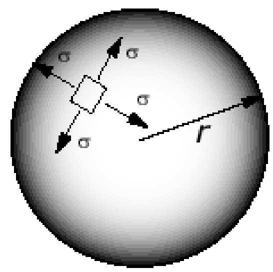
Pressure Vessels Stresses Under Combined Loads Yield Criteria for Ductile Materials and Fracture Criteria for Brittle Materials

Pressure Vessels:

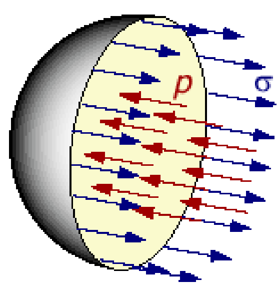
In the previous lectures we have discussed elements subjected to plane stress where $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. Thin-walled pressure vessels are one of the most typical examples of plane stress. When the wall thickness is thin relative to the radius of the vessel, plane stress equations are valid. In addition, since no shear stresses exist, the state of stress can be further classified as a biaxial state of stress.

Spherical Pressure Vessel

Let's begin by considering a spherical pressure vessel with radius "*r*" and wall thickness "*t*" subjected to an internal gage pressure "*p*".



For reasons of symmetry, all the normal stresses on a small stress element in the wall must be identical. Furthermore, there can be no shear stress. The normal stresses σ can be related to the pressure *p* by inspecting a free body diagram of the pressure vessel. To simplify the analysis, we cut the vessel in half as illustrated.



Since the vessel is under static equilibrium, it must satisfy Newton's first law of motion. In other words, the stress around the wall must have a net resultant to balance the internal pressure across the cross-section. Summing forces we obtain:

$$2\pi r t \sigma = p \pi r^2$$

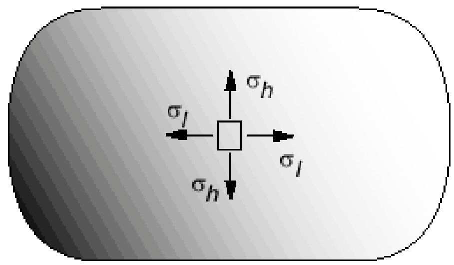
Solving for stress we obtain:

$$\sigma = \frac{pr}{2t}$$

This normal stress is known as the axial, longitudinal or meridional stress.

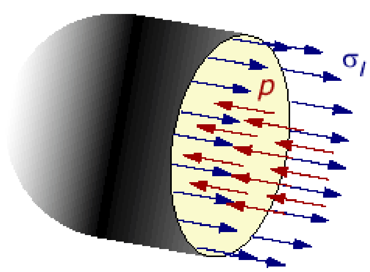
Cylindrical Pressure Vessel

Now let's consider a cylindrical pressure vessel with radius "*r*" and wall thickness "*t*" subjected to an internal gage pressure "*p*".



The coordinates used to describe the cylindrical vessel can take advantage of its axial symmetry. It is natural to align one coordinate along the axis of the vessel (*i.e. in the longitudinal or axial direction*). To analyze the stress state in the vessel wall, a second coordinate is then aligned along the hoop direction (*i.e. tangential or circumferential direction*).

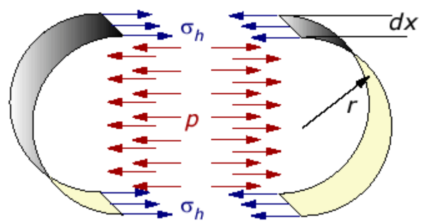
With this choice of axisymmetric coordinates, there is no shear stress. The hoop stress σ_h and the longitudinal stress σ_l are the principal stresses. To determine the longitudinal stress σ_l, we make a cut across the cylinder similar to analyzing the spherical pressure vessel. The free body, shown on the next page, is in static equilibrium. This implies that the stress around the wall must have a resultant to balance the internal pressure across the cross-section.



