Opti521 Report #1

Basic Concept and a simple example of FEM

Michihisa Onishi

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1. Introduction

The Finite Element Method (FEM) was developed in 1950' for solving complex structural analysis problem in engineering, especially for aeronautical engineering, then the use of FEM have been spread out to various fields of engineering.

In solving a structural problem, the fundamental continuum equation is set up for infinitesimal small elements of a bulk. Since this fundamental equation usually results in the differential equations or integral equations with some boundary condition, it is not easy to get analytical solutions. For this case, the discrete analysis can be used to approximate the continuum problems with infinite degree of freedom (DOF) by using only finite degree of freedom. The discrete analysis includes Rayleigh-Ritz Method, Method of Weighted Residuals (MWR), Finite Differential Method (FDM) and Boundary Element Method (BDM) as typical examples. FEM is also categorized in the discrete analysis.

The basic idea of discrete analysis is to replace the infinite dimensional linear problem with a finite dimensional linear problem using a finite dimensional subspace. For the Finite Element Method, a space of piecewise linear functions is taken to approximate the solutions. An appropriate set of basis is usually referred to an "element".

2. Formulation of small displacement elastic problem

Although the materials covered in this section is out of scope of the OPTI-521 class, we should discuss the basic concept of elastic problem. For small deformation, the basic equations for elastic problem are given by following equations.

(a) Equation of Equilibrium

$$\sigma_{ij,j} + \overline{F_i} = 0$$

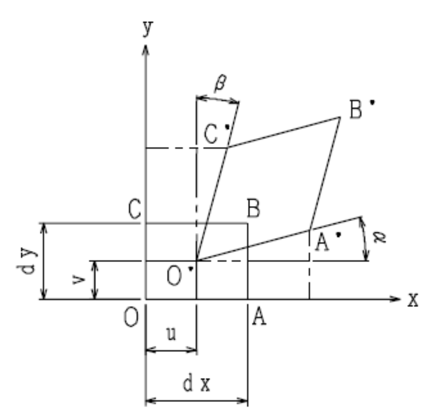
where F is the body force per unit volume. This equation simply represents the equilibrium of the forces applied to the material.

(b) Strain-Displacement Relationship

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For the two dimensional plane stress problem with homogeneous isotropic material,

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \gamma_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$



(c) Stress-Strain Relationship

$$\sigma_{ij} = d_{ijkl}\varepsilon_{kl}$$

For a homogeneous and isotropic material, the stress-strain relationship can be greatly simplified. Starting from Hook's theorem,

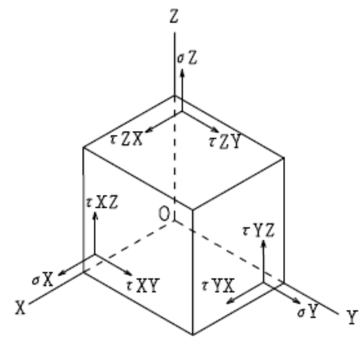
$$\varepsilon_x = \frac{1}{E}\{\sigma_x - \nu(\sigma_y + \sigma_z)\} \qquad \varepsilon_y = \frac{1}{E}\{\sigma_y - \nu(\sigma_z + \sigma_x)\} \qquad \varepsilon_z = \frac{1}{E}\{\sigma_z - \nu(\sigma_x + \sigma_y)\}$$

the stress-strain relationship is given by

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\{(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\{(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)\}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\{(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\}$$



$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx} \qquad \text{(where } G = \frac{E}{2(1+\nu)}\text{)}$$

Matrix expression greatly simplifies the expression.

$$\vec{\sigma} = \mathbf{D} \cdot \vec{\varepsilon} \qquad \text{(\{D\} is called as D-matrix)}$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \qquad \vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \qquad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}$$

This is the most basic stress-strain relationship for homogeneous and isotropic materials. For the two dimensional case (plane stress problem), the stress-strain relationship can be expressed as follows. Starting from Hook's theorem,

$$\varepsilon_x = \frac{1}{E}\{\sigma_x - \nu\sigma_y\} \qquad \varepsilon_y = \frac{1}{E}\{\sigma_y - \nu\sigma_x\}$$

Then

$$\sigma_x = \frac{E}{(1+\nu)(1-\nu)}(\varepsilon_x + \nu\varepsilon_y) \qquad \sigma_y = \frac{E}{(1+\nu)(1-\nu)}(\varepsilon_y + \nu\varepsilon_x)$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{(1+\nu)(1-\nu)}\left(\frac{1-\nu}{2} \cdot \gamma_{xy}\right)$$

Using matrix expression

$$\vec{\sigma} = \mathbf{D} \cdot \vec{\varepsilon} \qquad (\ \{D\}\text{: D-matrix})$$

$$\mathbf{D} = \frac{E}{(1+\nu)(1-\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

(d) Dynamic Boundary Condition

$T_i = \overline{T}_i$; on S_S where $T_i = \sigma_{ij} n_j$, n_j is the surface normal.

