Tutorial of Hertzian Contact Stress Analysis

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ABSTRACT

Stresses formed by the contact of two radii can cause extremely high surface stresses. The application of Hertzian Contact stress equations can estimate maximum stresses produced. These stresses can then be analyzed in context of the application. In many cases, the resultant stresses are not of design significance, but in some cases failure can occur. Ball bearings or kinematic mounts that are repeatedly removed and remounted can start to show damage from fatigue caused by the Hertzian stresses. The Hertzian contact stress equations come in two forms, spherical and cylindrical. The details and applications of these equations will be explained.

INTRODUCTION

Any time there is a radius in contact with another radius or flat, contact stresses will occur. In the case of two spheres contacting each other, the entire force will be imparted into a theoretical point. Due to elastic properties of the materials this point will deform to a contact area. The deformation that occurs will produce high tensile and compressive stresses in the materials. Even if a singular loading does not produce a failure, it can lead to future fatigue or surface damage.

Cai and Burge have shown that for the case of a glass lens mounted against a sharp corner with calculated contact tensile stresses approaching 20Ksi no failure was observed. 20Ksi is far beyond the practical 1Ksi tensile stress design limit. Due to the fact the depth of the tensile stresses did not exceed 4 μm, a critical flaw was not created.[i] This is good news for using sharp edge contacts to mount glass lenses. Though the stress created did not damage the lens in this testing, the stress is still present. For systems that need to survive extreme environments or have no room for failure would need to carefully examine risks and benefits of allowing these high contact stresses.

The stress field created by the contact stresses was first introduced by Hendrick Hertz in 1881. His equations work well using a computer spreadsheet format, allowing a plot of the stress field to be quickly created. These equations assume the system has no friction are elastic, isotropic, homogeneous and do not account for surface roughness.[ii] If a frictionless assumption does not apply with the system being analyzed the classical Hertzian equations cannot be used. The Hertzian equations will provide lower shear stresses than actually exist. The Smith-Liu equations must be used in this case. These equations will not be covered in this paper but are worth investigating for contact stresses of a rolling contact or a shear surface load.[iii]

Once the stresses have been calculated they must be analyzed for fatigue or implications in stress flaw propagation. Often the Hertzian stresses calculated with be overly conservative and will call for a design change when not necessary as shown by experimentation. In many cases it is best to err on the side of caution and make a simple change to reduce the contact stresses. This may be as simple as increasing a contact radius or reducing the forces in the joint. Ease of manufacture, ease of assembly and system performance must be weighted against potential for system failure. Since contact stress failures consist mostly of surface damage, evaluations must be made on the allowable surface indentation or surface fatigue. [ii]

SPHERICAL CONTACT STRESSES

For spherical contacts like a ball and socket or ball on a flat plate the pressure distribution is circular and extends out as shown in the hatched region of figure 1. The size of this region depends on the elastic properties and geometries of the parts in contact. Figure 2 gives the equation for calculating the radius of the contact area produced by the deformation of the two spheres from force F. Where E1, v1, R1 is the elastic modulus, Poisson's Ratio and radius respectively of sphere 1. The same is true respectively for sphere 2.

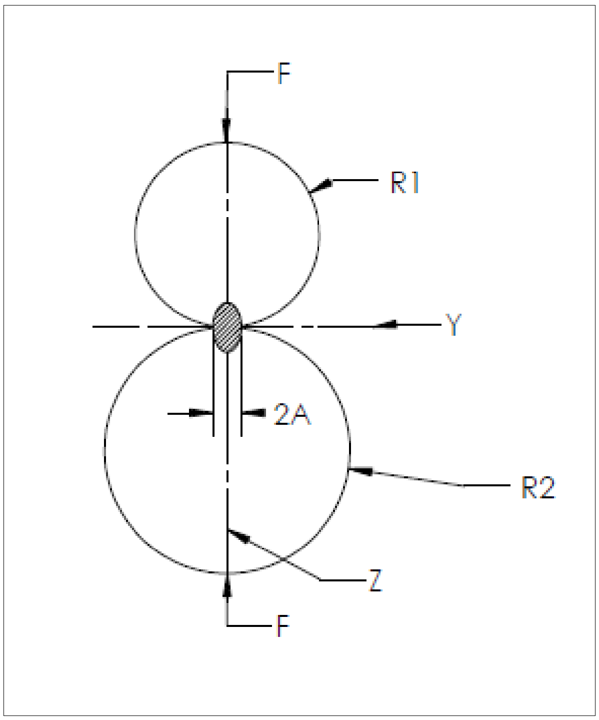

Figure 1: Contact stress between two spheres. (Cross hatched area is the pressure zone)

$$a = \sqrt[3]{\frac{3\cdot F}{8}\cdot\frac{\frac{1-\nu 1^2}{E1}+\frac{1-\nu 2^2}{E2}}{\frac{1}{2\cdot R1}+\frac{1}{2R2}}}$$