Summing forces in the longitudinal direction we obtain the same result as with the spherical pressure vessel.

$$\sigma_l = \frac{pr}{2t}$$

To determine the circumferential or hoop stress σ_h, we make a cut along the longitudinal axis and construct a small slice as illustrated below.



Summing forces in the hoop direction we obtain:

$$2\sigma_h t d_x = p 2 r d_x$$

Solving for the hoop stress we obtain:

$$\sigma_h = \frac{pr}{t}$$

In summary we have:

Longitudinal Stress

$$\sigma_l = \frac{pr}{2t}$$

Hoop Stress

$$\sigma_h = \frac{pr}{t}$$

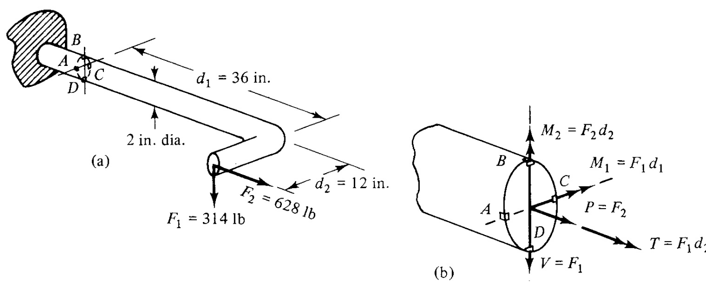
Note: The above formulas are good for thin-walled pressure vessels. Generally, a pressure vessel is considered to be "thin-walled" if its radius r is larger than 5 times its wall thickness t ($r > 5t$).

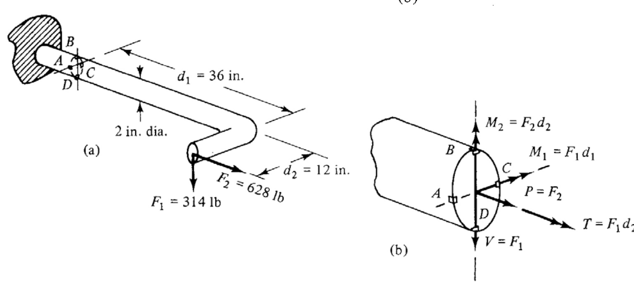
When a pressure vessel is subjected to *external pressure*, the above formulas are still valid. However, the stresses are now *negative* since the wall is now in compression instead of tension.

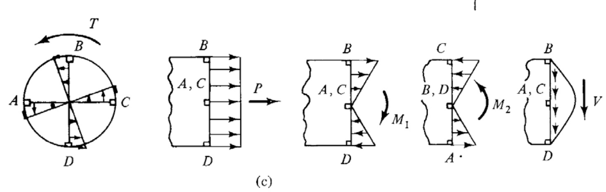
Stresses Under Combined Loads:

To this point we have considered the response of members subjected to the separate effects of axial loads, torsion, bending and uniform pressure. However, in many cases structural members are required to resist more than one type of loading. The stress analysis of a member subjected to **combined loadings** can usually be performed by superimposing the stresses due to each load acting separately. Superposition is permissible if the stresses are linear functions of the loads and if there is no interaction effect between the various loads (i.e. the stresses due to one load are not affected by the presence of any other loads).

Let's consider the following example.







Torsion	$\tau = \frac{Tr}{J}$	2,400 psi
Bending (M_1)	$\sigma = \frac{M_1}{S}$	14,400 psi
Bending (M_2)	$\sigma = \frac{M_2}{S}$	9,600 psi
Axial	$\sigma = \frac{P}{A}$	200 psi
Transverse Shear	$\tau = \frac{4V}{3A}$	133 psi

We can now combine the individual load cases to obtain the stress elements show below:

