Tutorial Report on Hertzian Contact Stress Theory

Jacob Etter
OPTI 521, University of Arizona, College of Optical Sciences, Tucson, AZ

Introduction

High stresses are induced when a load is applied to two elastic solids in contact. This can cause serious issues in the optomechanical design if not considered and addressed properly. Hertz developed a theory to calculate the contact area and pressure between the two surfaces and predict the resulting compression and stress induced in the objects. This tutorial addresses the basics of Hertzian contact stress theory and it relates to certain aspects of optomechanical design.

Background [1]

Contact between two continuous, non-conforming solids is initially a point or line. Under the action of a load the solids deform and a contact area is formed as shown in Figure 1. Hertz contact stress theory allows for the prediction of the resulting contact area, contact pressure, compression of the bodies, and the induced stress in the bodies. In 1880 Heinrich Hertz developed his theory for contact stress after studying Newton's rings with two glass lenses. He became concerned about the effect of contact pressure between the two lenses and set out to analyze the effects. The result was the first satisfactory theory for contact mechanics and is still in use today.

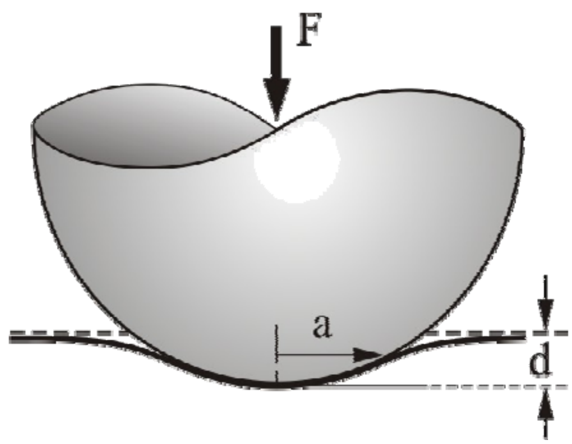

Figure 1. Depiction of contact area under applied load[2].

In the course of developing his theory Hertz made some simplifying assumptions which are summarized as follows:

a) Surfaces are continuous and non-conforming (i.e. initial contact is a point or a line)

b) Strains are small

c) Solids are elastic

d) Surfaces are frictionless

With the exception of d), these assumptions imply that $a \ll R$ where a is the contact radius and R is the effective radius of curvature of the two solids.

The effective radius of curvature is defined by the radii of curvature of the two solids as,

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{1}$$

In a similar manner the effective modulus of elasticity can also be defined by the modulus of elasticity and Poisson ratio of the individual solids as,

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}. \tag{2}$$

These equations serve to make the notation in subsequent equations simpler.

Contact Mechanics

With the previous definitions established, we can begin understanding the mechanics of contact stress. First is the prediction of the contact area. This discussion will consider cylindrical contact only although the methodology can be applied to any generalized profile.

As previously stated, a normal load applied to two bodies in contact will give rise to a rectangular contact area due to the deformation of the two bodies. The force, F, applied over the area gives rise to a pressure distribution in the materials with a maximum (p_{max}) along a line at the center of the area. The maximum pressure is given as,

$$p_{max} = \frac{2F}{\pi a} \tag{3}$$

Relating the parameters $\bar{u}_{z1}$ and $\bar{u}_{z2}$ shown in Figure 2 to δ and R^*, the contact area formed by under p_{max} can be shown to be,

$$a = \left[\frac{4FR^*}{\pi E^*}\right]^{1/2} \tag{4}$$

The dashed line in Figure 2 indicates overlap of the two surfaces if deformation did not occur depicting the geometry necessary for these calculations.

