Combined Loading

To this point, we have considered the response of members subjected to the separate effects of axial load, direct shear, bending, torsion and transverse shear. The stressed given by these formulas, shown below, act over the cross section of the member and provide the stresses acting upon certain planes, usually parallel and perpendicular to the longitudinal axis of the member.

$$\sigma = \frac{P}{A} \qquad \tau = \frac{V}{A} \qquad \sigma_b = \frac{My}{I} \qquad \tau = \frac{Tr}{J} \qquad \tau = \frac{VQ}{Ib}$$

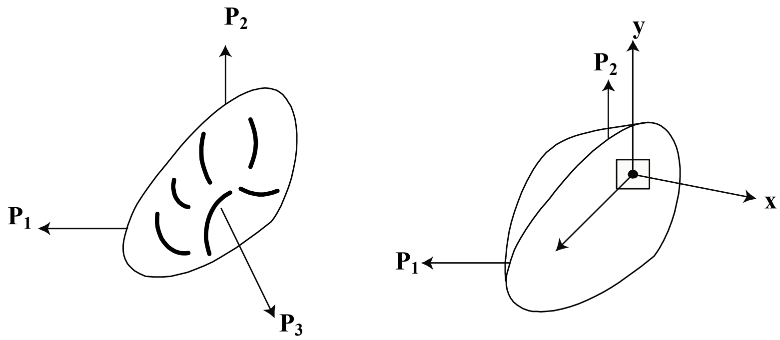
However, in practice loads often act simultaneously on a structural member. Depending on the point of interest in the body, the stresses may combine to produce a higher or lower state of stress at the point. For example, consider a cantilevered beam subjected to a transverse bending load and an axial tension load. Bending stresses vary linearly over the cross section with the maximum stresses occurring at the outer fibers near the support. One side of the beam will be in tension and the opposite side will be in compression. In addition, the direction of the bending stresses is aligned with the longitudinal axis of the beam. The stress due to the axial load is positive and uniform over the cross section and its direction is also aligned with the longitudinal axis of the beam. Therefore:

$$\sigma_{point} = \frac{My}{I} \pm \frac{P}{A}$$

As stated above, these stresses act on a plane normal to the cross section of the member and are directed along the length of the member. How does one determine the stresses on planes other than those for which these basic equations apply? Our approach uses **stress elements** to represent the state of stress at a point in a body. The element is an infinitesimally small cube of material at the point of interest in the body. Typically, the edges of the element are parallel to the x, y, and z axes of the structure. Our objective is to derive the transformation relationships that give the stress components for any orientation of these axes. In other words, we wish to determine the stresses acting on the sides of a stress element rotated to any desired position. This process is referred to as a **transformation of axes** or a **stress transformation**.

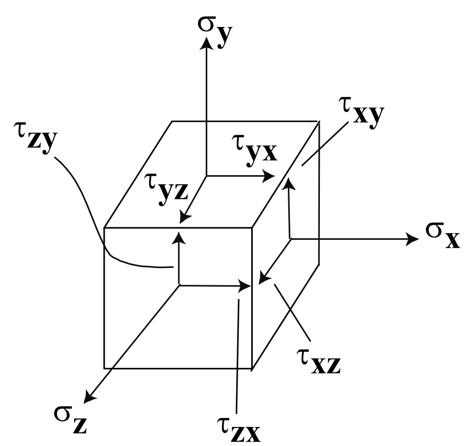
3D Stress At A Point

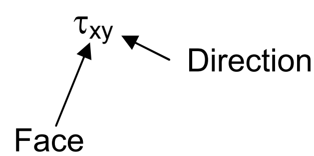
Let us now consider an arbitrary body with loads P_1, P_2 and P_3 acting on the body. The point of interest is inside the body with a reference Cartesian coordinate system positioned at the point.



As shown below, three (3) normal stresses may act on faces of the cube, as well as, six (6) components of shear stress. Equilibrium of the cube requires that:

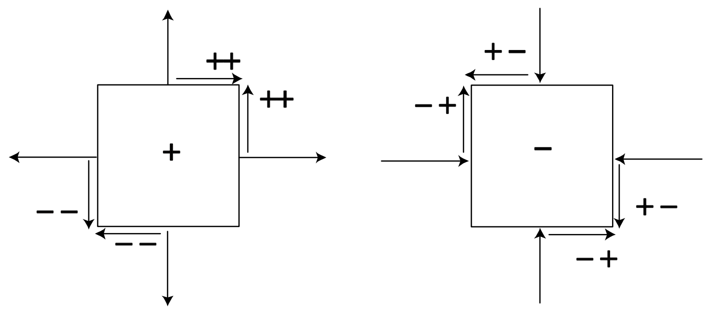
$$\tau_{xy} = \tau_{yx}$$
$$\tau_{xz} = \tau_{zx}$$
$$\tau_{zy} = \tau_{yz}$$





