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Principles of vibration isolation

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Abstract

Basic principles of vibration isolation are presented using a simple single-degree-of-freedom-model. This model is used to explain the design of linear spring and damper isolation systems. Governing equations for air spring isolators are reviewed. The relationship between system isolation efficiency and table or platform performance is shown. Some problems with current configurations of vibration isolation systems are analyzed.

Introduction

Vibration amplitude exceeding 100 nm can damage the performance of many types of sensitive equipment used in optics and metrology such as auto collimators and interference microscopes.[1] In 1964, the Instrument Society of America (ISA) published a recommended standard for allowable vibration in optics and metrology laboratories.[2] This standard limits accelerations to 0.01 m/s^2 and displacements to 25 nm, for frequencies below 200 Hz. Typical machinery and building vibration occurs between 10 and 100 Hz, with displacement amplitudes of less than 100 μm. Effective isolation of sensitive equipment from vibrations therefore requires an isolation system that reduces the level of input vibration by at least 1000.

Single degree of freedom model

Any system with elasticity has a fundamental frequency. The fundamental or natural frequency is that at which the system vibrates if displaced from equilibrium and allowed to freely oscillate. Anisotropic elasticity and mass in distributed systems induces complex dynamic response to vibration. Analysis of such complex systems is outside the scope of this paper. However, insight into the behavior of vibration isolation systems is possible by analysis of a simple single-degree-of-freedom (SDOF) model.

Consider a system comprising a concentrated mass, supported by a massless, linear spring, and constrained by frictionless guides to allow movement only in the vertical direction as shown in Figure 1. This SDOF system has a natural frequency given by

$$f_n = \frac{1}{2\pi} \frac{\sqrt{k}}{m} \qquad (1)$$

where
- f_n is the natural frequency in Hz
- k is the spring constant or stiffness
- m is the concentrated mass.

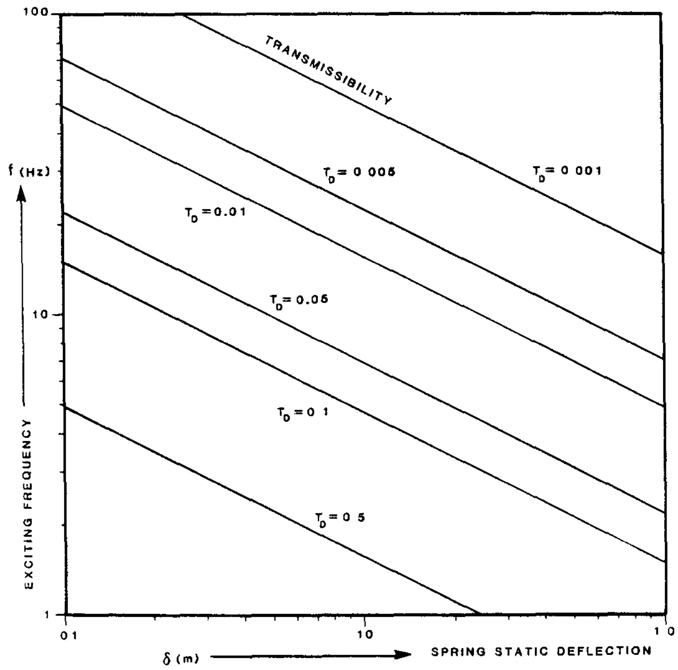

Figure 2. Transmissibility of a linear, single-degree-of-freedom system in terms of forcing frequency and static definition.

where

C_R is the critical damping ratio

C is the system damping coefficient.

When the critical damping ratio equals 1, there is no oscillation. Oscillation of the SDOF is possible when the critical damping ratio is less than 1. The damped natural frequency of a single-degree-of-freedom system is

$$f_d = f_n \sqrt{1 - C_R^2} \tag{9}$$

where

f_d is the damped natural frequency.

Normally, the critical damping ratio is below 0.1, so the damped natural frequency is within 1% of the undamped natural frequency.

The displacement transmissibility of a damped single degree-of-freedom system is given by

$$T_D = \left[\frac{1 + \left[\frac{2C_R f}{f_n}\right]^2}{\left[1 - \frac{f^2}{f_n^2}\right]^2 + \left[\frac{2C_R f}{f_n}\right]^2} \right]^{1/2} \tag{10}$$

If the exciting frequency is less than the fundamental frequency, the SDOF is spring controlled, and response is determined by spring stiffness.

