefficient A_i is given in units of wavelength ($\epsilon = 0$).

Table 5. Aberration Coefficient A_i, Absolute Peak Value $|W_p|$, and Peak-to-Peak Value W_{p-p}, All in Units of Wavelength, for a Strehl Ratio of 0.80 ($\epsilon = 0$)

Aberration	A_i (λ)	$|W_p|$ (λ)	W_{p-p} (λ)
Spherical	0.25	0.25	0.25
Balanced spherical	1	0.25	0.25
Coma	0.21	0.21	0.42
Balanced coma	0.63	0.21	0.42
Astigmatism	0.30	0.30	0.30
Balanced astigmatism	0.37	0.18	0.37

to-peak values W_{p-p}, all in units of wavelength, for a Strehl ratio of 0.8. It is evident from the data in Tables 3–5 that, except for spherical aberration (the example considered by Rayleigh[14]) and balanced spherical aberration, a Strehl ratio of 0.8 is not obtained for a quarter-wave of peak absolute value or peak-to-peak value of a primary aberration. Thus the Rayleigh quarter-wave rule does not provide a quantitative measure of the quality of an aberrated image. At best, it provides an indication of a reasonably good image. Similar conclusions generally hold when the pupil is obscured.

6. STREHL RATIO FOR NONOPTIMALLY BALANCED ABERRATIONS

It should be noted that, when a classical aberration is balanced with other aberrations to minimize its variance, the balanced aberration does not necessarily yield a higher or the highest possible Strehl ratio. For small aberrations, a maximum Strehl ratio should be obtained according to formula (8) when the variance is minimum. For large aberrations, however, there is no simple relationship between the Strehl ratio and the aberration variance.

Let us consider spherical aberration balanced with an arbitrary amount of defocus. The Strehl ratio can again be written in terms of the Fresnel integrals. For example, if

$$\Phi(\rho) = A_s\rho^4 - A_d\rho^2, \tag{13}$$

the Strehl ratio can be written as

$$S = [\pi/2A_s(1-\epsilon^2)^2] \times \{[C(a_+) + C(a_-)]^2 + [S(a_+) + S(a_-)]^2\}, \tag{14}$$

where

$$a_\pm = [(1-\epsilon^2)A_s \pm \Delta]/(2\pi A_s)^{1/2} \tag{15}$$

and

$$\Delta = A_d - A_s(1+\epsilon^2). \tag{16}$$

It is evident that the Strehl ratio is independent of the sign of Δ, the deviation of defocus from its optimum (in the sense of minimum variance) value. Thus the axial irradiance of a spherically aberrated wave is symmetric about the axial point with respect to which the aberration variance is minimum. This fact was pointed out by Nijboer[19] for circular pupils, but it holds for annular pupils as well. Figure 10 shows how S varies with Δ in the case of circular pupils for several typical values of A_s. It is seen that, for large Strehl ratios, S is maximum when $\Delta = 0$, i.e., minimum variance leads to a maximum of Strehl ratio. For small Strehl ratios, however, minimum variance gives a minimum of Strehl ratio. The Strehl ratio is maximum for a nonoptimally balanced aberration. For example, when $A_s = 3\lambda$, the optimum amount of defocus is $A_d = 3\lambda$, but the Strehl ratio is a minimum and equal to 0.12. The Strehl ratio is maximum and equal to 0.26 for $A_d \simeq 4\lambda$, 2λ. For $A_s \lesssim 2.3\lambda$, the axial irradiance is maximum at a point with respect to which the aberration variance is minimum.