Therefore the previous nine (9) components of stress reduce to the six (6) components of stress listed on the page 90. It should be noted that a shear stress, τ, has two subscripts; the first denotes the face on which the stress acts and the second gives the direction on that face. For example, the stress τ_{xy} acts on the x face in the direction of the y axis. In addition, shear stresses have a sign, which must be considered. The sign convention, shown below, is easily remembered.

Shear Stress Sign Convention

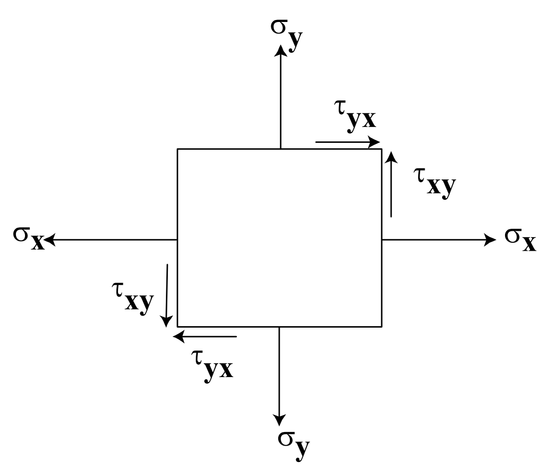


Positive Shear   Negative Shear

Positive Shear = positive face, positive direction or negative face, negative direction
Negative Shear = positive face, negative direction or negative face, positive direction

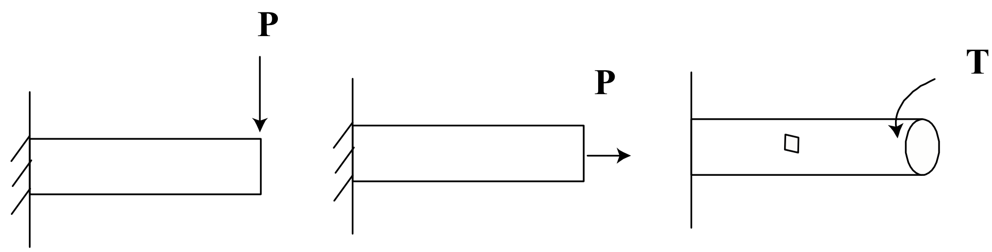
3D Stress Components

9 Components			6 Components
σ_x	Outward Normals	→	σ_x
σ_y			σ_y
σ_z			σ_z
τ_{xy}			τ_{xy}
τ_{xz}			τ_{yz}
τ_{yx}			τ_{zx}
τ_{yz}			
τ_{zx}			
τ_{zy}			



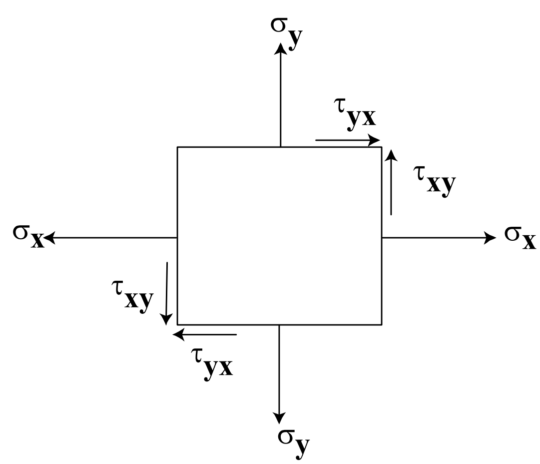
Plane Stress Element

The stress conditions encountered in axially loaded bars, beams, shafts in torsion, beams and thin plates are examples of a state of stress called plane stress.



$$\sigma_z = \tau_{xz} = \tau_{zy} = 0$$

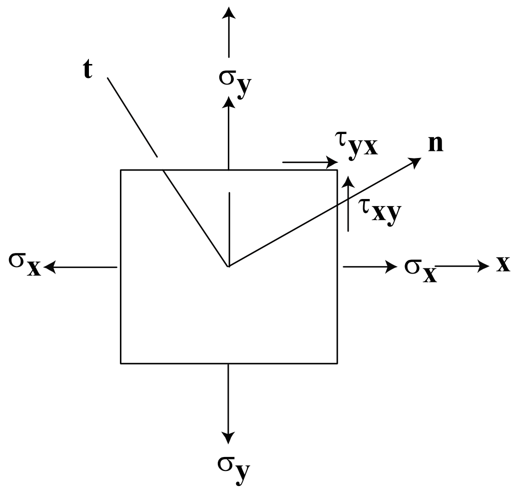
In plane stress all stresses act in the xy plane, parallel to the x and y axes. All out of plane stresses (σ_z, τ_{zx} and τ_{zy}) are considered to be zero.