If the fundamental frequency is the same as the exciting frequency, the system is at resonance and is damper controlled. Transmissibility at resonance is approximately equal to the inverse of twice the critical damping ratio. Since transmissibility at resonance is greater than 1, amplification of the exciting displacement occurs. Amplification at resonance is defined in shock and vibration literature as the Q of the system and is related to the critical damping ratio by

$$Q \cong \frac{1}{2C_R} \quad . \tag{11}$$

If the exciting frequency is greater than the fundamental frequency, the SDOF is mass controlled. When the exciting frequency is 1.414 times higher than the fundamental frequency, the transmissibility is less than 1, and the system is isolated. Assuming viscous damping, displacement response is inversely proportional to the square of the ratio of the exciting and fundamental frequencies. In this region, damping increases transmissibility as shown in Figure 3.

Directly coupled viscous damping provides a damping force that is proportional to velocity. Coulomb or friction dampers provide a damping force that is constant. Defining the Coulomb damping parameter

$$\gamma_0 = \frac{F_f}{K y_0} \tag{12}$$

where

γ_0 is the Coulomb damping parameter
F_f is the Coulomb damping force
y_0 is the base displacement amplitude
K is the system stiffness.

Finite response at resonance for a Coulomb damped system requires

$$\gamma_0 > \frac{\pi}{4} \quad .$$

Use of Coulomb damping provides control of response at resonance. However, Coulomb damping can degrade vibration isolation at high frequencies. The optimum value of Coulomb damping parameter required to limit response at resonance while still providing high frequency vibration isolation is

$$(\gamma_0)_{Optimum} = \frac{\pi}{2} \quad .$$

Displacement transmissibility of a Coulomb damped single-degree-of freedom system is given by:[3]

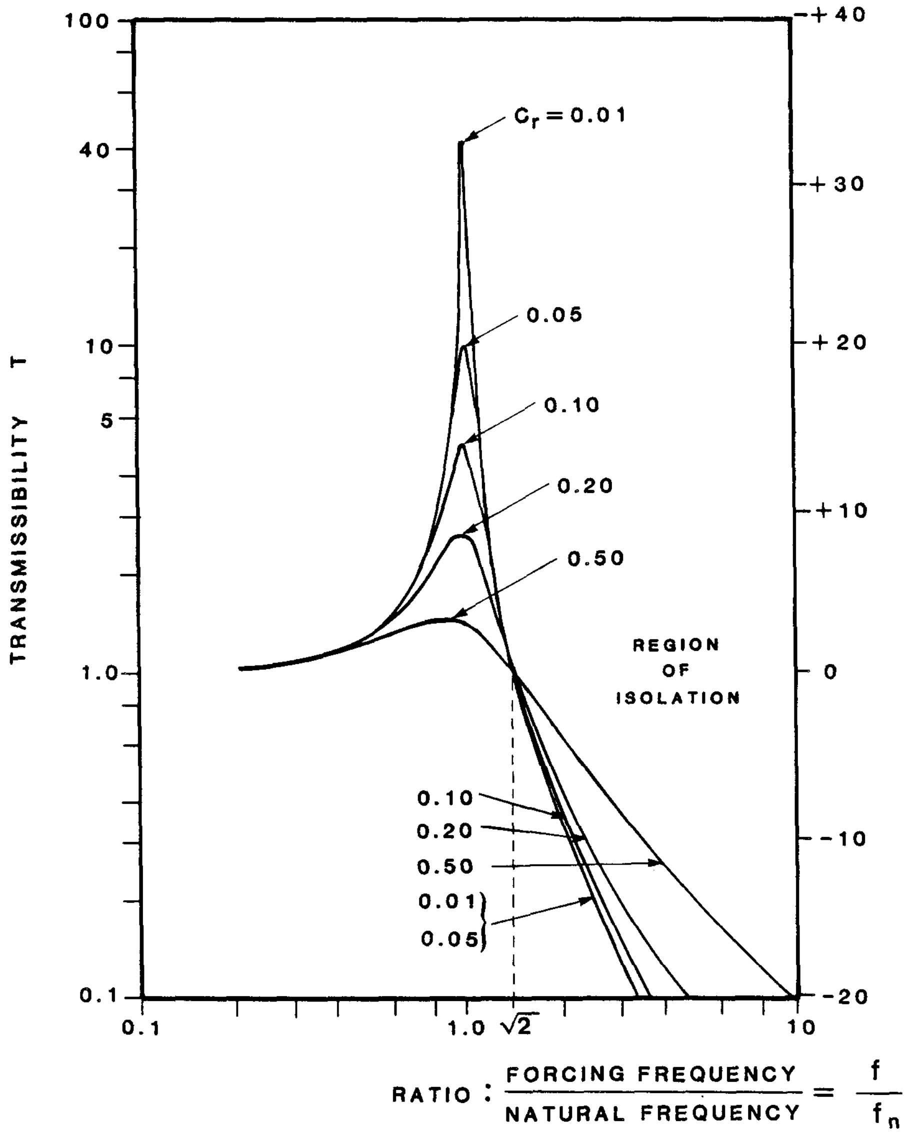

Figure 3. Transmissibility of a directly coupled viscous-damped linear spring single-degree-of-freedom system.