The total compression between the two bodies is given by,

$$\delta = \frac{a^2}{R^*} \tag{5}$$

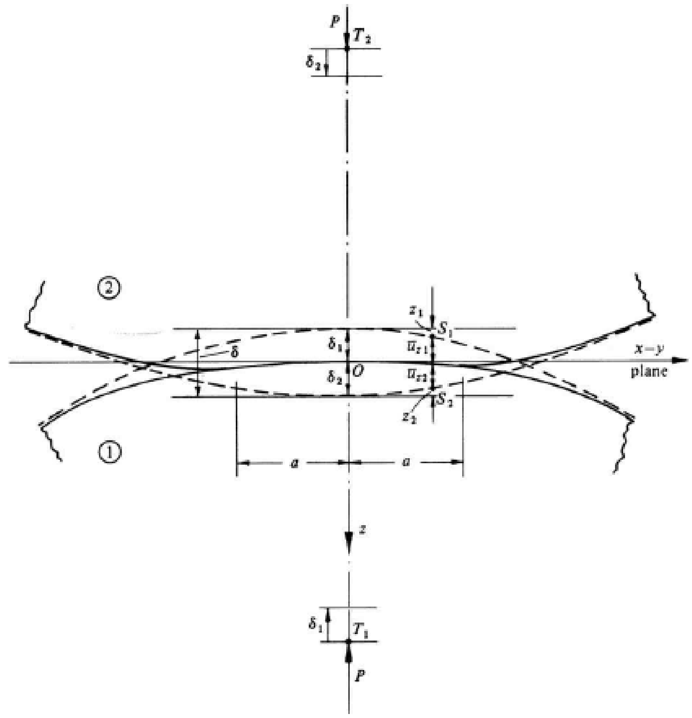

Figure 2. Cross section of two solids after deformation[1].

With the loads and area now established the stresses in the materials can be now be predicted. Given the geometry and proper choice of coordinate frame the principle stresses in the material align with the major coordinate axes. That is[3],

$$\sigma_3 = \sigma_z, \text{ and}$$

$$\sigma_1 = \begin{cases} \sigma_x & \text{for} \quad 0 \le {}^{z}/_{a} \le 0.436 \\ \sigma_y & \text{for} \quad 0.436 \le {}^{z}/_{a} \end{cases}$$

The dominant stress occurs in the z-axis where the load is applied and can be shown to be,

$$\sigma_z = -p_{max} \frac{1}{\sqrt{1 + ({}^{z}/_{a})^2}} \tag{6}$$

Stress in the orthogonal directions is shown to be,

$$\sigma_x = -2\nu p_{max} \left[\sqrt{1 + ({}^{z}/_{a})^2} - \left| \frac{z}{a} \right| \right], and \tag{7}$$

$$\sigma_y = -p_{max} \left[\left(\frac{1 + 2({}^{z}/_{a})^2}{\sqrt{1 + ({}^{z}/_{a})^2}} \right) - 2 \left| \frac{z}{a} \right| \right], and \tag{8}$$

Using Mohr's circle the maximum shear stress, τ_{max}, can be determined and is shown to be,

$$\tau_{max} = \begin{cases} \dfrac{\sigma_z - \sigma_x}{2} & \text{for} \quad 0 \le {}^{z}/_{a} \le 0.436 \\ \dfrac{\sigma_z - \sigma_y}{2} & \text{for} \quad 0.436 \le {}^{z}/_{a} \end{cases} \tag{8}$$

Relation to Optics

Several situations arise in optical design where contact stress must be considered and addressed. The two particular situations illustrated here are lens mounts and kinematic constraint applied to precision motion. An example of each will be described along with the potential issues involved.

Lens Mounts

For the case of lens mounting, the discussion is limited to sharp corner lens seats for a spherical lens as shown in Figure 3. In this case, the initial contact is a ring and when a preload is applied for say, alignment retention under acceleration, high stresses can be induced in the glass due to the small contact area. The specified acceleration and the lens weight are used to determine the required preload. The preload creates compressive and tensile stress in the glass. The tensile stress is the primary concern in glass as flaws will propagate under tensile load. The maximum tensile stress is[4],

$$\sigma_T = \frac{1 - 2\nu}{3}\sigma_z$$

It should be assured that the glass strength is less than this stress including sufficient margin.

