

Flexural Stresses In Beams (Derivation of Bending Stress Equation)

General:

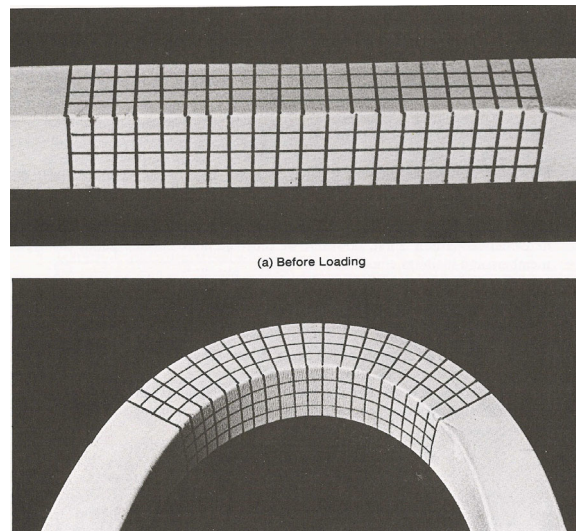
A beam is a structural member whose length is large compared to its cross sectional area which is loaded and supported in the direction transverse to its axis. Lateral loads acting on the beam cause the beam to bend or flex, thereby deforming the axis of the beam into a curved line. We shall now consider the stresses and strains associated with bending moments.

Pure Bending Assumptions:

1. Beam is straight before loads are applied and has a constant cross-sectional area.
2. Beam has a longitudinal plane of symmetry and the bending moment lies within this plane.
3. Beam is subjected to pure bending (bending moment does not change along the length).
4. Beam material is homogeneous and isotropic.

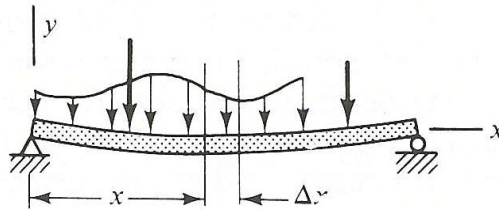
Experiments show that beams subjected to pure bending (*see above*) deform in such a way that plane sections remain plane. In other words, planes perpendicular to the longitudinal axis before loading remain plane and perpendicular to the axis after loading. Beam sections rotate relative to one another when the beam deforms.

Consider the following rubber beam with grid lines subjected to pure bending.

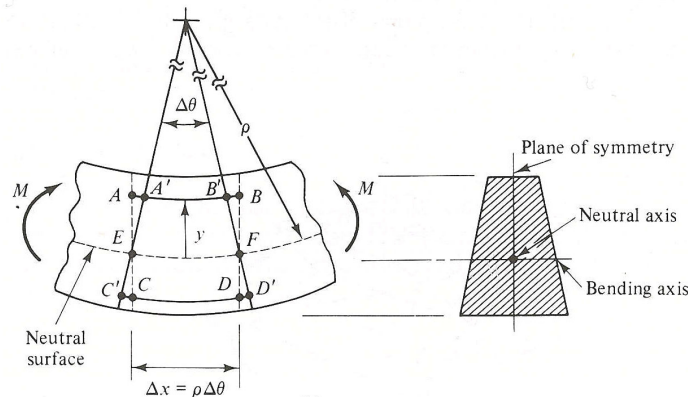


Strain – Deformation Relationship:

Now let's consider a beam subjected to an arbitrary transverse load.



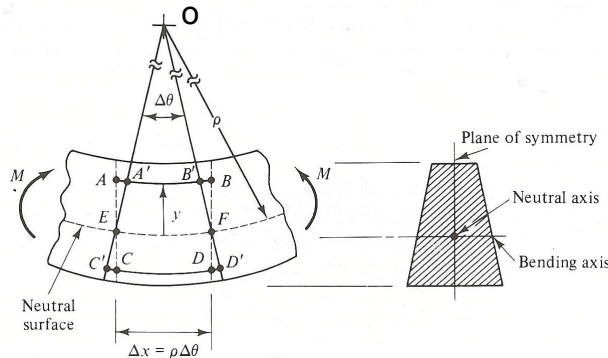
The deformations can be related to the strains by considering a small element of the beam Δx . It is helpful to view the beam as consisting of a number of longitudinal fibers, or line elements, laid side by side.



