Generalized Hooke’s Law

3D Mohr’s Circle:

As discussed in the previous lecture, it is important not to lose sight that the material element is a three-dimensional body and we have only been considering a two-dimensional view of it.

It some engineering texts, the maximum shear stress determined by viewing the element from the other principal axes is called the **Absolute Maximum Shear Stress**. The absolute maximum shear stress is the largest of the following three values.

$$\frac{\left|\sigma_1 - \sigma_2\right|}{2}$$

$$\frac{\left|\sigma_1 - \sigma_3\right|}{2}$$

$$\frac{\left|\sigma_2 - \sigma_3\right|}{2}$$

**Hooke’s Law for Plane Stress:**

In the previous lecture we examined the stresses acting on inclined planes for an element in plane stress. In those discussions we used statics only – the properties of the material were not considered. Now, let’s assume that the material is homogeneous and isotropic. Furthermore, let’s assume that the material behaves in a linear elastic manner (i.e. Hooke’s law holds). Under these conditions, we can readily obtain the relationships between stresses and the strains in the body. Normal strains for a biaxial state of stress may be determined as follows:
Shown above are the normal strain equations for a biaxial state of stress. The shear stress, $\tau_{xy}$, causes a distortion of the element such that each $z$ face becomes a rhombus. As you may recall, there is no volume change associated with shear strains.

\[
\varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right)
\]
\[
\varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right)
\]
\[
\varepsilon_z = -\frac{\nu}{E} \left( \sigma_x + \sigma_y \right)
\]
\[ \gamma_{xy} = \frac{\tau_{xy}}{G} \]

but,

\[ G = \frac{E}{2(1+\nu)} \]

Substituting we obtain:

\[ \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E} \]

In many engineering texts you will see the above relationships expressed in matrix form. Note \( e_{xy} = \gamma_{xy} \).

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\nu & 0 \\
-\nu & 1 & 0 \\
0 & 0 & 2(1+\nu)
\end{bmatrix} \begin{bmatrix}
e_x \\
e_y \\
e_{xy}
\end{bmatrix}
\]

In many cases, we require an expression for stress. The first two equations for normal strain, \( \varepsilon_x \) and \( \varepsilon_y \), may be solved simultaneously for the stresses in terms of the strains.

\[
\sigma_x = \frac{E}{1-\nu^2} (\varepsilon_x + \nu\varepsilon_y)
\]

\[
\sigma_y = \frac{E}{1-\nu^2} (\varepsilon_y + \nu\varepsilon_x)
\]

\[
\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}
\]

If we consider the inverted form of the previous matrix we obtain:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix} \begin{bmatrix}
e_x \\
e_y \\
e_{xy}
\end{bmatrix}
\]

where:

\( e_{xy} = \gamma_{xy} \)
In many cases it is important to consider thermal effects. If we do so, the previous matrices become:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
= \frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
-\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_y \\
e_{xy}
\end{bmatrix}
- \frac{E\alpha \Delta T}{1-\nu}
\begin{bmatrix}
e_x \\
e_y \\
e_{xy}
\end{bmatrix}
\]

This analysis can be readily extended to a triaxial state of stress.

\[
\epsilon_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right]
\]

\[
\epsilon_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right]
\]

\[
\epsilon_z = \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right]
\]

In matrix form we obtain:

\[
\begin{bmatrix}
e_x \\
e_y \\
e_{xy} \\
e_{xz} \\
e_{yz} \\
e_{zx}
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
+ \alpha \Delta T
\begin{bmatrix}
e_x \\
e_y \\
e_{xy} \\
e_{xz} \\
e_{yz} \\
e_{zx}
\end{bmatrix}
\]
Example Problem:

Problem Statement: At a point on the surface of a titanium alloy machine part subjected to a biaxial state of stress, the measured strains were:

\[ \varepsilon_x = +1250 \ \mu \text{in./in.} \]
\[ \varepsilon_y = +600 \ \mu \text{in./in.} \]
\[ \tau_{xy} = +650 \ \mu \text{rad.} \]

Titanium Specifications:

\[ E = 14,000,000 \ \text{psi} \]
\[ G = 5,300,000 \ \text{psi} \]

Determine:

1. The stresses \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) at the point.
2. Determine the principal stresses and the maximum shear stress at the point.