Barakat and Houston[20] have calculated the Strehl ratio for annular pupils aberrated by one wave of spherical aberration optimally balanced with defocus for annular pupils and circular pupils. Thus they use two aberrations,

$$\Phi_A(\rho;\epsilon) = 2\pi[\rho^4 - (1+\epsilon^2)\rho^2] \tag{17}$$

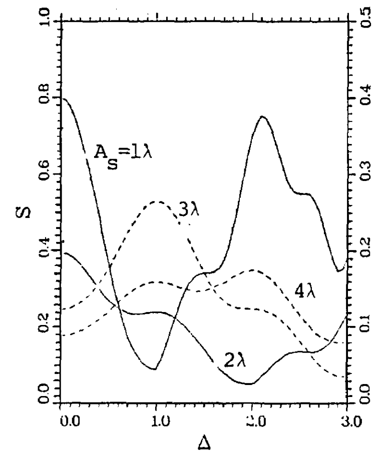

Fig. 10. Strehl ratio for circular pupils ($\epsilon = 0$) aberrated by spherical aberration as a function of the deviation of focus from its optimum balancing value ($\Delta = A_d - A_s$) for several values of the aberration coefficient A_s. The curves are symmetric about the origin. The right-hand-side vertical scale is for $A_s = 3\lambda, 4\lambda$.

Table 6. Strehl Ratio for Annular Pupils Aberrated with One Wave of Spherical Aberration Optimally Balanced with Defocus for Annular and Circular Pupils

ϵ	S_A	S_B
0	0.8003	0.8003
0.1	0.8074	0.8069
0.2	0.8279	0.8239
0.3	0.8589	0.8407
0.4	0.8957	0.8452
0.5	0.9326	0.8315
0.6	0.9637	0.8082
0.7	0.9852	0.7993
0.8	0.9962	0.8340
0.9	0.9995	0.9240

and

$$\Phi_B(\rho;\epsilon) = 2\pi(\rho^4 - \rho^2), \tag{18}$$

to calculate the Strehl ratios given in columns A and B, respectively, of their Table II. Letting $A_s = 2\pi$ and $A_d = (1+\epsilon^2)A_s$ and A_s, we obtain from Eq. (14)

$$S_A(\epsilon) = [C^2(1-\epsilon^2) + S^2(1-\epsilon^2)]/(1-\epsilon^2)^2 \tag{19}$$

and

$$S_B(\epsilon) = \{[C(1) + C(1-2\epsilon^2)]^2 + [S(1) + S(1-2\epsilon^2)]^2\}/4(1-\epsilon^2)^2. \tag{20}$$

By using Eqs. (19) and (20), we obtain the Strehl ratios given in Table 6. Comparing these numbers with those given by Barakat and Houston,[20] we find that the numbers in their column A need to be divided by $(1-\epsilon^2)$, and those in column B by $(1-\epsilon^2)^2$. That the Strehl ratio should increase with ϵ for a given small amount of optimally balanced spherical aberration is evident from the fact that the corresponding standard deviation decreases as ϵ increases. Note that $S_B(\epsilon)$ fluctuates as ϵ increases.

In the case of coma, we note from Table 1 that it is balanced with tilt to minimize its variance. For circular pupils, for example, the coefficient of tilt is equal to two thirds of the coma coefficient. Thus the point with respect to which the aberration variance is minimum lies in the image plane at a distance of $2A_cF/3$ from the origin. From Barakat's work,[21] we find that maximum irradiance occurs at this point only if $A_c \lesssim 0.7$, which in turn corresponds to $S \gtrsim 0.76$. For larger values of A_c, the distance of the point of maximum irradiance does not increase linearly with its value and even fluctuates in some regions. Thus, only for large Strehl ratios, the irradiance is maximum at the point associated with minimum aberration variance. Moreover, we note from Figs. 3 and 4 that, when $\epsilon = 0$ and $A_c \geq 2.3\lambda$, the classical coma gives a larger Strehl ratio than the balanced coma, i.e., the irradiance at the origin is larger than at the point with respect to which the aberration variance is minimum.

For secondary spherical aberration and secondary coma, King[22] has concluded that, when these aberrations are balanced with lower-order aberrations to minimize their variance, a maximum of Strehl ratio is obtained only if its value comes out to be greater than about 0.5. Otherwise, a mixture of aberrations yielding a larger-than-minimum possible variance gives a higher Strehl ratio than the one provided by a minimum-variance mixture.