This condition should be applied to the forces on the surface of the bulk considered.

(e) Geometric Boundary Condition

$u_i = \overline{u}_i$; on S_u

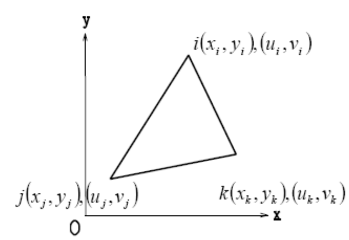
3. Element

In the process of discretization, an appropriate basis of piecewise functions is used. The term ‘element’ usually refers to a set of basis used in FEM. Although there are many types of elements correspond to the various types of problems, the simplest element, triangle linear plane element is introduced here. Inside the triangle element, the displacement u is approximated by primary expression.

$$u(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$$

$$v(x,y) = \beta_0 + \beta_1 x + \beta_2 y$$

Let the displacement at each node of the element be (u_i, v_i), (u_j, v_j) and (u_k, v_k), then the following relations should be satisfied.



$$\begin{cases} u_1 = \alpha_0 + \alpha_1 x_1 + \alpha_2 y_1 \\ u_2 = \alpha_0 + \alpha_1 x_2 + \alpha_2 y_2 \\ u_3 = \alpha_0 + \alpha_1 x_3 + \alpha_2 y_3 \end{cases} \qquad \begin{cases} v_1 = \beta_0 + \beta_1 x_1 + \beta_2 y_1 \\ v_2 = \beta_0 + \beta_1 x_2 + \beta_2 y_2 \\ v_3 = \beta_0 + \beta_1 x_3 + \beta_2 y_3 \end{cases}$$

Solving the equations for a_1, a_2, b_1 and b_2, we get

$$\begin{cases} \alpha_1 = [u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2)]/\Delta \\ \alpha_2 = [u_1(x_2 - x_3) + u_2(x_3 - x_1) + u_3(x_1 - x_2)]/\Delta \\ \beta_1 = [v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)]/\Delta \\ \beta_2 = [v_1(x_2 - x_3) + v_2(x_3 - x_1) + v_3(x_1 - x_2)]/\Delta \end{cases} \qquad \text{where } \Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{cases} \alpha_0 = [u_1(y_3 x_2 - y_2 x_3) + u_2(y_1 x_3 - y_3 x_1) + u_3(y_2 x_1 - y_1 x_2)]/\Delta \\ \beta_0 = [v_1(y_3 x_2 - y_2 x_3) + v_2(y_1 x_3 - y_3 x_1) + v_3(y_2 x_1 - y_1 x_2)]/\Delta \end{cases}$$

Using the results, the strain of the element can be expressed by

$$\varepsilon_x = \frac{\partial u}{\partial x} = \alpha_1, \qquad \varepsilon_y = \frac{\partial v}{\partial y} = \beta_2, \qquad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \alpha_2 + \beta_1$$

Then the strain-displacement relation is given by

$$\vec{\varepsilon} = \mathbf{B} \cdot \vec{u} \qquad (\{B\} \text{ is called as B-matrix})$$

$$\mathbf{B} = \frac{1}{\Delta}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & -(x_2 - x_3) & 0 & -(x_3 - x_1) & 0 & -(x_1 - x_2) \\ -(x_2 - x_3) & y_2 - y_3 & -(x_3 - x_1) & y_3 - y_1 & -(x_1 - x_2) & y_1 - y_2 \end{bmatrix} \qquad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\vec{u} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3]^T$$

The most important concept here is the B-matrix which gives us the matrix of coefficient for the strain-displacement relationship. The final simultaneous equations can be obtained by this B-matrix working with the D-matrix and the principle of virtual work.

Similarly to the above development, a physical quantity A can be approximated by

$$A(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y$$

$$= N_1 A_1 + N_2 A_2 + N_3 A_3 = \sum_i N_i A_i = [N] \cdot \{A\}$$

$$\begin{cases} N_1 = [(y_2 - y_3)x + (x_3 - x_2)y + (y_3 x_2 - y_2 x_3)]/\Delta \\ N_2 = [(y_3 - y_1)x + (x_1 - x_3)y + (y_1 x_3 - y_3 x_1)]/\Delta \\ N_3 = [(y_1 - y_2)x + (x_2 - x_1)y + (y_2 x_1 - y_1 x_2)]/\Delta \end{cases}$$

The matrix [N] is called the Shape Function.

