Analysis of Stress and Strain - Principal Stresses

In the previous lecture we derived the stress transformation equations for stresses aligned with the nt coordinate system. Let's now determine the maximum and minimum normal stresses, which are known as the **principal stresses**.

We begin with the stress transformation equation for normal stress.

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

By taking the derivative of σ_n with respect to θ and setting it equal to zero, we obtain an equation, which can be solved for the values of θ at which σ_n is a maximum or minimum.

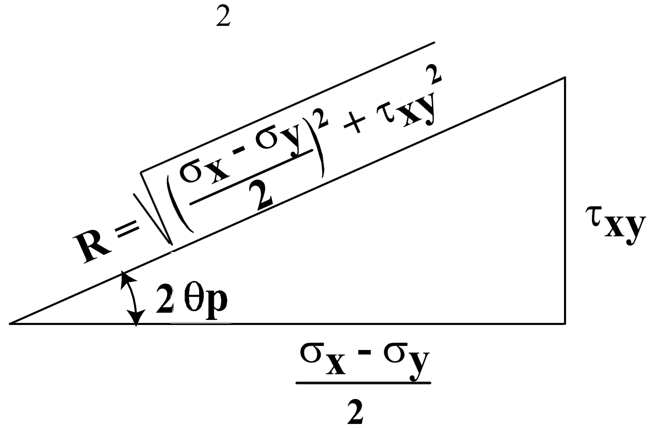
$$\frac{d\sigma_n}{d\theta} = -\left(\sigma_x - \sigma_y\right) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

$$\boxed{\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}}$$

It should be noted that we obtain 2 values for $2\theta_p$ between 0° and 360°. The angles differ by 180°. Therefore θ_p has 2 values that differ by 90°. For one angle we obtain a maximum, for the other a minimum. **Principal stresses occur on mutually perpendicular planes.**

Consider:

$$\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$$



Now:

$$Cos\ 2\theta_p = \frac{\sigma_x - \sigma_y}{2R} \qquad Sin\ 2\theta_p = \frac{\tau_{xy}}{R}$$

Recall:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\ Cos\,2\theta \quad \tau_{xy}\ Sin\,2\theta$$

Substituting we obtain:

$$\boxed{\sigma_{1,2} = \frac{(\sigma x + \sigma y)}{2} \pm \sqrt{\left(\frac{\sigma x - \sigma y}{2}\right)^2 + \tau_{xy}{}^2}}$$

Where:

- σ_1 = Maximum principal stress
- σ_2 = Minimum principal stress

Repeating for shear stresses we obtain:

$$\tau_{n\tau} = \frac{-(\sigma_x - \sigma_y)}{2}\ Sin\ 2\theta + \tau_{xy}\ Cos\ 2\theta$$

$$\frac{d\ \tau_{n\tau}}{d\theta} = -(\sigma_x - \sigma_y) Cos\ 2\theta - 2\ \tau_{xy}\ Sin\,2\theta = 0$$

$$\boxed{\tan 2\theta_\tau = -\frac{\sigma_x - \sigma_y}{2\ \tau_{xy}}}$$

Substituting we obtain:

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

or

$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}}$$

Note:

1) Shear stresses are zero on principal planes
2) Principal stresses occur on mutually perpendicular planes.
3) Planes of maximum shear stress occur at 45° to the principal planes.
4) The maximum shear stress is equal to ½ the difference of the principal stresses

In summary:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$