Deformation of Axial Members

For a prismatic bar of length L in tension by axial forces P we have determined:

$$\sigma = \frac{P}{A} \qquad\qquad \varepsilon = \frac{\delta}{L}$$

It is important to recall that the load P must act on the centroid of the cross section. Now, let us assume that the bar is made of a homogenous material and that the material is linearly elastic so that Hooke's law applies.

$$E = \sigma/\varepsilon$$

Combining and solving for displacement, we obtain the following equation for the elongation (*deformation*) of the bar.

$$\delta = \frac{PL}{AE}$$

The above equation shows that deformation is proportional to the load and the length and inversely proportional to the cross sectional area and the elastic modulus of the material.

The product **AE** is known as the **axial rigidity** of the bar.

We can see that a bar is tension is analogous to an axially loaded spring. Recall for a spring P = Kδ, where K is the spring stiffness. Likewise, the above equation can be expressed as follows:

$$P = \frac{AE}{L}\delta$$

The quantity AE/L is the **stiffness K** of an axially loaded bar and is defined as the force required to produce a unit deflection.

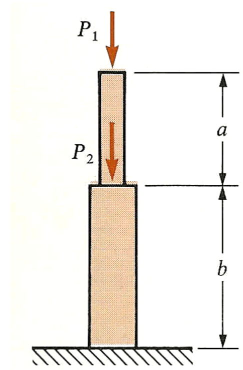
In an analogous manner, the **flexibility** f is defined as the deformation due to a unit load. Thus the flexibility of a axially loaded bar is:

$$f = \frac{L}{AE}$$

In general, the total elongation (deformation) of a bar consisting of several parts having different axial forces and cross sectional areas may be obtained as follows:

$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{P_i L_i}{A_i E_i}$$

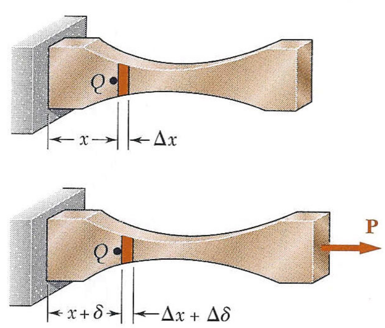
Consider the following example:



Where:

a = 10 in $A_{(a)}$ = 1 in^2
b = 15 in $A_{(b)}$ = 2.0 in^2
$P_1 = P_2$ = 1000 lbs
$E_1 = E_2$ = 10,000,000 psi

When the axial force or the cross-sectional area varies continuously along the axis of the bar, the previous equation is no longer suitable. Recall the strain at point Q is defined as follows:



$$\varepsilon_Q = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx}$$

Therefore:

$$d\delta = \varepsilon_Q dx$$

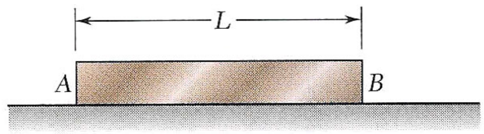
But $\varepsilon = \sigma/E$ and $\sigma = P/A$

$$d\delta = \frac{P_x}{A_x E} dx$$

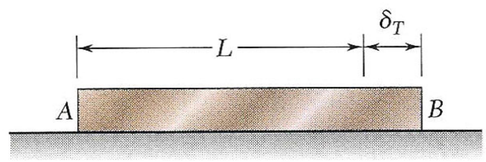
$$\delta = \int_0^L \frac{P_x dx}{A_x E}$$

Thermal Strains & Design Concepts

Thermal Strains – All the members and structures we have considered so far were assumed to be at a constant uniform temperature. Let us consider a homogeneous bar AB, which rest freely on a smooth horizontal surface. If we raise the temperature by ΔT, we observe the bar elongates by an amount δ_T.



(a)



(b)

$$\delta_T = \alpha(\Delta T)L$$

Where α is a material characteristic called the ***coefficient of thermal expansion***. Therefore, we conclude that the thermal strain is:

$$\varepsilon_T = \alpha \Delta T$$

α = in/in/°F or in/in/°C or mm/mm/°C

In many cases α is stated as parts/million/°F or parts/million/°C.