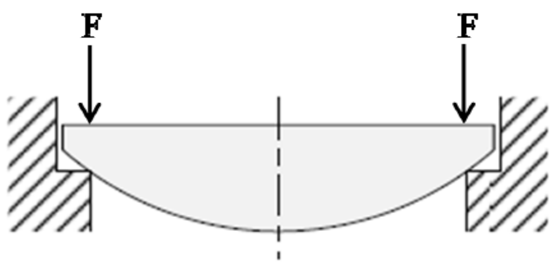

Figure 3. Cross section of a spherical lens with sharp corner seat under applied load, F[5].

Kinematic Constraint[6]

Kinematic constraints can be used to provide precision motion in a device generally for alignment purposes. Utilizing spheres, all 6 degrees of freedom can be constrained and by moving one sphere at a time a high degree of precision motion can be achieved. However, a preload is required to avoid the non-linearity in stiffness with no load. The preload causes the initial point contact of the sphere to deform to a circular area but again results in high stress at the interfaces.

Figure 4 shows the effects of damage due to contact stress. Through repetitive motion and the high stresses involved, the metallic spheres and the tangential interface are subject to fretting damage. Over time the damage will seriously degrade the precision that can be achieved. Often this leads to alternate design forms such as semi-kinematic constraint.

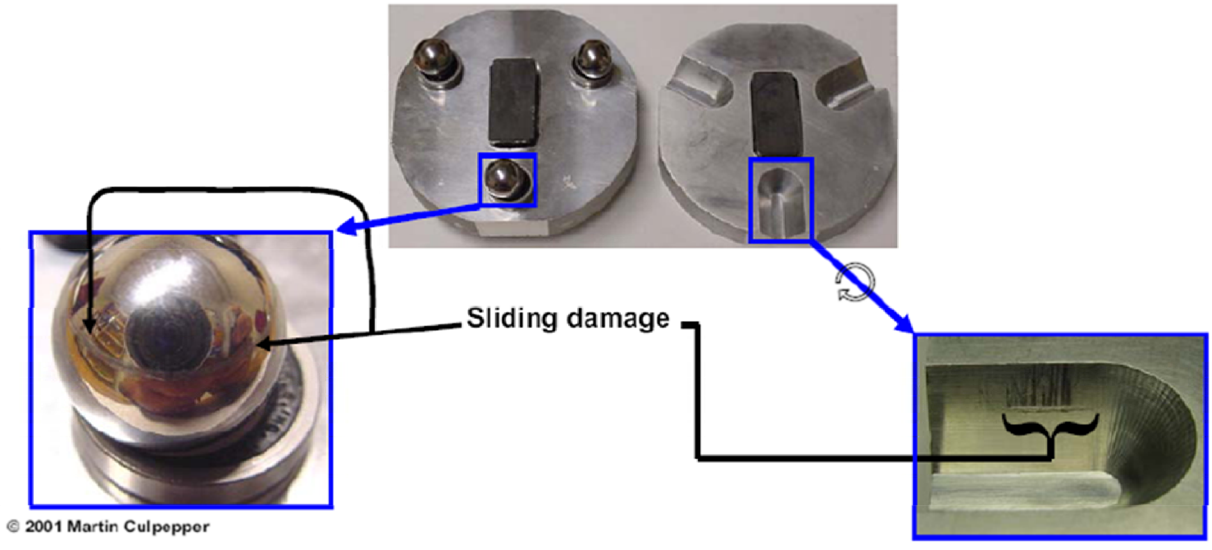

Figure 4. Damage in kinematic constraint using spheres[5].

Conclusion

Hertz theory for contact stress is an important tool for analysis in optomechanical design. It allows for the prediction of contact area, pressure, compression and the resulting stresses between to continuous non-conforming solids under a load. The small contact areas that result often cause design difficulties that may require alternate design forms or methods.

References

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