Opti 521 Tutorial

The Weibull distribution in the strength of glass

Eugene Salamin

1. Introduction

This tutorial was originally intended to be a rather complete survey on the strength of glass. But it soon became clear that the amount of information would be overwhelming. In particular, I do not want to just parrot lengthy explanations without actually understanding the material I present.

Keeping the totality manageable, the major part of this tutorial consists of a discussion of the Weibull distribution, its application to the strength of glass, and a very careful analysis of the statistical process for determining the parameters of that distribution. I show that it is not sufficient to simply calculate a best fit. Consideration must be given to the uncertainty in the parameter estimation. If this is not done, there is a risk of underestimating the failure probability at low stress.

2. Inherent strength of glass and surface flaws

The strength of glass can be theoretically estimated on the basis of breaking atomic bonds [Stansworth 1950, p. 75], and this gives about 10^6 psi. This far exceeds the measured strength, and the discrepancy is due to the presence of small flaws within the glass. Stress concentrates at the edges of the cracks, and while fracture may occur where the atomic-scale local stress is 10^6 psi, the average stress in the bulk of the glass may be more like 10^4 psi. Flaws that extend to the surface are of particular significance since they provide an entry point for water, which is known to catalyze the breaking of bonds in glass.

For a crack of length *L*, the stress concentration is proportional to $\sqrt{L}$. We write [Doyle 2003]

$$K_I = Y \sigma \sqrt{L}$$

for the *stress intensity factor*. Fracture occurs when K_I exceeds a critical value K_{IC}, which is called the *fracture toughness*. Here, *Y* is a numerical factor. For cracks extending to the surface [Menčík 1992, p. 240] gives $Y = 1.22\sqrt{\pi}$.

Here are some tabulated values of fracture toughness [Doyle 2003, p. 16], [Harris 1999, p. 118], [Yoder 2005, p.738]. Note: $1.0\ \text{MPa}\ \text{m}^{1/2} = 910\ \text{psi}\ \text{in}^{1/2}$.

Material	K_{IC}, psi in$^{1/2}$	K_{IC}, MPa m$^{1/2}$
Zinc Selenide	455	0.500
Corning 7940 fused silica	674	0.741
BK7 glass	774	0.851
LaK10 glass	865	0.951
Zinc Sulfide	910	1.0
Sapphire	1820	2.0
Diamond CVD	2548	2.8
Diamond single crystal	3094	3.4
Silicon carbide, Silicon nitride	3640	4.0

The subscript "I" in K_I designates the mode I of fracture opening. There also exist modes II and III. These are illustrated in [Harris 1999, p. 118] and [Menčík 1992, p. 100].

3. Weibull statistics

For a particular type of glass, prepared and processed by a specified procedure, it is reasonable to suppose that surface fractures can be statistically characterized by a function $f(L)$ such that for any small area dA, the probability that it contains flaws of length *L* or greater is $f(L)\,dA$.

In order to calculate the probability that a finite area *A* contains flaws of length *L* or greater, divide the area into a large number *N* of small areas A/N. The area *A* has no flaws of length *L* or greater if and only if none of the areas A/N does. The probability of this last event is $1-f(L)A/N$. Since separate areas should have independent flaw distributions, the probability that the entire area *A* has no flaw of length $\geqslant L$ is

$$\left(1-f(L)A/N\right)^N \rightarrow e^{-A\,f(L)} \quad .$$

Finally, the probability that area *A* does possess a flaw of length $\geqslant L$ is

$$1-e^{-A\,f(L)} \quad .$$

Now suppose also that under some specified stress loading, a sample of the glass will fracture if it contains a flaw of length $\geqslant L_C$. Then the probability of failure is

$$P_f = 1-e^{-A\,f(L_C)} \quad . \qquad (3\text{-}1)$$

As discussed earlier, fracture occurs when the stress intensity factor $K_I = Y\sigma\sqrt{L}$ exceeds the critical value K_{IC}. Setting $K_{IC} = Y\sigma\sqrt{L_C}$, we see that the stress and critical length are related by

$$L_C = \left(\frac{K_{IC}}{Y\sigma}\right)^2, \quad \sigma = \frac{K_{IC}}{Y\sqrt{L_C}} \quad . \qquad (3\text{-}2)$$