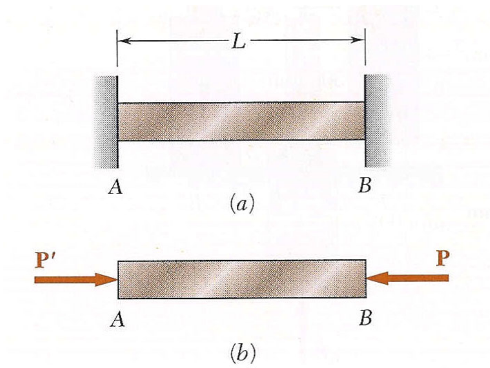
Strains caused by temperature changes and strains caused by applied loads are essentially independent. Therefore, the total amount of strain may be expressed as follows.

$$\varepsilon_{total} = \varepsilon_\sigma + \varepsilon_T$$

$$\varepsilon_{total} = \frac{\sigma}{E} + \alpha \Delta T$$

Note: Since homogeneous, isotropic materials expand uniformly in all directions when heated (and contract uniformly when cooled), neither shear stresses nor shear stains are affected by temperature changes

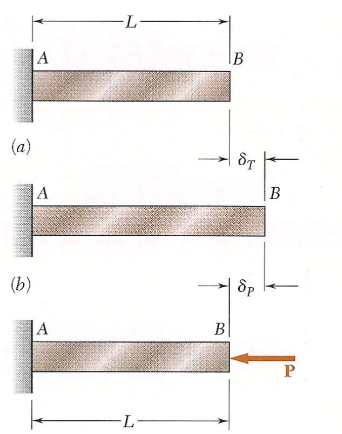
Now let us consider that the same rod is placed between two fixed supports as shown below. It is assumed that there is no stress or strain in the rod at this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the supports. Therefore, $\delta T = 0$.



Our problem is to determine the stress in the bar caused by the temperature change ΔT. From previous examples, we observe that this problem is statically indeterminate.

Superposition Method

1. Designate one of the unknown reactions as redundant and eliminate the corresponding support.
2. Treat the redundant reaction as an unknown load, which together with the other loads must produce deformations that are compatible with the original constraints.
3. Solve by considering separately the deformations caused by the given loads and the redundant reactions and by adding (superposing) the results obtained.



$$\delta = \delta_T + \delta_P = 0$$

Substituting:

$$\delta = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

Therefore:

$$P = -AE\alpha(\Delta T)$$

And:

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Coefficient of Thermal Expansion for Common Materials

Material	Coefficient of thermal expansion z	
	$10^{-6}/°F$	$10^{-6}/°C$
Aluminum and aluminum alloys	**13**	**23**
Brass	**10.6-11.8**	**19.1-21.2**
Red brass	**10.6**	**19.1**
Naval brass	**11.7**	**21.1**
Brick	**3-4**	**5-7**
Bronze	**9.9-11.6**	**18-21**
Manganese bronze	**11**	**20**
Cast Iron	**5.5-6.6**	**9.9-12.0**
Gray cast iron	**5.6**	**10.0**
Concrete	**4-8**	**7-14**
Medium strength	**6**	**11**
Copper	**9.2-9.8**	**16.6-17.6**
Beryllium copper	**9.4**	**17.0**
Glass	**3-6**	**5-11**
Magnesium (pure)	**14.0**	**25.2**
Alloys	**14.5-16.0**	**26.1 28.8**
Monel (67% Ni, 30% Cu)	**7.7**	**14**
Nickel	**7.2**	**13**
Nylon	**40-60**	**75-100**
Rubber	**70-110**	**130-200**
Steel	**5.5-9.9**	**10-18**
High-strength	**8.0**	**14**
Stainless	**9.6**	**17**
Structural	**6.5**	**12**
Stone	**3-5**	**5-9**
Titanium (Alloys)	**4.5-5.5**	**8 10**
Tungsten	**2.4**	**4.3**
Wrought Iron	**6.5**	**12**

Design Concepts

Failure – State or condition in which the member or structure no longer functions as intended.

Modes of Failure

- Yielding
- Fracture
- Excessive deformation
- Creep
- Buckling
- Fatigue
- Brittle Fracture (Fracture Mechanics)
- Mathematical Analysis
- Probabilistic mechanical design
- Allowable stress design (ASD)