As shown above, before bending:

$$AB = CD = EF = \Delta x$$

After bending (*deformations greatly exaggerated for clarity*) line segment AB shortened, line segment CD lengthened and line segment EF does not change. Line segment EF is the edge of the surface extending over the width and length of the beam and is referred to as the **neutral surface**. The intersection of the neutral surface with the longitudinal plane of symmetry is called the **neutral axis** and coincides with the centroidal axis of the cross section.



As shown above,

o = center of curvature

ρ = radius of curvature

From calculus and analytic geometry

$$\kappa = \frac{1}{\rho} = \frac{\Delta\theta}{\Delta x}$$

κ = curvature

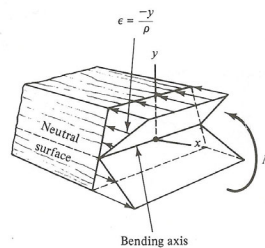
Therefore:

$$\rho = \frac{\Delta x}{\Delta\theta} \quad \text{and} \quad \Delta x = \rho\Delta\theta$$

Now, the normal strain in the longitudinal direction along line segment AB may be determined as follows:

$$\begin{aligned} \epsilon_{AB} &= \frac{\delta}{L} = \frac{L_f - L_i}{L_i} \\ \epsilon_{AB} &= \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \\ \epsilon_{AB} &= \frac{-y}{\rho} \end{aligned} \quad (1)$$

The strain is proportional to the distance from the neutral axis and, as shown below is the greatest at the top and bottom of the beam.



Stress – Strain Relationship:

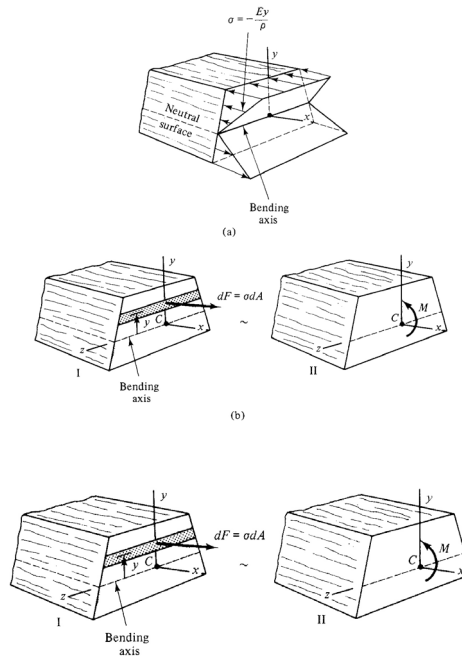
Recall:

$$\sigma = E\epsilon$$

Substituting equation 1:

$$\sigma = \frac{-Ey}{\rho} \quad (2)$$

The distribution of stress is identical to the distribution of strain shown on the previous page. The maximum values occur at the top and bottom surfaces of the beam.

Moment – Stress Relationship:

From the figure above, we can see that internal stresses are statically equivalent to the external forces and moment. Therefore:

$$\sum F_{x(I)} = \sum F_{x(II)} \Rightarrow \int \sigma dA = 0$$

$$\sum M_{z(I)} = \sum M_{z(II)} \Rightarrow -\int y \sigma dA = M$$

Rewriting we have:

$$M = -\int y \sigma dA$$

Substituting equation (2) yields:

$$M = -\int y \frac{-Ey}{\rho} dA$$

$$M = \frac{E}{\rho} \int y^2 dA$$

Let:

$$I = \int y^2 dA$$

Where I is the second moment of area (*moment of inertia*).

Therefore:

$$M = \frac{EI}{\rho} \quad (3)$$

Recall that curvature, κ , equals $1/\rho$.

$$M = EI\kappa$$

and

$$y = f(x)$$

$$y' = \text{slope}$$

$$y'' = \text{curvature}$$

$$M = Ely''$$

Which is the basic differential equation of the deflection curve of a beam.

Finally, combining equations (2) and (3) we have:

$$\sigma = \frac{-My}{I}$$

Dropping the sign we have:

$$\sigma = \frac{My}{I}$$

M = bending moment

I = moment of inertia (in^4 or mm^4)

y = distance from neutral axis