It has become established practice [TIE-33] to model the failure probability as the *two-parameter Weibull distribution*, [Weibull 1951]

$$P_f = 1-\exp\left(-\left(\sigma/\sigma_0\right)^m\right) \quad . \qquad (3\text{-}3)$$

This formula gives the probability that the specimen will fail if it is loaded to stress σ. The quantities σ_0 and *m* are model parameters that must be experimentally determined. While *m* is characteristic of the glass and its surface preparation, σ_0 is in addition dependent on the surface area.

The stress σ_0 leads to failure probability $P_f = 1-e^{-1} = 0.63$, while *m*, the *Weibull modulus*, measures the scatter of fracture stress about σ_0. A large modulus implies less scatter since P_f more quickly transits between 0 and 1. More specifically [Mathworld], the mean and standard deviation are

$$\text{Mean} = \Gamma(1+1/m)\sigma_0 \approx (1-\gamma/m)\sigma_0 \quad ,$$

$$\text{Standard deviation} = \sqrt{\Gamma(1+2/m)-\Gamma^2(1+1/m)}\,\sigma_0 \approx (\pi/\sqrt{6})(\sigma_0/m) \quad .$$

Here, Γ is the gamma function, $\gamma \approx 0.577$ is Euler's constant, and the approximations, which I calculated using Maple, are valid for $m \gg 1$, and accurate with error of order $1/m^2$.

Equating the failure probabilities from Equations (3-1) and (3-3),

$$A\ f(L_C) = \left(\sigma/\sigma_0\right)^m \quad ,$$

and then using (3-2), we find that

$$f(L) = \frac{1}{A}\left(\frac{K_{IC}}{Y\,\sigma_0\sqrt{L}}\right)^m \quad ,$$

in which L_C can be replaced by L since it is just the argument of function *f*.

The right hand side must be independent of *A*, so that σ_0 scales with area as

$$\sigma_0(A) = \sigma_0(A_0)\left(\frac{A}{A_0}\right)^{-1/m} \quad .$$

This choice of Weibull model is equivalent to the assumption that the probability per unit area of finding a flaw of length $\geqslant L$ varies as $L^{-m/2}$. Stated another way: The probability that an area dA contains a flaw of length between *L* and $L+dL$ is

$$C\ K_{IC}^m\ L^{-m/2-1} \quad ,$$

where the constant *C* is independent of material and geometry. This would seen to be amenable to experimental check, although I am not aware of any such measurement having been done.

4. Bayesian statistical methods

I will discuss Bayesian statistics in order to apply this principle to an example in the next section. A clear and detailed reference to these methods is [Jaynes 2003].

Let $P(A|B)$ denote the probability that event *A* is true given that *B* is true, and let $P(AB)$ denote the probability that events *A* and *B* are both true. As we know from probability theory,

$$P(A)P(B|A) = P(AB) = P(B)P(A|B) \quad .$$

Written in the form

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \quad ,$$

it is known as *Bayes' Theorem*.

A frequent problem in scientific reasoning is to decide among several alternative hypotheses $H_1,\ H_2,\ \cdots,\ H_n$, given some observed data. *D*. From Bayes' theorem,

$$P(H_i|D) = \frac{P(H_i)P(D|H_i)}{P(D)} \quad .$$

Here, $P(H_i)$ is the *prior* probability of H_i ; it expresses our knowledge of the situation prior to performing the experiment. Sometimes, even under ignorance, symmetry arguments can be used to assign the prior. In situations involving fitting data to a model with free parameters, it is usual to just ignore the prior and assign equal probability to each hypothesis. This is acceptable when the evidence $P(D|H_i)$ overwhelms the prior, as a good experiment should do.