$$T_D = \left\{ \frac{1 + \left[\frac{4}{\pi}\gamma_0\right]^2 B^2 \left[1 - 2\left[\frac{f_n}{f}\right]^2\right]}{\left[1 - \left[\frac{f}{f_n}\right]^2\right]^2} \right\}^{1/2} \tag{13}$$

where the Coulomb frequency ratio parameter B is given by

$$B = \frac{\pi}{4} \left| \frac{1 - \left[\frac{f}{f_n}\right]^2}{\frac{f}{f_n}} \right| \tan \left[\frac{\pi}{2} \frac{f_n}{f}\right] . \qquad (14)$$

Displacement transmissibility in a Coulomb damped system decreases by a factor of 100 for an increase of exciting frequency by a factor of 10 However, acceleration transmissibility approaches a limiting constant value at high frequency. Acceleration transmissibility is the limiting factor in the performance of Coulomb damped vibration isolation systems.

Multiple-degree-of-freedom model

An elastically supported rigid body has one to six natural modes of vibration. These modes will depend on the geometry and stiffness of the elastic supports. There are three translational modes along three mutually perpendicular axes, and three rotational modes with respect to the same three axes. If the elastic supports or isolators are located unsymmetrically with respect to the body's center of gravity, translational and rotational modes may be coupled.

Vibration isolation tables or platforms consist of a rectangular solid platform with isolators located at the bottom corners. The center of gravity of the isolation system is normally above the bottom. A horizontal displacement of the platform will elastically deform the isolators in the horizontal plane. The isolators will provide both horizontal and vertical restoring forces. Because the isolators are below the center of gravity, the restoring forces and inertial forces induce a moment in the platform. This moment causes rotation of the platform about the horizontal axis.

The uncoupled natural frequency of a translational mode for a base supported vibration isolation system is given by

$$f_{nt} = \frac{1}{2\pi} \left[\frac{K_t}{m}\right]^{1/2} \qquad (15)$$

where
f_{nt} is the natural frequency (Hz)
K_t is the total linear stiffness is the specified direction
m is the seismic mass of the system.

The total stiffness is given by

$$K_t = \sum_{i=1}^{n} k_{it} \qquad (16)$$

where:
k_{it} is the stiffness of each isolator
n is the number of isolators.

The uncoupled natural frequency of a rotational mode for a base supported isolation system is given by

$$f_{nr} = \frac{1}{2\pi}\left[\frac{K_r}{I_r}\right]^{1/2} \qquad (17)$$

where

f_{nr} is the natural frequency
K_r is the angular stiffness about the specified axis
I_r is the system mass moment of inertia of the seismic mass about the specified axis.

The angular stiffness is given by:

$$K_r = \sum_{i=1}^{n} a_i^2 k_{ir} \qquad (18)$$

where:

K_{ir} is the stiffness of each isolator
a_i is the distance from the system center of gravity to each isolator
n is the number of isolators.

The coupled natural frequencies of a four-isolator base type vibration isolation system with two planes of symmetry can be determined by a method developed by Crede[4] and extended by Macinante.[5] This method assumes that the isolated platform is a rectangular solid with four isolators supporting it. Each isolator is located near one of the corners, and is assumed to respond linearly. See Figure 4.