Figure 2: Equation for the radius of the contact area of two spheres[iv]

The maximum pressure within the contact area occurs as a compression in the center. For the equation in figure 2 use R=∞ for a sphere against a flat surface. For an internal radius like a ball and socket joint a negative radius would be used for the socket. Figure 3 shows the equation for calculating this pressure. Knowing the maximum pressure then allows you to calculate out the principle stresses along the z axis. Figure 4 shows the equations for the principle/normal stresses and the maximum shear stress with in the contact region. Note: be sure to use the Poisson's ratio for the side of the contact you are interested in.

$$Pmax = \frac{3\cdot F}{2\cdot\pi\cdot a^2}$$

Figure 3: Maximum pressure within the contact area. [iv]

$$\sigma 1 = \sigma 2 = \sigma x = \sigma y = -Pmax\left[\left(1-\left|\frac{z}{a}\right|\cdot\operatorname{atan}\left(\frac{1}{\left|\frac{z}{a}\right|}\right)\right)\cdot(1+\nu)-\frac{1}{2\cdot\left(1+\frac{z^2}{a^2}\right)}\right]$$

$$\sigma 3 = \sigma z = \frac{-Pmax}{1+\frac{z^2}{a^2}}$$

$$\tau max = \tau 13 = \tau 23 = \frac{\sigma 1-\sigma 3}{2} = \frac{\sigma 2-\sigma 3}{2}$$

Figure 4: Principle stresses along the Z axis. [iv]

With the principle stresses and the shear stresses known, evaluations must made. The first evaluation should be to compare the maximum stresses to the yield or shear strength of the material. Permanent plastic deformation will occur if these values are approached. If the contact stresses are in a precision alignment system it may be wiser to use the micro yield and micro shear strengths. Fatigue due to multiple mounting cycles or cyclic operation should be evaluated.

Appendix 1 shows an example of a Hertzian stress analysis of a micrometer ball pushing against an aluminum plate. Here the aluminum plate being R2 and having no radius was set to infinity. The depth of maximum shear stress was calculated only .48a = 0.06mm below the surface. The calculated stresses far exceed anything 6061 can withstand. This leads to the prediction that the material will yield affecting the precision of the 10 μm micrometer. This result should lend well to anyone who has setup a quick prototype fixture with a micrometer pushing on aluminum. You will get a surface fatigue and indentation in the aluminum if your loads are to high.

CYLINDRICAL CONTACT STRESSES

Cylindrical contacts stresses undergo the same procedure as spherical contact stresses with the addition of a length turning the contact area into an ellipse. Figure 5 shows the calculation of the half width of the contact area. In the case of cylindrical contact we call this half with "b" instead of "a" as in the spherical equations.

The extra length component allows for a larger contact area reducing the resultant stresses. Note any time you have a sharp edge pushing against a flat or radius, the cylindrical contact stress equations can be used. In the example Cai and Burge finite element analysis was used to evaluate a round tube ground to a sharp edge pressed against a flat plate of glass. To use the cylindrical contact stress equations the best fit radius of the sharp edge, and use the circumference of the tube for length would be used for "R1" and "l" respectively.

$$b = \sqrt{\frac{2 \cdot F}{\pi \cdot l} \cdot \frac{\frac{\left(1 - \nu 1^2\right)}{E1} + \frac{\left(1 - \nu 2^2\right)}{E2}}{\frac{1}{d1} + \frac{1}{d2}}}$$

Figure 5: Equation for the half width of the contact area of two cylinders. [iv]

$$Pmax = \frac{2 \cdot F}{\pi \cdot b \cdot l}$$

Figure 6: Maximum pressure within the contact area. [iv]

Unlike the spherical contact equations the principle stresses do not always equal the normal stress components. There is a distance below the surface were the principle stresses reverse. 0.436b below the surface is the switching point.