The quantity $P(H_i|D)$ is the *posterior* probability of H_i ; it expresses our knowledge of the situation subsequent to performing the experiment. This process by which we update our knowledge from the prior to the posterior is described as *Bayesian*.

The variable of interest here is the hypothesis H_i , and so the denominator $P(D)$ is merely a normalization. This normalization can be deferred until the end of the calculation, so that

$$P(H_i|D) = \frac{P(H_i)P(D|H_i)}{P(H_1)P(D|H_1) + \cdots + P(H_n)P(D|H_n)} \approx \frac{P(D|H_i)}{P(D|H_1) + \cdots + P(D|H_n)} \quad . \quad (4\text{-}1)$$

Indeed, any factor appearing in $P(D|H_i)$ that is independent of H_i can likewise be ignored until the end. If the totality of data consists of several independent observations $D_1, D_2, \cdots, D_r$, we have

$$P(D|H_i) = P(D_1|H_i)\, P(D_2|H_i) \cdots P(D_r|H_i) \quad . \qquad (4\text{-}2)$$

The approximation in equation (4-1) consists in setting all the priors equal.

5. Example from D. C. Harris

This example, [Harris 1999, p. 99] and repeated in [Yoder 2005, p. 741], concerns 13 disks of standard grade zinc sulfide, of a certain size, and processed similarly, subjected to stress in a ring-on-ring fixture. The observed stresses at fracture took the following values, in MPa.

62	89	110
69	90	125
73	93	126
76	100	
87	107	

To the *i*-th stress value, Harris assigns a failure probability $P_i = (i - \frac{1}{2})/13$. From equation (3-3), note that

$$\log\left(\log\frac{1}{1-P_f}\right) = m\log\sigma - m\log\sigma_0 \quad .$$

Harris fits a straight line to a plot of $\log\left(\log(1-P_f)^{-1}\right)$ *vs.* $\log\sigma$, obtaining $m = 5.4338$ as the slope, and then $\sigma_0 = 100.6$ MPa from the intercept.

This is a very *ad hoc* procedure. I will give a proper Bayesian analysis of this data, and demonstrate that the Harris analysis underestimates the failure probability at low stress.

The underlying assumption is that the ZnS disks possess a flaw length distribution leading to the Weibull distribution of equation (3-3). The hypothesis $H(\sigma_0, m)$ asserts that these disks, as prepared, are characterized by Weibull parameters σ_0 and *m*. The data D_i are the observed stresses σ_i at failure. The probability $P(\sigma|\sigma_0, m)\, d\sigma$ that failure occurs at stress between σ and $\sigma + d\sigma$ is given by

$$P(\sigma|\sigma_0, m) = \frac{d}{d\sigma} P_f(\sigma|\sigma_0, m) = \frac{d}{d\sigma}\left[1 - \exp\left(-(\sigma/\sigma_0)^m\right)\right] \ .$$

Applying Bayes' theorem, as in equations (4-1,2) gives the posterior probability for the Weibull parameters.

$$P(\sigma_0, m|\sigma_1, \cdots, \sigma_{13}) = N \prod_{i=1}^{13} P(\sigma_i|\sigma_0, m) \ .$$