Torsion	$\tau = \dfrac{Tr}{J}$	2,400 psi
Bending (M_1)	$\sigma = \dfrac{M_1}{S}$	14,400 psi
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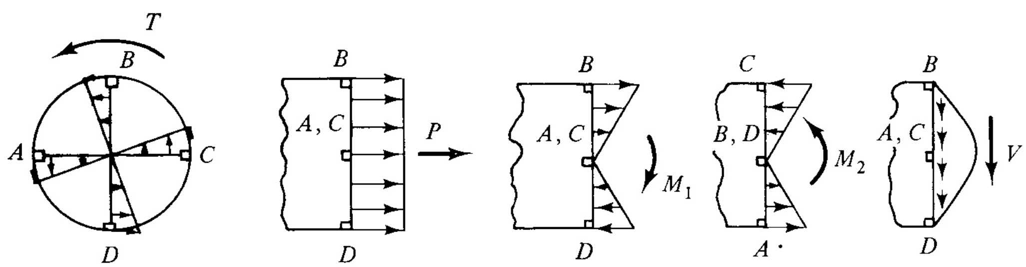

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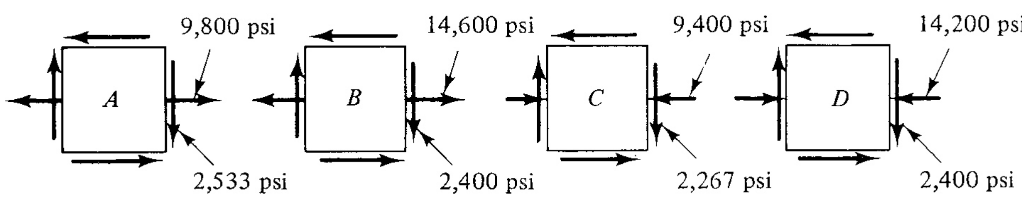


Failure Criteria:

The purpose of failure criteria is to predict or estimate the failure/yield of structural members subjected biaxial or triaxial states of stress.

A considerable number of theories have been proposed. However, only the most common and well-tested theories applicable to isotropic materials are discussed here. These theories, dependent on the nature of the material in question (i.e. brittle or ductile), are listed in the following table:

Material Type	Failure Theories
Ductile	Maximum shear stress criterion, Von Mises criterion
Brittle	Maximum normal stress criterion, Mohr's theory

1. Whether a material is *brittle* or *ductile* could be a subjective guess, and often depends on temperature, strain levels, and other environmental conditions. However, a *5% elongation* criterion at break is a reasonable dividing line. Materials with a larger elongation can be considered ductile and those with a lower value brittle.
   Another distinction is a brittle material's compression strength is usually significantly larger than its tensile strength.
2. All popular failure criteria rely on only a handful of basic tests (such as uniaxial tensile and/or compression strength), even though most machine parts and structural members are typically subjected to multi-axial loading. This disparity is usually driven by cost, since complete multi-axial failure testing requires extensive, complicated, and expensive tests.

Non Stress-Based Criteria:

The success of all machine parts and structural members are not necessarily determined by their strength. Whether a part succeeds or fails may depend on other factors, such as stiffness, vibrational characteristics, fatigue resistance, and/or creep resistance.

For example, the automobile industry has endeavored many years to increase the rigidity of passenger cages and install additional safety equipment. The bicycle industry continues to decrease the weight and increase the stiffness of bicycles to enhance their performance.

In civil engineering, a patio deck only needs to be strong enough to carry the weight of several people. However, a design based on the "strong enough" precept will often result a bouncy deck that most people will find objectionable. Rather, the *stiffness* of the deck determines the success of the design.

Many factors, in addition to stress, may contribute to the design requirements of a part. Together, these requirements are intended to increase the sense of security, safety, and quality of service of the part.

Maximum Shear Stress Criterion:

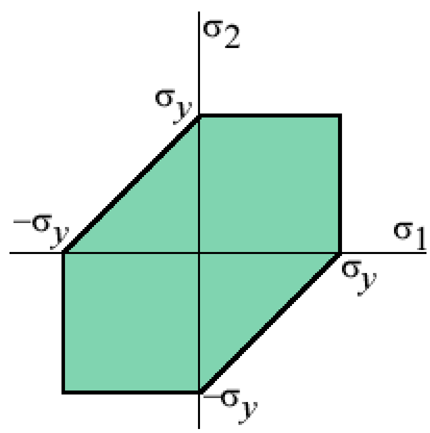
The maximum shear stress criterion, also known as Tresca's or Guest's criterion, is often used to predict the yielding of ductile materials.