4. Variational Principle

Although there are many methods for discretization such as collocation method and Galerkin method, the principle of virtual work is widely used to formulate the FEM for continuum elastic problems. It requires that the energy of the system in equilibrium should be minimized or at least locally minimized. For the FEM, this principle states that

$$\int_V \sigma_{ij} \delta\varepsilon_{ij} dV - \int_V \overline{F}_i \delta u_i dV - \int_{S_\sigma} \overline{T}_i \delta u_i dS = 0$$

$$\delta u_i = 0 \text{ on } S_u$$

While the expression of the principle of virtual work seems difficult and complicated, what the equation means is really simple. It requires that the energy stored in the body should be equal to the energy provided by the applied body force and the surface force. Working with the element introduced in section 3 and using the equations

$$\vec{\sigma} = \mathbf{D} \cdot \vec{\varepsilon}$$

$$\vec{\varepsilon} = \mathbf{B} \cdot \vec{u}$$

The equation above can be rewrite as

$$\delta U = \delta W$$

$$\delta U = \int_V \delta\{\varepsilon\}^T \{\sigma\} dV = \int_V \delta\{\varepsilon\}^T [D]\{\varepsilon\} dV$$

$$\delta W = \int_V \delta\{u\}^T \{\overline{F}\} dV + \int_{S_\sigma} \delta\{u\}^T \{\overline{T}\} dS$$

If we assume that the total variational strain energy and the total virtual work can be given by summing the variational strain energy and the virtual work of each element respectively, i.e,

$$\delta U = \sum_m \delta U^{(m)} \text{ , and } \delta W = \sum_m \delta W^{(m)} \qquad \text{(suffix m represent the m-th element)}$$

where

$$\delta U^{(m)} = \int_{V^{(m)}} \delta\{\varepsilon\}^T [D]\{\varepsilon\} dV$$

$$\delta W^{(m)} = \int_{V^{(m)}} \delta\{u\}^T \{\overline{F}\} dV + \int_{S^{(m)}_{\sigma}} \delta\{u\}^T \{\overline{T}\} dS$$

Then the following equations can be obtained.

$$\delta U^{(m)} = (\delta\{u\})^T ([K]\{u\})$$

$$\delta W^{(m)} = (\delta\{u\})^T (\{F_v\} + \{F_s\}) = (\delta\{u\})^T \{F\}$$

where

$$[K] = \int_{V^{(m)}} [B]^T [D][B] dV$$

$$\{F_v\} = \int_{V^{(m)}} [N]^T \{\overline{F}\} dV , \qquad \{F_s\} = \int_{S^{(m)}_{\sigma}} [N]^T \{\overline{T}\} dS$$

Since, in above equations, d{u} can be arbitrary chosen, the governing equation of each element is given by

$$[K] \cdot \{u\} = \{F\}$$

This expression represents exactly the Hook's law which is extended to the multi dimensions. The final equation for the system considered can be constructed by the sum of the governing equations of all elements.

5. Simple Example

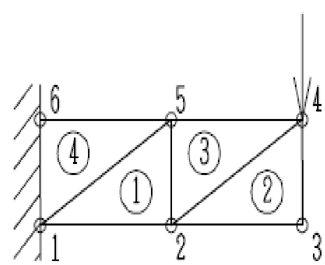
Now consider the simple example shown on the right.

The coordinates of each point are given by

Node #1; (0, 0)	Node #2; (1, 0)
Node #3; (2, 0)	Node #4; (2, 1)
Node #5; (1, 1)	Node #2; (0, 1)

Node #1 and Node #6 are fixed on the wall.

The force A (1N) is applied to the Node #4 as shown in the figure.



<Step 1> Preparation & calculating D-matrix

The triangle plane strain linear element should be chosen since it represents the physics considered here better than the other elements.