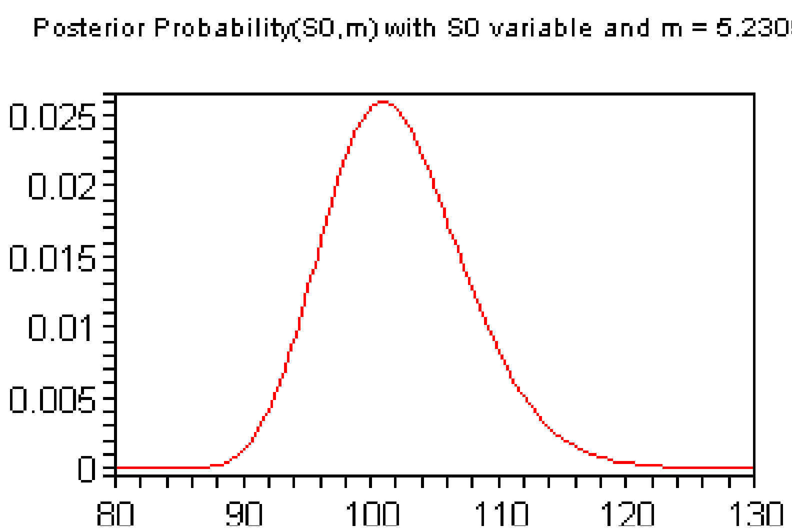
Here, N is a normalization constant chosen so that

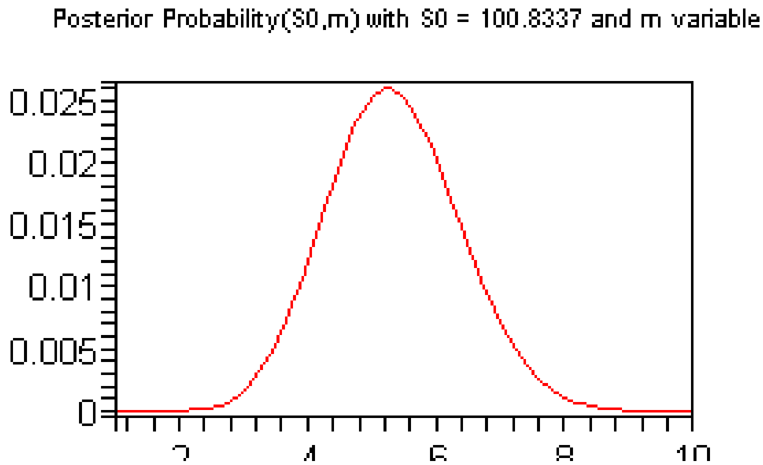
$$\int d\sigma_0 \, dm \, P(\sigma_0, m|\sigma_1, \cdots, \sigma_{13}) = 1$$

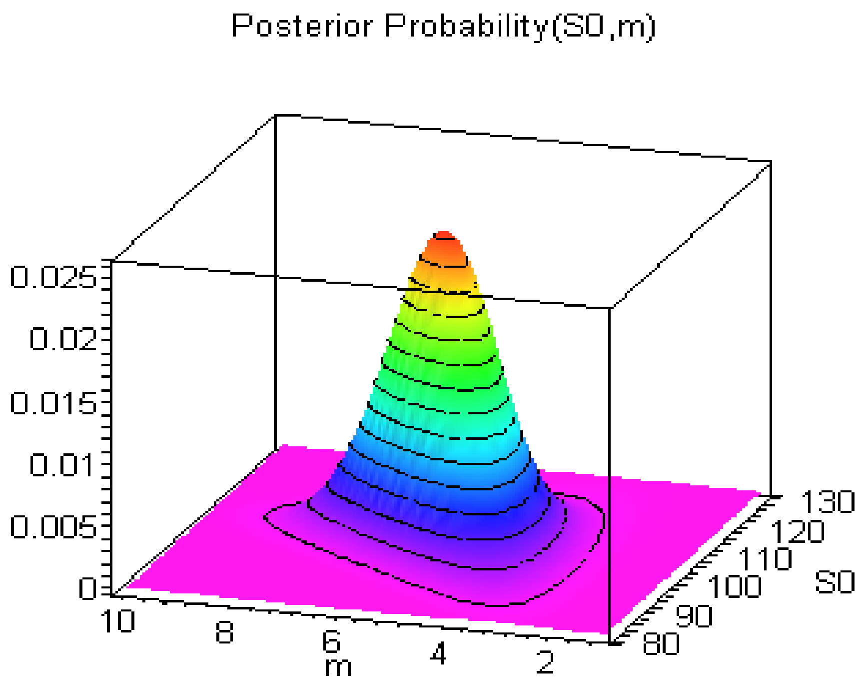
At this point, computer assistance is most helpful. I constructed the posterior $P(\sigma_0, m|\sigma_1, \cdots, \sigma_{13})$ using both Maple and Excel. The former [Maple] is a symbolic mathematics software package. Because the maximization function in Maple (version 9.5) is broken, I used Excel to locate the peak of the posterior. The slight differences between the Harris solution and the peak of the Bayesian posterior will be seen to be insignificant.