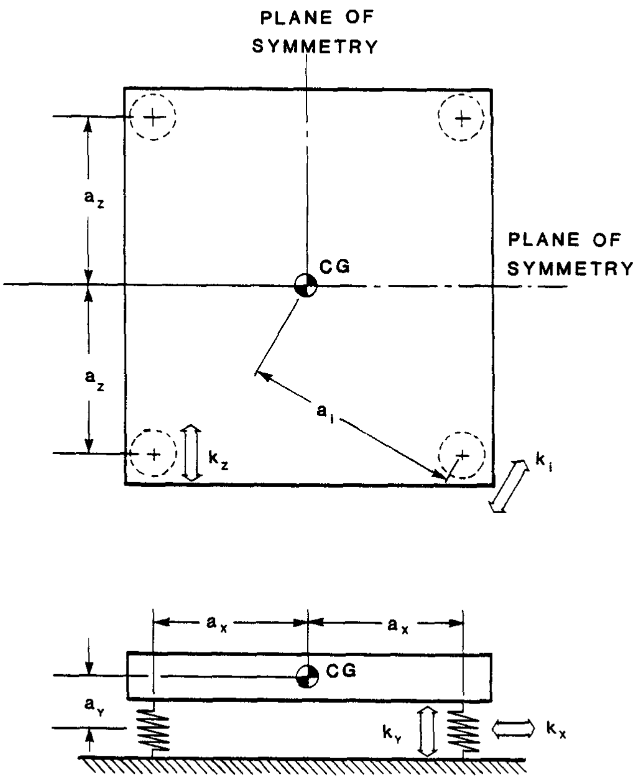

Figure 4. Four-isolator base type vibration isolation system with two planes of symmetry.

Determination of the coupled natural frequency about the horizontal axis requires that the uncoupled vertical and horizontal stiffnesses, isolator locations, and radius of gyration of the platform be known. The distance between the point of applied load that does not induce a coupled rotation of the system (defined as the elastic plane) and the center of gravity is the vertical isolator distance. If the height of the isolator is small in comparison with the overall height of the center of gravity, this distance can be measured to the mid-height of the isolators.

The following relationship determines the coupled natural frequency:

$$\left[\frac{f_c}{f_y}\right]\left[\frac{P_z}{a_x}\right] = \frac{1}{\sqrt{2}}\left[\left[\frac{k_x}{k_y}\right]\frac{P_z^2}{a_x^2}\left[1+\frac{a_y^2}{P_z^2}\right] + 1 \pm \left\{\left[\left[\frac{k_x}{k_y}\right]\frac{P_z^2}{a_x^2}\left[1+\frac{a_y^2}{P_z^2}\right]+1\right]^2 - 4\left[\frac{k_x}{k_y}\right]\frac{P_z^2}{a_x^2}\right\}^{1/2}\right]^{1/2} \qquad (19)$$

where

f_c is the coupled natural frequency of the system about the horizontal axis
f_y is the uncoupled vertical natural frequency
a_x is the horizontal distance from the center of gravity to the center of the isolators
a_y is the vertical distance between the center of gravity and horizontal elastic plane of the isolators
P_z is the radius of gyration about the horizontal axis
k_x is the horizontal isolator stiffness
k_y is the vertical isolator stiffness

This relationship is illustrated in Figure 5.

Equation (9) can be used with the equations developed for total and angular stiffness to find the coupled modes about any axis. Further insight into coupled modes can be gained by considering the special case of a rectangular solid. For a uniform mass distribution rectangular solid, on a four-isolator base type vibration isolation system with two planes of symmetry, Equation (9) becomes:

$$\frac{f_c}{f_y} = \frac{1}{\sqrt{2}}\left\{\frac{4e\left[\frac{h}{b}\right]^b + e + 3}{\left[\frac{h}{b}\right]^2 + 1} \pm \left[\left[\frac{4e\left[\frac{h}{b}\right]^b + e + 3}{\left[\frac{h}{b}\right]^2 + 1}\right]^2 - \frac{12e}{1 + \left[\frac{n}{b}\right]^2}\right]^{1/2}\right\}^{1/2} \qquad (20)$$