For simplicity, assume that the D-matrix is given by

$$[D] = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* note that this D-matrix does not seem to be realistic.
For aluminum (E=70GPa and n=0.23) D-matrix is given

$$[D] = \frac{7\times10^{10}}{(1+0.23)(1-0.23)}_{[N]} \begin{bmatrix} 1 & 0.23 & 0 \\ 0.23 & 1 & 0 \\ 0 & 0 & \frac{1-0.23}{2} \end{bmatrix} = 7.4\times10^{10}{}_{[N]} \begin{bmatrix} 1 & 0.23 & 0 \\ 0.23 & 1 & 0 \\ 0 & 0 & 0.385 \end{bmatrix}$$

It is also assumed that the governing equation for the system is given by $[K]\{u\} = \{F\}$.

<Step 2> Constructing the elements

Four element can be constructed by given six nodes.

Element #1; Node #1 ⇨ #2 ⇨ #5 Element #2; Node #2 ⇨ #3 ⇨ #4

Element #3; Node #4 ⇨ #5 ⇨ #2 Element #2; Node #5 ⇨ #6 ⇨ #1

<Step 3> Calculating the B-matrix and the K-matrix

The B-matrix is given by

$$\mathbf{B} = \frac{1}{\Delta}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & -(x_2 - x_3) & 0 & -(x_3 - x_1) & 0 & -(x_1 - x_2) \\ -(x_2 - x_3) & y_2 - y_3 & -(x_3 - x_1) & y_3 - y_1 & -(x_1 - x_2) & y_1 - y_2 \end{bmatrix}$$

For the element #1, the B-matrix can be expressed as follows;

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \qquad \mathbf{B} = [B] = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{D} = [D] = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K} = [K] = [B]^T \cdot [D] \cdot [B] = \begin{bmatrix} 1 & 0 & -1 & 0.5 & 0 & -0.5 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & 1 & 2 & -1.5 & -1 & 0.5 \\ 0.5 & -1 & -1.5 & 2 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ -0.5 & 0 & 0.5 & -1 & 0 & 1 \end{bmatrix}$$

Therefore the governing equation ([K]{u}={f}) for the element #1 can be expressed by

$$\begin{bmatrix} 1 & 0 & -1 & 0.5 & 0 & -0.5 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & 1 & 2 & -1.5 & -1 & 0.5 \\ 0.5 & -1 & -1.5 & 2 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ -0.5 & 0 & 0.5 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_5 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \\ f_5 \\ g_5 \end{Bmatrix}$$

By repeating the same procedure, the governing equations for each element can be obtained. The followings shown below are the result.

<Element #1>

u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6	
1	0	-1	0.5					0	-0.5			u1
0	1	1	-1					-1	0			v1
-1	1	2	-1.5					-1	0.5			u2
0.5	-1	-1.5	2					1	-1			v2
												u3
												v3
												u4
												v4
0	-1	-1	1					1	0			u5
-0.5	0	0.5	-1					0	1			v5
												u6
												v6

<Element #2>

u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6	
												u1
												v1
		1	0	-1	0.5	0	-0.5					u2
		0	1	1	-1	-1	0					v2
		-1	1	2	-1.5	-1	0.5					u3
		0.5	-1	-1.5	2	1	-1					v3
		0	-1	-1	1	1	0					u4
		-0.5	0	0.5	-1	0	1					v4
												u5
												v5
												u6
												v6

<Element #3>

u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6	
												u1
												v1
		1	0			0	-1	-1	1			u2
		0	1			-0.5	0	0.5	-1			v2
												u3
												v3
		0	-0.5			1	0	-1	0.5			u4
		-1	0			0	1	1	-1			v4
		-1	0.5			-1	1	2	-1.5			u5
		1	-1			0.5	-1	-1.5	2			v5
												u6
												v6

<Element #4>

u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6	
1	0							0	-1	-1	1	u1
0	1							-0.5	0	0.5	-1	v1
												u2
												v2
												u3
												v3
												u4
												v4
0	-0.5							1	0	-1	0.5	u5
-1	0							0	1	1	-1	v5
-1	0.5							-1	1	2	-1.5	u6
1	-1							0.5	-1	-1.5	2	v6

<Step 4> Adding up all elements

As a result of adding up all equations, the following equation can be obtained.