	Harris solution	Location of Bayesian peak
Characteristic failure stress σ_0	100.6	100.8337
Weibull Modulus m	5.4338	5.230907

However, when the posterior is plotted in full, we can immediately see a problem with the "best fit" concept.







These plots of the posterior probability illustrate that, contrary to the best fit concept, no unique pair (σ_0, m) is actually singled out. Though it may be true that unique Weibull parameters characterize these ZnS disks, the limited data available limits our ability to discern their values. A wide range of values is consistent with the experimental data. This is especially significant for the Weibull modulus *m*, because small values imply a large failure stress scatter about σ_0 .

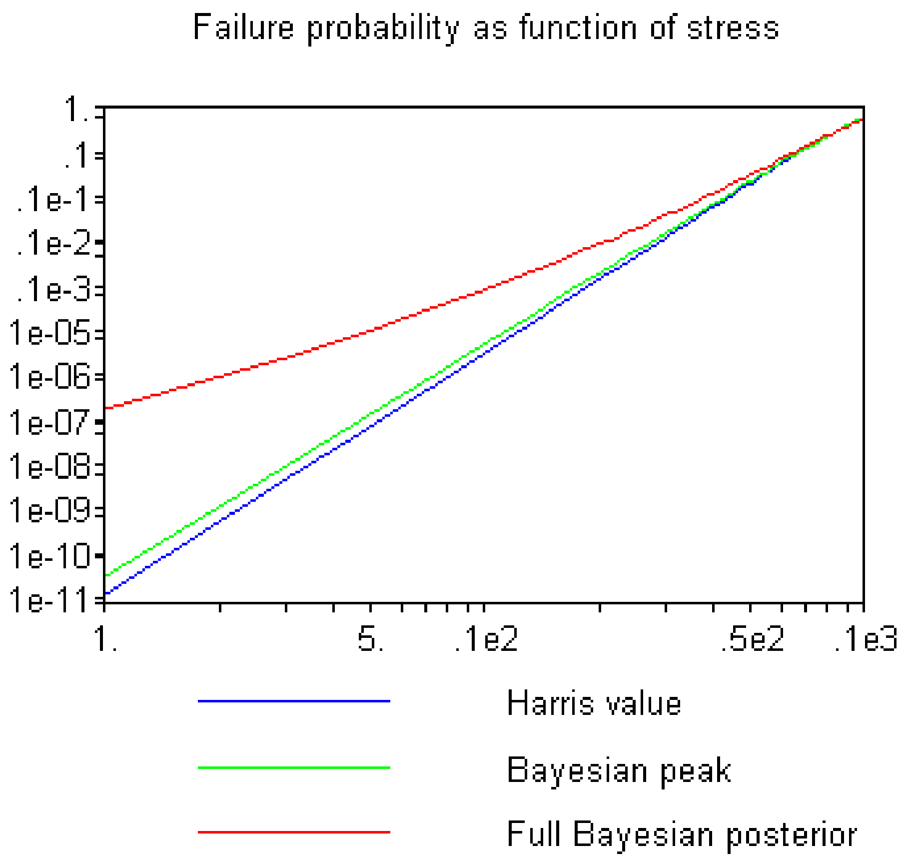
What can we say about failure probability in view of the experimental data? The Weibull formula gives the failure when σ_0 and *m* are already known accurately.