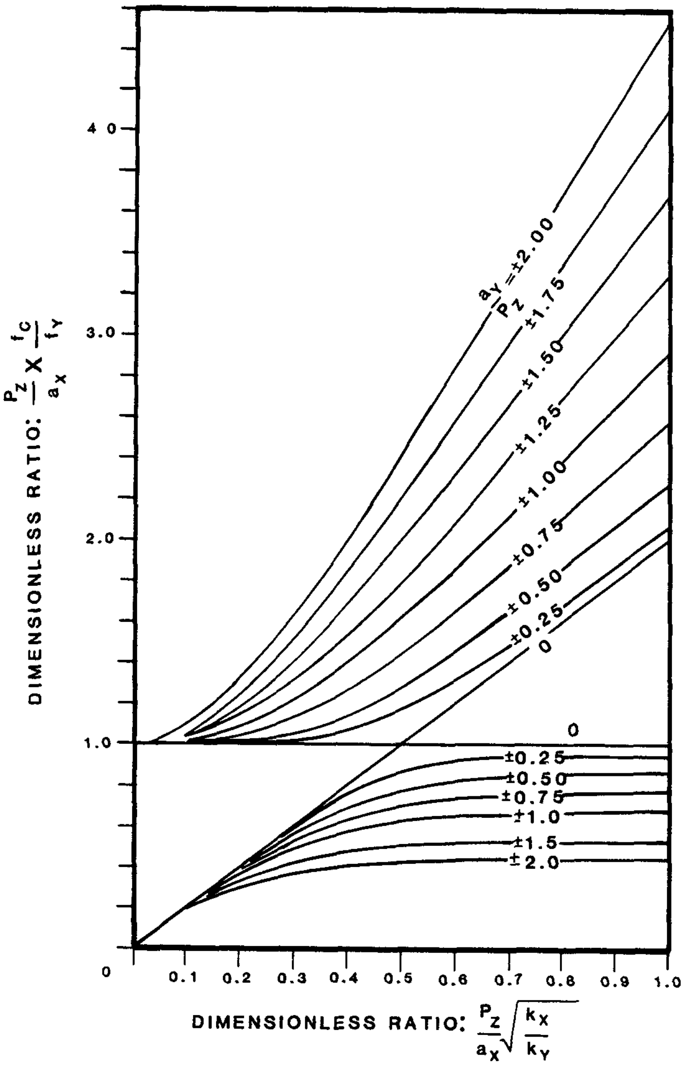

Figure 5. Coupled natural frequencies of a four-isolator base type vibration isolation system with two horizonatal planes of symmetry.

where

f_c is the coupled natural frequency of the system about the horizontal axis

f_y is the vertical natural frequency

e is the stiffness ratio, $e = \frac{k_y}{k_y}$

h/b is the ratio of height to width of the isolated solid.

This relationship is illustrated in Figure 6. The coupled natural frequencies are minimal when the ratio of horizontal to vertical stiffness is low. When vibration isolators are base mounted, low natural frequencies are obtained by using isolators whose horizontal stiffness is less than their vertical stiffness.

Increasing the height of the mounted body decreases the coupled frequency for the lower set of curves. For the upper set of curves in Figure 6, the coupled natural frequency becomes high when the height of the body exceeds the width, and when the horizontal stiffness exceeds the vertical stiffness of the isolators.

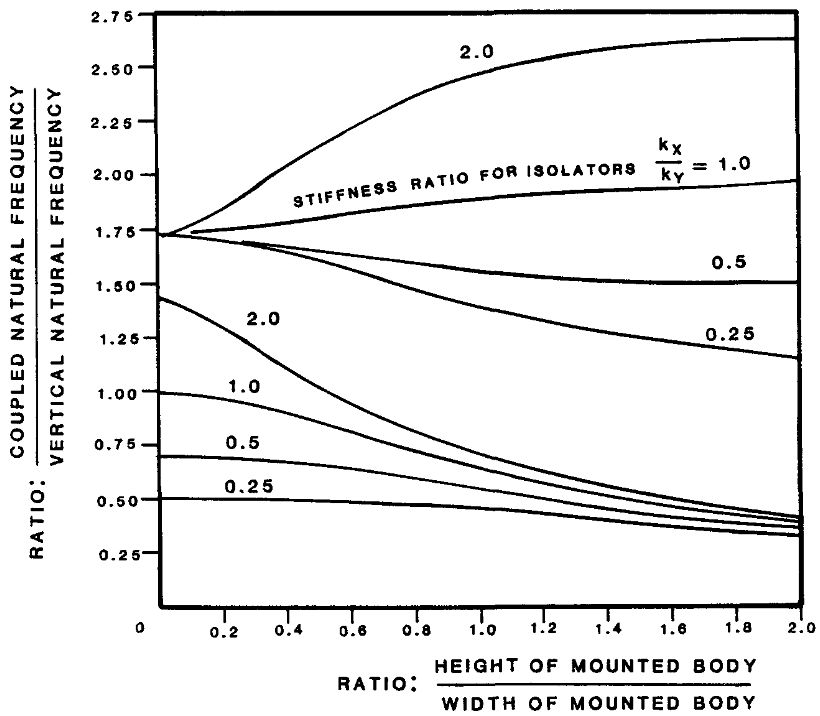

Figure 6. Coupled natural frequencies of a four-isolator base type vibration isolation system with two horizontal planes of symmetry and uniform distribution of platform mass.

Coupled modes are eliminated when the elastic plane passes through the center of gravity of the system. This requires raising the isolators by attachment to brackets on the sides of the platform, or lowering the center of gravity by adding mass to the bottom of the platform. If the principal (axial) elastic axes of the individual isolators pass through the center of gravity, coupled modes are eliminated. This is done by inclining the isolators.