Factor of Safety

$$FS = \frac{Strength}{Stress}$$

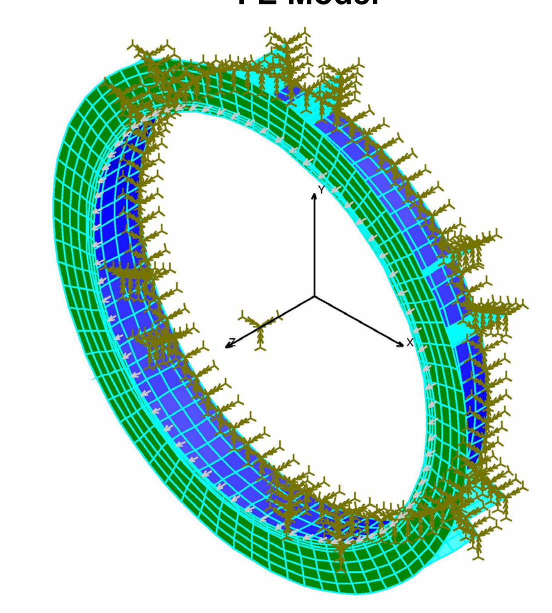
Margin of Safety

$$MS = \frac{Allowable stress}{Stress} - 1$$

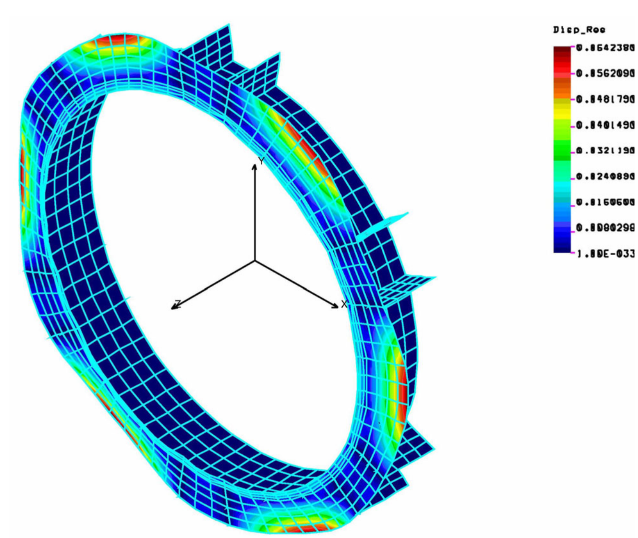
Knife Edge Support Example

- FEA Analysis of Knife Edge Support
- 4 Node Shell Elements
  - Cell Structure
  - Knife Edge
- 3 Node Beam Elements
  - Bolts
- 2g Gravity Load Applied as Pressure to Knife Edge

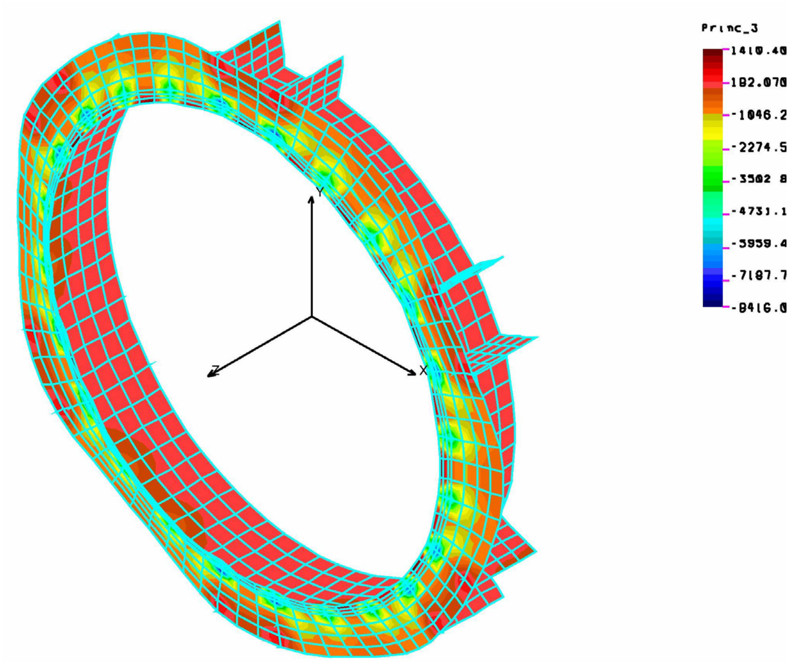
FE Model



Max Displacement. = 0.064 in.



<u>Max Stress = 8416 psi</u>



Margin of Safety
(Ball Aerospace Equation)

$$MS = \frac{AllowableStress}{LimitStress * FS} - 1$$

Gravity Load	Ring Thickness (in)	Principal Stress (psi)	Stress Concentration Factor	Ring Material	Yield Strength (psi)	Ultimate Strength (psi)	Factor of Safety (yield)	Factor of Safety (ultimate)	Margin of Safety (yield)	Margin of Safety (ultimate)
5215	0.500	2207	1.0	6061-T6	40000	45000	3	5	5.04	3.08
5215	0.375	3933	1.0	6061-T6	40000	45000	3	5	2.39	1.29
5215	0.313	5582	1.0	6061-T6	40000	45000	3	5	1.39	0.61
5215	0.250	8416	1.0	6061-T6	40000	45000	3	5	0.58	0.07