$$P_f(\sigma|\sigma_0, m) = 1 - \exp\left(-(\sigma/\sigma_0)^m\right) \quad .$$

Lacking this precise knowledge, we must weight the Weibull function by the posterior, and integrate over the parameter space. The resulting failure probability becomes

$$P_f(\sigma|\sigma_1, \cdots, \sigma_{13}) = \int d\,\sigma_0\, dm\; P(\sigma_0, m|\sigma_1, \cdots, \sigma_{13})\; P_f(\sigma|\sigma_0, m) \quad .$$

The following plot captures the significance of this statistical investigation by demonstrating the risk of relying upon "best fit" parameters, rather than doing a Bayesian analysis and taking into account the full posterior probability.



Suppose we wish to use the 13 measured failure stress values as a basis for specifying a stress level that provides a given failure probability, for disks similar to those used in the experiment. The three methods yield the following results.

Failure probability	Weibull at Harris value	Weibull at Bayesian peak	Weibull weighted by full Bayesian posterior
10^{-3}	28.2 MPa	26.9 MPa	20.4 MPa
10^{-6}	7.91 MPa	7.19 MPa	2.02 MPa

Another way to exhibit this idea is to start from a desired failure probability, and use one of the pure Weibull distributions to calculate the corresponding permitted stress. Then use the full Bayesian analysis to calculate the failure probability at that stress.

Desired failure probability	Using Weibull at Harris value	Using Weibull at Bayesian peak
10^{-3}	3.38×10^{-3}	2.82×10^{-3}
10^{-4}	6.99×10^{-4}	5.59×10^{-4}
10^{-5}	16.3×10^{-5}	12.6×10^{-5}
10^{-6}	42.4×10^{-6}	31.7×10^{-6}

This need to use the full posterior arises because of its large spread. If we accept the Weibull model, then the material, together with its preparatory process, is characterized by a unique (σ_0, m). However, the experimental data are insufficient to pin them down. When the full posterior leads to conclusions not consistent with a single Weibull distribution, the preferred mitigation, cost permitting, is to conduct further measurements, until the new posterior is sufficiently narrow that its width is inconsequential.

6. Conclusion

A stress-induced fracture example from [Harris 1999] was extensively analyzed for the purpose of illustrating a general principle. When estimating model parameters from experimental data in circumstances where these parameters will be used to make predictions involving very small probabilities, one needs to be wary of accepting a single best-fit value. It may be necessary to take into account the spread about the best fit due to uncertainty of knowledge of the parameters.

The example from Harris involved the Weibull modulus and the failure probability at low stress. The same situation can also occur when estimating the standard deviation in a normal distribution. If the posterior encompasses values much above the single, best fit value, then the probability of large deviations will be underestimated if the full posterior is ignored.

Perhaps it bears mentioning one last time. The parameter uncertainty considered here is not in any way a physical, or even a statistical, property of the material under test. Rather, it is state of uncertainty of knowledge, and is a consequence of an insufficiency of observational data.

References

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The author suggests the normal and Weibull distributions are equally satisfactory models for the strength of glass.

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http://bayes.wustl.edu/

This webste contains a nearly complete collection of the published and unpublished works of Edwin T. Jaynes (1922-1998), together with related material by other authors.

This is a wonderful book for anyone who needs to use statistical reasoning in scientific work. Jaynes develops probability theory as an extension of deductive logic, for the purpose of dealing quantitatively with propositions that are neither true nor false with certainty, but rather have degrees of plausibility. The Bayesian methodology naturally follows as a consequence. History, amusing anecdotes, and a clear explanation of Bayesian statistics, are all to be found in this 700 page book. I never understood the mystical recipes found in statistics books. Then after reading Jaynes' publications, I still don't understand these recipes, but now I can do my job without them.

[Maple] http://www.maplesoft.com/

Symbolic mathematics software.

[Mathworld] http://mathworld.wolfram.com/WeibullDistribution.html

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Appendix has table of *Y* factors for various crack configurations.

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The hallmark paper, and the discussion paper.

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