Isolator design

Helical compression springs are widely used as isolators. Axial stiffness of helical springs is determined using standard equations from any spring design handbook. Determining the transverse stiffness of helical compression springs is more difficult; however, a method developed by Crede can be used. The ratio of the axial spring stiffness to transverse or lateral spring stiffness is given by

$$\frac{k_y}{k_x} = 1.44\,\phi\left[0.256 + 0.204\left[\frac{h_s}{d}\right]^2\right] \quad (21)$$

where

k_y is the axial spring stiffness
k_x is the transverse spring stiffness
h_s is the working height of the spring
d is the mean coil diameter of the spring
ϕ is the Rausch correction factor.

where

- b is the width of the passage connecting the surge tank to the cylinder
- L is the length of the passage connecting the surge tank to the cylinder
- t is the thickness of the passage connecting the surge tank to the cylinder
- μ is the dynamic viscosity of the gas
- C is the effective damping
- C_R is the critical damping.

For the lowest possible amplification at resonance, an optimum value of damping must be selected. This optimum damping is given in terms of the ratio of the cylinder to surge tank volume

$$\left[\frac{C}{C_R}\right] = \left[\frac{(N + 1)(N + 2)}{8N}\right]^{1/2} . \qquad (27)$$

Transmissibility at resonance of a optimum damped air spring system is given by

$$T_C = \frac{2}{N} + 1 \qquad (28)$$

where

T_C is the transmissibility at resonance.

As the exciting frequency increases, there is less time for air to flow from the cylinder to the surge tank. At some high frequency, air from the cylinder will not reach the surge tank during one cycle, and the surge tank will become isolated from the cylinder. At higher frequencies, the air spring response is undamped. The transmissibility for this case is given by

$$T_P = \frac{1}{\left[\frac{\omega^2}{N\omega_n^2}\right]} \qquad (29)$$

where

T_P is the high frequency transmissibility.

In contrast, the high frequency transmissibility of a linear spring and damper system is given by

$$T = 2\left[\frac{C}{C_R}\right]\left[\frac{\omega_n}{\omega}\right] . \qquad (30)$$

Good isolation of high frequency vibration is possible for air spring systems, while amplification at resonance is limited. Addition of damping to control response at resonance in linear spring isolation systems degrades performance. Low natural frequencies are readily attained in air spring systems without excessive static deflection.

Current air spring isolators employ rolling diaphragm seals between the piston and the cylinder to reduce friction. The surge tank is normally built into the base of the isolator. Both cylinder and surge tank are insulated to ensure isothermal compression of the air. Isothermal compression reduces the ratios of specific heats of air from 1.4 to 1.0, which reduces stiffness and system natural frequency as shown in Figure 8.

Changes in pressure and temperature can alter the response of an air spring isolator. Low vertical stiffness results in considerable variation in platform height. Servo controls of the isolator is used to control variations in performance and to maintain platform height. Design of active servo controls is a complex subject, and is outside the scope of this paper.

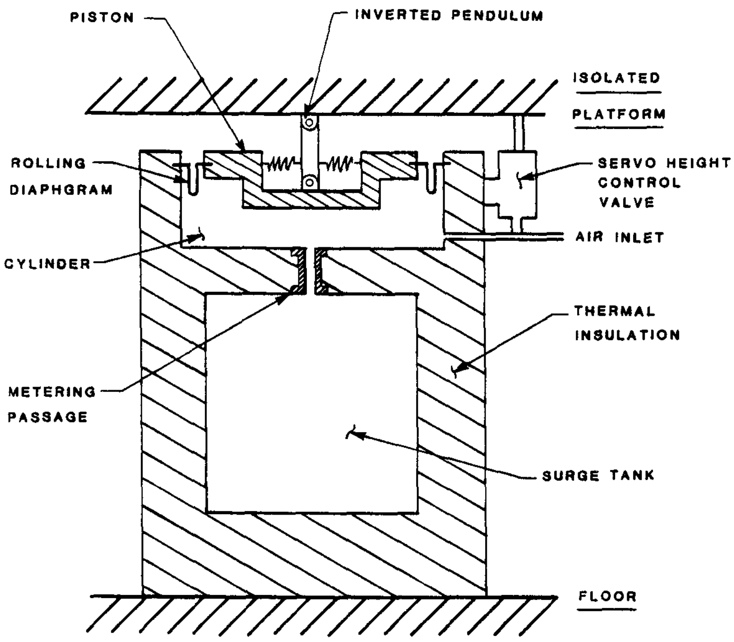

Figure 8. Schematic air spring isolator.