Yield in ductile materials is usually caused by the *slippage* of crystal planes along the maximum shear stress surface. Therefore, a given point in the body is considered safe as long as the maximum shear stress at that point is under the yield shear stress obtained from a uniaxial tensile test.

With respect to plane stress, the maximum shear stress is related to the difference in the two principal stresses. Therefore, the criterion requires the principal stress difference, along with the principal stresses themselves, to be less than the **yield shear stress**,

$$|\sigma_1| < \sigma_y, \quad |\sigma_2| < \sigma_y, \quad |\sigma_1-\sigma_2| < \sigma_y$$

As shown below, the maximum shear stress criterion requires that the two principal stresses be within the green zone.



Where:

σ_y = Yield strength of material of uniaxial tension test

Maximum Distortion-Energy (Von Mises) Criterion:

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield of ductile materials.

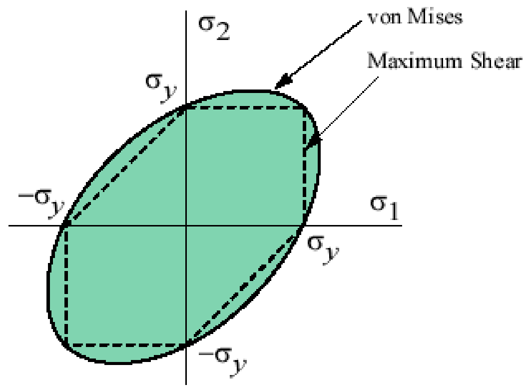
The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension. Mathematically, this is expressed as:

$$\sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} \le \sigma_y$$

In the cases of plane stress, $\sigma_3 = 0$, the Von Mises criterion reduces to:

$$\sqrt{\sigma_1^{\ 2} - \sigma_1\sigma_2 + \sigma_2^{\ 2}} \le \sigma_y$$

As shown below, this equation represents a principal stress ellipse.



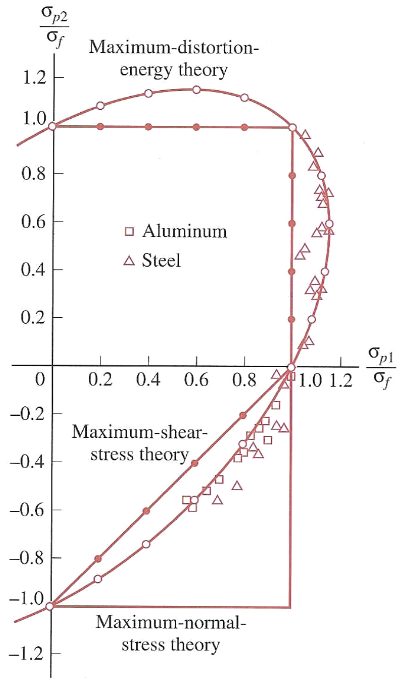
Substituting σ_x, σ_y and τ_{xy} for $\sigma_1$and σ_2 we obtain:

$$\sqrt{\sigma_x^{\ 2} - \sigma_x\sigma_y + \sigma_y^{\ 2} + 3\tau^2} \le \sigma_y$$

Also shown on the previous figure is the maximum shear stress criterion (dashed line). This theory is more conservative than the von Mises criterion since it lies inside the von Mises ellipse.

In addition to bounding the principal stresses to prevent ductile failure, the von Mises criterion also gives a reasonable estimation of fatigue failure, especially in cases of repeated tensile and tensile-shear loading.