<Sum>														
u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6			
2	0	-1	0.5	0	0	0	0	0	-1.5	-1	1	u1		f1
0	2	1	-1	0	0	0	0	-1.5	0	0.5	-1	v1		g1
-1	1	4	-1.5	-1	0.5	0	-1.5	-2	1.5	0	0	u2		f2
0.5	-1	-1.5	4	1	-1	-1.5	0	1.5	-2	0	0	v2		g2
0	0	-1	1	2	-1.5	-1	0.5	0	0	0	0	u3		f3
0	0	0.5	-1	-1.5	2	1	-1	0	0	0	0	v3	=	g3
0	0	0	-1.5	-1	1	2	0	-1	0.5	0	0	u4		f4
0	0	-1.5	0	0.5	-1	0	2	1	-1	0	0	v4		g4
0	-1.5	-2	1.5	0	0	-1	1	4	-1.5	-1	0.5	u5		f5
-1.5	0	1.5	-2	0	0	0.5	-1	-1.5	4	1	-1	v5		g5
-1	0.5	0	0	0	0	0	0	-1	1	2	-1.5	u6		f6
1	-1	0	0	0	0	0	0	0.5	-1	-1.5	2	v6		g6

<Step 5> Boundary conditions

The boundary conditions for this example are

Constraints; $(u_1, v_1) = (u_6, v_6) = (0, 0)$

Force; $(f_4, g_4) = (0, -1)$

Then the system equation can be rewrite as follows;

<Sum>														
u1	v1	u2	v2	u3	v3	u4	v4	u5	v5	u6	v6			
2	0	-1	0.5	0	0	0	0	0	-1.5	-1	1	0		f1
0	2	1	-1	0	0	0	0	-1.5	0	0.5	-1	0		g1
-1	1	4	-1.5	-1	0.5	0	-1.5	-2	1.5	0	0	u2		0
0.5	-1	-1.5	4	1	-1	-1.5	0	1.5	-2	0	0	v2		0
0	0	-1	1	2	-1.5	-1	0.5	0	0	0	0	u3		0
0	0	0.5	-1	-1.5	2	1	-1	0	0	0	0	v3	=	0
0	0	0	-1.5	-1	1	2	0	-1	0.5	0	0	u4		0
0	0	-1.5	0	0.5	-1	0	2	1	-1	0	0	v4		-1
0	-1.5	-2	1.5	0	0	-1	1	4	-1.5	-1	0.5	u5		0
-1.5	0	1.5	-2	0	0	0.5	-1	-1.5	4	1	-1	v5		0
-1	0.5	0	0	0	0	0	0	-1	1	2	-1.5	0		f6
1	-1	0	0	0	0	0	0	0.5	-1	-1.5	2	0		g6

where (f_1, g_1) and (f_6, g_6) are the reaction forces due to the constraints applied on Node #1 and #6.

<Step 6> Formulation of the system equations

Following the procedure explained above, the final equations for the system can be rewrite as follows;

For the displacement {u};

$$\begin{bmatrix} 4 & -1.5 & -1 & 0.5 & 0 & -1.5 & -2 & 1.5 \\ -1.5 & 4 & 1 & -1 & -1.5 & 0 & 1.5 & -2 \\ -1 & 1 & 2 & -1.5 & -1 & 0.5 & 0 & 0 \\ 0.5 & -1 & -1.5 & 2 & 1 & -1 & 0 & 0 \\ 0 & -1.5 & -1 & 1 & 2 & 0 & -1 & 0.5 \\ -1.5 & 0 & 0.5 & -1 & 0 & 2 & 1 & -1 \\ -2 & 1.5 & 0 & 0 & -1 & 1 & 4 & -1.5 \\ 1.5 & -2 & 0 & 0 & 0.5 & -1 & -1.5 & 4 \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

For the reaction force {F};

$$\begin{bmatrix} -1 & 0.5 & 0 & 0 & 0 & 0 & 0 & -1.5 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -1 \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ g_1 \\ f_6 \\ g_6 \end{Bmatrix}$$

For the strain {e};

$$\vec{\varepsilon} = \mathbf{B} \cdot \vec{u} = [B] \cdot \{u\}$$

For the stress {s};

$$\vec{\sigma} = \mathbf{D} \cdot \vec{\varepsilon} = [D] \cdot \{\varepsilon\}$$

6. Summary

While only the basic concept and the simplest example are introduced in this report, the FEM is widely used in various engineering fields, such as nonlinear problem, natural vibration analysis, large displacement region, thermal analysis, fluid dynamics and even an electromagnetic field analysis, and then there are many types of elements related to the specific physical quantities. Even though it is impossible to give a full explanation of FEM in this report, good opto-mechanical engineer should know the basic concept and various applications of FEM which enable us to expand the capability of dealing opto-mechanical issues.

Reference;

1. K. Washizuka, H. Miyamoto, Y. Yamada, Y, Yamamoto, T. Kawai, "Handbook of the Finite Element Method", Baifukan, October 2001.
2. J. Burge. Class note of OPTI-521, College of Optical Science, University of Arizona, 2007