Although air springs provide good vertical vibration isolation, a separate isolation mechanism is required for horizontal vibration isolation. In current designs, some form of "inverted pendulum" is employed to provide horizontal isolation. Natural frequencies as low as 0.01 Hz have been attained with inverted pendulums. The natural frequency of an inverted pendulum is given by

$$f_n = \frac{1}{2\pi}\left[\frac{ka^2}{mL^2} - \frac{g}{L}\right]^{1/2} \qquad (31)$$

where:

- f_n is the natural frequency
- L is the pendulum length
- m is the pendulum mass
- k is the spring stiffness
- a is the distance between the spring and the center of mass of the pendulum
- g is the acceleration due to gravity.

Platform design

Overall effectiveness of a vibration isolation system is determined by isolator performance and platform characteristics. To avoid coupled vibration between the platform and isolator, the octave rule requires that the natural frequency of the platform be at least twice as high as the natural frequency of the isolation system. This establishes the lower limit on platform stiffness.

Maximum deflection of a uniformly loaded rectangular plate supported at its corners is at the center and is given approximately by

$$\delta \cong \frac{1}{144} e^{1.267\left\{\frac{b}{\ell}\right\}} \frac{\rho_0}{D} \ell^4 \qquad (32)$$

where:

δ is the maximum deflection
ℓ is the plate length
b is the plate width
ρ_0 is the area density of the plate
D is the flexural rigidity of the plate.

For a solid plate, the flexural rigidity is given by

$$D_{solid} = \frac{E\ h^3}{12(1 - \nu^2)} \qquad (33)$$

where

E is the elastic modulus of the plate material
h is the plate thickness
ν is Poisson's ratio for the plate material.

For a sandwich plate, the flexural rigidity is given approximately by

$$D_{sand} = \frac{E\ t_1\ t_2\ h_c^2}{(t_1 + t_2)(1-\nu^2)} \qquad (34)$$

where

h_c is the centroid distance from top to bottom facings
t_1 is the thickness of the upper facing
t_2 is the thickness of the lower facing.

Combining Equations (32) and (33), the self-weight induced deflection of a rectangular solid platform is given by

$$\delta_{solid} \cong \frac{1}{12} \frac{\rho}{E} \frac{\ell^4}{h^2} (1 - \nu^2)\ e^{1.297(b/\ell)} \qquad (35)$$

where

ρ is the material density.

The specific stiffness, the ratio of density to elastic modulus, governs deflection. Most common structural materials have about the same specific stiffness. To reduce deflection, the table must be made thicker, with resulting weight increase.

An alternative to making the platform thicker is sandwich construction. A sandwich panel consists of top and bottom facings attached to a low density shear core. Current commercial sandwich panels employed as vibration isolation system platforms use aluminum or steel facings with a cellular aluminum or steel core. For a platform thickness of 0.3 m, current sandwich platforms have an area density of about 120 kg/m². The same thickness for a solid steel table would be 20 times heavier. The sandwich panel has about 7 times the deflection of a solid steel platform. Assuming equal weight, a sandwich platform has less self-weight induced deflection than a solid platform.

Disturbances on the working surface of the platform must be rapidly damped. Logarithmic decrement is the natural logarithm of the ratio of the amplitudes of two successive cycles of damped free vibration. Logarithmic decrement is related to the loss coefficient by

$$\Lambda \cong \pi \eta_s \tag{36}$$

where
η_s is the material loss coefficient
Λ is the logarithmic decrement.

Table 1 lists loss coefficients and other relevant material properties from some platform materials. Materials with high loss coefficients tend to have poor stiffness. As a rough rule of thumb, stiff structures have low loss coefficients and are poorly damped. Use of concrete or granite improves damping but imposes a severe weight penalty.

Damping of stiff sandwich structures is improved by adding damping materials to the structure. In Table 1, adding loose sand to an aluminum beam increases the loss coefficient by a factor of 100. More sophisticated damping materials are available and can be bonded to the facings of a sandwich platform. Another approach is to fill the cells of the shear core with damping material. Considerable ingenuity has been applied to damping in commercial tables.

Temperature changes can degrade the performance of the platform of a vibration isolation system. An isothermal temperature change alters platform size without changing geometry. The change is size is given by

$$\Delta x = \alpha x \, \Delta\theta \tag{37}$$

where:
Δx is the change in size
α is the thermal coefficient of expansion of the material
$\Delta\theta$ is the change in temperature
x is the platform dimension.

Low thermal coefficients of expansion materials are used to control dimensional changes with temperature in vibration isolation system platforms. Invar, a nickel-iron alloy, is the classic material for this purpose. Unfortunately, Invar is expensive, difficult to machine, and has poor specific stiffness.

Distortion of platform geometry can occur because of temperature gradients. A linear gradient from top to bottom distorts the platform into a spherical shape. The radius of curvature of this distorted shape is given by

$$R = \frac{h}{\alpha \, \Delta\theta} \tag{38}$$

where:
R is the radius of curvature
h is the platform thickness
$\Delta\theta$ is the temperature gradient
α is the thermal coefficient of expansion.