Failure Theory Comparison



Maximum Normal Stress Criterion:

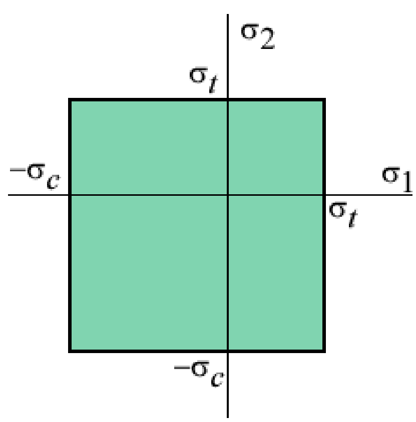
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials.

The maximum stress criterion states that failure occurs when the maximum (normal) principal stress reaches either the *uniaxial* tension strength σ_t, or the *uniaxial* compression strength σ_c,

$$-\sigma_c < \{\sigma_1, \sigma_2\} < \sigma_t$$

where σ_1 and σ_2 are the principal stresses for plane stress.

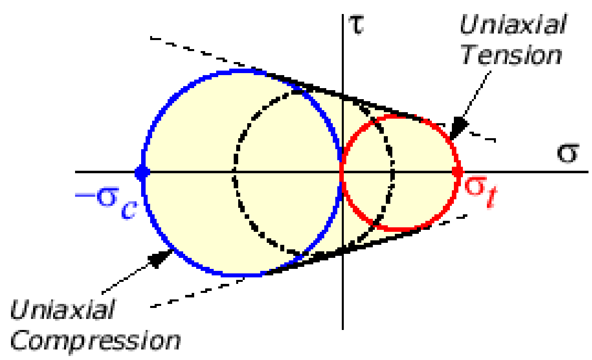
Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone.



Mohr's Theory:

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous Mohr's Circle. Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of plane stress.

Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown below,



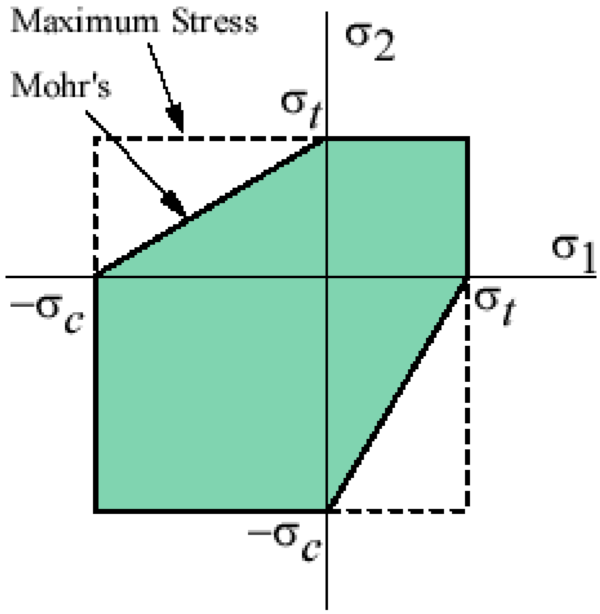
The left circle is for uniaxial compression at the limiting compression stress σ_c of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress σ_t.

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state.

All intermediate stress states fall into one of the four categories in the following table. Each case defines the maximum allowable values for the two principal stresses to avoid failure.

Case	Principal Stresses		Criterion requirements
1	Both in tension	$\sigma_1 > 0, \sigma_2 > 0$	$\sigma_1 < \sigma_t, \sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1 < 0, \sigma_2 < 0$	$\sigma_1 > -\sigma_c, \sigma_2 > -\sigma_c$
3	$\square_1$ in tension, $\square_2$ in compression	$\sigma_1 > 0, \sigma_2 < 0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	$\square_1$ in compression, $\square_2$ in tension	$\sigma_1 < 0, \sigma_2 > 0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

Graphically, Mohr's theory requires that the two principal stresses lie within the green zone depicted below,



Also shown above is the maximum stress criterion (dashed line). This theory is less conservative than Mohr's theory since it lies outside Mohr's boundary.

Failure Theory Comparison