Figure 7 below shows the equations for calculating all the principle and normal stresses of the material a distance “z” away from the surface. These equations lend themselves well to a spreadsheet to plot out the stress distribution.

$$\sigma x = -2 \cdot \nu \cdot Pmax \cdot \left(\sqrt{1 + \frac{z^2}{b^2}} - \left| \frac{z}{b} \right| \right)$$

$$\sigma y = -Pmax \cdot \left(\frac{1 + 2 \cdot \frac{z^2}{b^2}}{\sqrt{1 + \frac{z^2}{b^2}}} - 2 \cdot \left| \frac{z}{b} \right| \right)$$

$$\sigma 3 = \sigma z = \frac{-Pmax}{\sqrt{1 + \frac{z^2}{b^2}}}$$

$$\text{For} \quad 0 \le z \le 0.436b \quad \sigma 1 = \sigma x \quad \tau max = \frac{(\sigma 1 - \sigma 3)}{2} = \frac{(\sigma x - \sigma z)}{2}$$

$$\text{For} \quad z \ge 0.436b \quad \sigma 1 = \sigma y \quad \tau max = \frac{\sigma 1 - \sigma 3}{2} = \frac{\sigma y - \sigma z}{2}$$

Figure 7: Principle stresses along the Z axis. [iv]

Analysis of results

The equations above lend themselves well to sanity checks of finite element analysis. With a full understanding of the contact stresses present in the system, an analysis must be made of the failure modes. The first thing to look at is the maximum compressive stress versus the compressive strength, of the materials. If the maximum compressive stress exceeds the compressive strength plastic deformation of the part will occur. The basic Hertzian equations focus mainly on compressive stresses. If the Smith-Liu equations were used or finite element results are available, the resultant tensile stresses should be evaluated against the material yield strength. Finally, the shear stress should be evaluated in respect to the shear strength of the material. High shear stresses is believed to contribute heavily to subsurface fatigue crack initiation. [ii] Though the stresses calculated dissipate quickly though the material, surface stresses are high. High surfaces stresses can cause wear and surface fatigue. In many cases this surface damage would be inconsequential and not of concern for the system. Or in the case of a micro-positioner surface damage could lead to inaccurate positioning.

Summary

Stresses due to contact of spherical or cylindrical components can have extremely large magnitudes. The depth of the stress fields tends to be very shallow though. The low depth of the stresses tends to lead to purely surface damage. Cai and Burge have shown that even in the case of glass that fails from the propagation of defects, total structural failure is improbable. Surface damage or surface deformation can be of concern in precision alignment systems. The Hertzian contact equations can provide an analysis of the stress fields present. Though it has been shown through experimentation the Hertzian equations maybe on the conservative side but in many cases the stresses can be mitigated with simple design changes.

APPENDIX 1
Example of Hertzian Contact Analysis

Newport SM-25 Vernier Micrometer Contact Stress
Hardened steel ball bearing pushing against 6061-T6 flat plate at Micrometers 100N load capacity

$GPa := 1000MPa$

Micrometer
$F := 100N$
$\nu 1 := .3$
$E1 := 203GPa$
$R1 := 1.5mm$

6061 – T6
$\nu 2 := .33$
$E2 := 68.9GPa$
$R2 := \infty mm$

http://assets.newport.com/web900w-EN/images/8409.jpg

$\nu := .33$

$$a := \sqrt[3]{\frac{3 \cdot F}{8} \cdot \frac{\frac{1-\nu 1^2}{E1} + \frac{1-\nu 2^2}{E2}}{\frac{1}{2 \cdot R1} + \frac{1}{2R2}}} \qquad a = 0.125\,mm$$

$$Pmax := \frac{3 \cdot F}{2 \cdot \pi \cdot a^2} \qquad z := .48 \cdot a$$

$$\sigma 1 := -Pmax \cdot \left[\left(1 - \left|\frac{z}{a}\right| \cdot atan\left(\frac{1}{\left|\frac{z}{a}\right|}\right)\right) \cdot (1+\nu) - \frac{1}{2 \cdot \left(1 + \frac{z^2}{a^2}\right)}\right] \qquad \sigma 1 = -629.778\,MPa$$

$$\sigma 3 := \frac{-Pmax}{1 + \frac{z^2}{a^2}} \qquad \sigma 3 = -2.478 \times 10^3\,MPa$$

$$\tau max := \frac{\sigma 1 - \sigma 3}{2} \qquad \tau max = 924.282\,MPa$$

Extremely High principle stress and shear stress will cause indentation in the aluminum. This will effect the repeatability of the micrometer. Aluminum would need to be changed to a harder material or the micrometer force will need to be limited.

i W. Cai, B. Cuerden, R.E. Parks, J. Burge, "Strength of Glass From Hertzian Line Contact," Proc. SPIE **8125**,(2011)
ii R. C. Juvinall, K. M. Marshek *Fundamentals of Machine Component Design, 2nd Ed.,* pp 322-332, John Wiley & Sons, (1991).
iii J.O. Smith, Chang Keng Liu, "Stresses Due to Tangential and Normal Loads on an Elastic Solid with Application to Some Contact Stress Problems." *Journal of Applied Mechanics,* June 1953
iv J.E. Shigley, C.R. Mischke, R.G. Budynas *Mechanical Engineering Design, 7th Ed.* pp 161-166, McGraw Hill, (2004).