Table 1: Platform Materials Properties

Material	Loss Coefficient η_s	Density ρ (kg/m^3)	Elastic Modulus E (GPa)	Thermal Expansion α (10^{-6}/K)	Specific Stiffness E/ρ (m$\times 10^6$)	α/K (m/w$\times 10^{-9}$)	Diffusivity D (m^2/s$\times 10^{-6}$)
1. Concrete	0.024	2.5	21	11	0.86	12200	0.54
2. Granite	0.0005-0.01	2.8	48	7	1.7	2000	1.5
3. Cast Iron	0.006-0.01	6.9	105	11	1.6	229	12.0
4. Carbon Steel	0.0009-0.0014	8.0	200	15	2.5	319	14.0
5. Stainless Steel (18% Cr, 8% Ni)	0.002-0.015	8.0	193	17	2.4	816	4.0
6. Invar	- -	8.0	145	2.4	1.8	39	3.4
7. Aluminum (6061-T6)	5×10^{-4} -0.005	2.7	69	23	2.6	135	66.0
8. Aluminum Beam							
-Alone	0.002						
-50% Weight, Loose sand	0.08-0.2						
-100% Weight Loose sand	0.2-0.9						

If the platform is exposed to a steady heat flux from top to bottom, a temperature gradient will develop. This gradient will bow the table into a spherical shape of radius

$$R = \frac{\kappa}{\alpha} q \tag{39}$$

where

q is the heat flux absorbed per unit area of the platform

κ is the thermal conductivity of the platform material.

The time required for the platform to come into thermal equilibrium with its surroundings is the thermal time constant. The thermal time constant is given by:

$$\tau = \frac{h^2}{2D} \tag{40}$$

where

τ is the thermal time constant

h is the platform thickness

D is the platform material thermal diffusivity, given by:

$$D = \frac{\kappa}{\rho C_\rho} , \tag{41}$$

C_ρ is the platform specific heat capacity.

A 0.3-m thick concrete platform has a thermal time constant of about 24 hours. Sandwich platforms have poor thermal diffusivity because of low core density. Thermal conductivity of the core is reduced by the ratio of the density of the core to that of the solid material.

No ideal platform material exists. In addition to the physical parameters, cost is an important consideration. Commercial platform designs are normally of the sandwich type. Very large vibration isolation systems often use concrete platforms as an economy measure.

Discussion

Current commercial vibration isolation systems use air spring for vertical isolation and inverted pendulums for horizontal isolation. Typical vertical natural frequencies are about 2 ηz, and horizontal natural frequencies are typically about 1 Hz. Platforms are normally of metal sandwich construction, with a cellular core and additional damping materials.

Traditionally, precision laboratories have been located on the ground floor of buildings (or underground). High construction costs are forcing use of upper floors of buildings for laboratories. Suspended floors in the upper stories of buildings present a much more serious vibration environment than traditional locations. Very low frequency horizontal vibration, of 0.1 to 5 Hz, caused by building side sway, is typical of these environments.

Effective isolation of very low frequency vibration caused by horizontal building motion requires an isolation system with a horizontal natural frequency of 0.01 Hz. An inverted pendulum isolator with this natural frequency is very sensitive to local tilt of the building floor, and changes in load. No commercial isolator with this performance is available.

Interferometry is becoming an increasingly important tool not only in the laboratory but in production facilities as well. Interferometers require high efficiency vibration systems. The challenge is to supply effective systems at a reasonable price.

Many vibration isolation table applications require connections between the platform and the laboratory facilities. The stiffness of a compressed air line, laser water cooling hose, or power cable can compromise the performance of a vibration isolation system. Very little information is available on low stiffness connections, and no commercial products for this purpose exist.

New technologies offer possible solutions to the vibration isolation problem. Active vibration isolation systems may increase efficiency. Active systems may also allow cascading of systems for critical applications. Cascading is not possible with today's vibration isolation platforms because of coupled vibration between systems.

Use of new composite materials, such as the graphite-epoxy composites, offers potential improvement of platform development. Very lightweight, high stiffness, low thermal coefficients of expansion platform materials are possible using composite material technology. Current composites are very expensive and are dimensionally unstable.

Although the principles of vibration isolation have been understood and used for over a century, the vibration isolation platform industry is only about two decades old. At least one vibration isolation company has joined the "Fortune 500," which is an indication of the importance of this technology. Current isolation platform technology is well developed, but future developments through the use of new materials and new technology are possible.

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