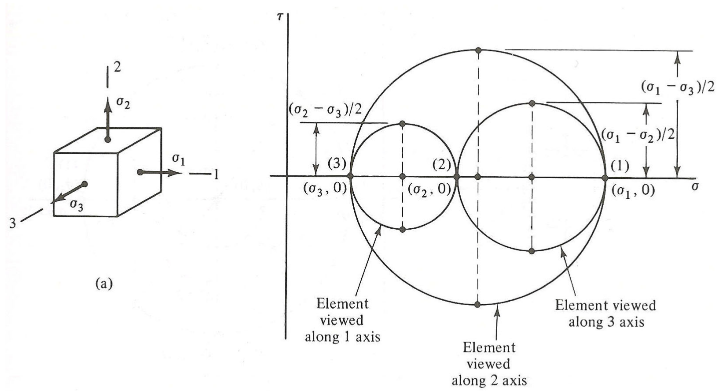
Generalized Hooke's Law

3D Mohr's Circle:

As discussed in the previous lecture, it is important not to lose sight that the material element is a three-dimensional body and we have only been considering a two-dimensional view of it.



It some engineering texts, the maximum shear stress determined by viewing the element from the other principal axes is called the **Absolute Maximum Shear Stress**. The absolute maximum shear stress is the largest of the following three values.

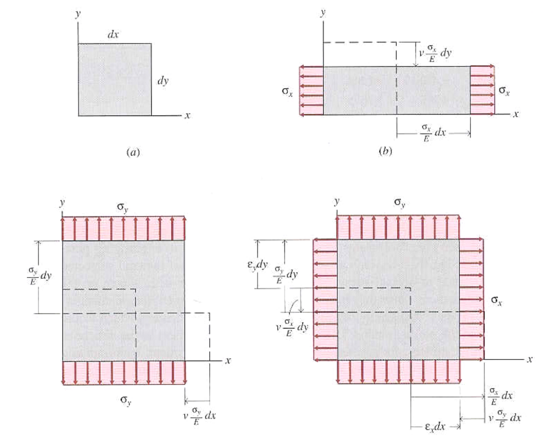
$$\frac{|\sigma_1 - \sigma_2|}{2}$$

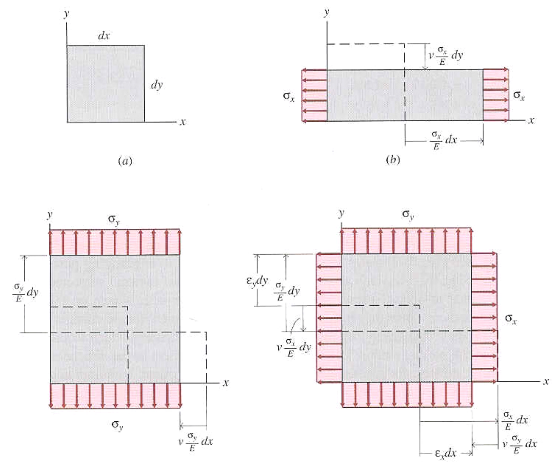
$$\frac{|\sigma_1 - \sigma_3|}{2}$$

$$\frac{|\sigma_2 - \sigma_3|}{2}$$

Hooke's Law for Plane Stress:

In the previous lecture we examined the stresses acting on inclined planes for an element in plane stress. In those discussions we used statics only – the properties of the material were not considered. Now, let's assume that the material is homogeneous and isotropic. Furthermore, let's assume that the material behaves in a linear elastic manner (i.e. Hooke's law holds). Under these conditions, we can readily obtain the relationships between stresses and the strains in the body. Normal strains for a biaxial state of stress may be determined as follows:



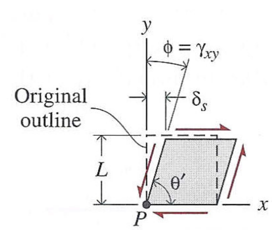


$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \upsilon\sigma_x\right)$$

$$\varepsilon_z = -\frac{\upsilon}{E}\left(\sigma_x + \sigma_y\right)$$

Shown above are the normal strain equations for a biaxial state of stress. The shear stress, τ_{xy}, causes a distortion of the element such that each z face becomes a rhombus. As you may recall, there is no volume change associated with shear strains.



$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

but,

$$G = \frac{E}{2(1+\upsilon)}$$

Substituting we obtain:

$$\gamma_{xy} = \frac{2(1+\upsilon)\tau_{xy}}{E}$$

In many engineering texts you will see the above relationships expressed in matrix form. Note $e_{xy} = \gamma_{xy}$.

$$\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

In many cases, we require an expression for stress. The first two equations for normal strain, ε_x and ε_y, may be solved simultaneously for the stresses in terms of the strains.

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_x + \upsilon\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_y + \upsilon\varepsilon_x\right)$$

$$\tau_{xy} = \frac{E}{2(1+\upsilon)}\gamma_{xy}$$

If we consider the inverted form of the previous matrix we obtain:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix}$$

where:

$$e_{xy} = \gamma_{xy}$$

In many cases it is important to consider thermal effects. If we do so, the previous matrices become.

$$\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} - \frac{E\alpha\Delta T}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

This analysis can be readily extended to a triaxial state of stress.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right]$$

In matrix form we obtain:

$$\begin{Bmatrix} e_x \\ e_y \\ e_z \\ e_{xy} \\ e_{yz} \\ e_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} + \alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}e_x\\ e_y\\ e_z\\ e_{xy}\\ e_{yz}\\ e_{zx}\end{Bmatrix}-\frac{E\alpha\Delta T}{1\text{-}2\nu}\begin{Bmatrix}1\\1\\1\\0\\0\\0\end{Bmatrix}$$

Example Problem:

Problem Statement: At a point on the surface of a titanium alloy machine part subjected to a biaxial state of stress, the measured strains were:

ε_x = +1250 μin./in.

ε_y = +600 μin./in.

τ_{xy} = +650 μrad.

Titanium Specifications:

E = 14,000,000 psi

G = 5,300,000 psi

Determine:

1. The stresses σ_x, σ_y, and τ_{xy} at the point.
2. Determine the principal stresses and the maximum shear stress at the point.