7. DISCUSSION AND CONCLUSIONS

The Maréchal formula for the Strehl ratio [formula (8)] gives approximate but reasonably accurate results for small aberrations. In the region of practical interest where Strehl ratio calculations are desirable (i.e., excluding the region of extremely small Strehl ratios and its secondary maxima), S_1 underestimates the value of the Strehl ratio (S_2 underestimates its value even more). Comparing the approximate results with the exact analytical results obtained in this paper, we have shown that, for a primary aberration, S_1 gives results with an error of less than 10% if the Strehl ratio is greater than 0.6. Thus, for less than 10% error, the standard deviation of the wave aberration must be less than $\lambda/9.4$.

Although the Rayleigh quarter-wave rule provides a criterion for a reasonably good image, it does not provide a quantitative measure of the image quality. A quarter-wave of a primary aberration, whether it is the absolute peak value or the peak-to-peak value of the aberration, does not necessarily give a Strehl ratio of 0.8. The Maréchal formula, on the other hand, gives a Strehl ratio of greater than or equal to 0.8 for $\sigma_w \lesssim \lambda/14$ with an error of less than a few percent.

The Maréchal formula, which is valid for small aberrations, shows that the Strehl ratio is maximum when the aberration variance is minimum. Thus the Strehl ratio of a slightly aberrated optical system can be improved if its aberration variance can be reduced. For highly aberrated systems, a reduction in aberration variance by balancing a higher-order aberration with lower-order aberrations does not necessarily lead to the highest possible Strehl ratio and may, in fact, decrease it.

APPENDIX A. STREHL RATIO FOR CLASSICAL ASTIGMATISM

The aberration function for classical astigmatism is

$$\Phi(\rho, \theta) = A\rho^2 \cos^2 \theta \tag{A1}$$

or

$$\Phi(\rho, \theta) = \tfrac{1}{2}A\rho^2(1 + \cos 2\theta). \tag{A2}$$

By substituting Eq. (A2) into Eq. (11), we obtain the Strehl ratio

$$S = \frac{1}{\pi^2(1-\epsilon^2)^2}\left|\int_\epsilon^1 \exp(iA\rho^2/2)\rho d\rho \times \int_0^{2\pi} \exp(\tfrac{1}{2}iA\rho^2 \cos 2\theta)d\theta\right|^2. \tag{A3}$$

By carrying out the integration over θ, we obtain

$$S = \frac{4}{(1-\epsilon^2)^2}\left|\int_\epsilon^1 \exp(iA\rho^2/2)J_0(A\rho^2/2)\rho d\rho\right|^2, \tag{A4}$$

where $J_0(\cdot)$ is the zero-order Bessel function of the first kind. Noting that[23]

$$\int_0^z \exp(it)J_0(t)dt = z\exp(iz)[J_0(z) - iJ_1(z)], \tag{A5}$$

Eq. (A4) can be written as

$$S = (1-\epsilon^2)^{-2}\{H^2(A/2) + \epsilon^2H^2(\epsilon^2A/2) - 2\epsilon^2H(A/2)H(\epsilon^2A/2)\cos[(1/2)(1-\epsilon^2)A - \alpha(A/2) + \alpha(\epsilon^2A/2]\}, \tag{A6}$$

where

$$H(a) = [J_0^2(a) + J_1^2(a)]^{1/2} \tag{A7}$$

and

$$\alpha(a) = \tan^{-1}[J_1(a)/J_0(a)]. \tag{A8}$$

In Eqs. (A7) and (A8), $J_1(\cdot)$ is the first-order Bessel function of the first kind. Equation (A6) is the result given in Table 1. For circular pupils, i.e., when $\epsilon = 0$, Eq. (A6) reduces to a simple relation:

$$S = J_0^2(A/2) + J_1^2(A/2). \tag{A9}$$

Note that, as in Table 1, the units of the aberration coefficient A are radians.

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