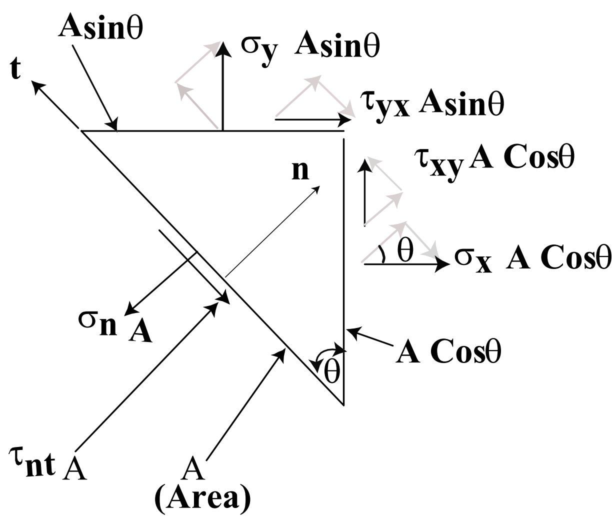
Stress Transformation

As mentioned above, when combined loads action simultaneously on a structural element, the maximum stress may occur on a different plane than the one associated with our basic equations. In other words, the stress on an inclined plane may be larger than the stresses acting on the cross section. Our job is to determine the maximum stress and the plane on which it occurs. It should be emphasized that only one intrinsic state of stress exists at a point regardless of the orientation of stress element being used to describe that state of stress. As the element is rotated from one orientation to another, the stresses acting on the faces of the element are different, but they still represent the same state of stress. This situation is analogous to the representation of a force vector by its components. When the axes are rotated the force is represented by different components, but the force itself is unchanged.

Consider the following plane stress element with the stresses acting as shown. All stresses are parallel to the x and y axes. We will now derive the stresses associated with the nt coordinate system shown below.



F = σA



Summing forces in the n direction we obtain:

$$\sum F_n = -\sigma_n A + \left(\sigma_x A C\text{os}\theta\right) C\text{os}\theta + \left(\tau_{xy} A C\text{os}\theta\right) S\text{in}\theta$$
$$+ \left(\tau_{xy} A S\text{in}\theta\right) C\text{os}\theta + \left(\sigma_y A S\text{in}\theta\right) S\text{in}\theta = 0$$

$$\boxed{\sigma_n = \sigma_x C\text{os}^2\theta + \sigma_y\ Sin^2\theta + 2\,\tau_{xy}\ S\text{in}\theta\ C\text{os}\theta}$$

Summing forces in the t direction we obtain:

$$\sum F_t = -\tau_{nt}A - (\sigma_x\ A\ Cos\theta)\ Sin\theta + (\tau_{xy}\ A\ Cos\theta)Cos\theta$$
$$-(\tau_{xy}\ A\ Sin\theta)Sin\theta + (\sigma_y\ A\ Sin\theta\ )Cos\theta = 0$$

$$\boxed{\tau_{nt} = -(\sigma_x - \sigma_y)Sin\theta\ \tau_{xy}\left(Cos^2\theta - Sin^2\theta\right)}$$

Applying Double Angle Identities we obtain:

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\ Cos2\theta\ +\ \tau_{xy}\ Sin2\theta$$
$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)Sin2\theta + \tau_{xy}\ Cos2\theta$$

Uniaxial

$$\sigma_y = \tau_{xy} = 0$$
$$\sigma_n = \sigma_x\ Cos^2\theta$$
$$\tau_{nt} = -\sigma_x\ Sin\theta\ Cos\theta$$

But $Cos^2\theta = \frac{1}{2} + \frac{1}{2}\ Cos2\theta\ \ + Sin\ Cos\theta = \frac{1}{2}\ Sin2\theta$

$$\therefore\ \sigma_n\ =\ \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\ \ Cos2\theta$$

$$\boxed{\sigma_n = \frac{\sigma_x}{2}(1 + Cos2\theta)}$$

$$\boxed{\tau_{nt} = \frac{\sigma_x}{2}Sin2\theta}$$

Pure Shear

$$\sigma_x = \sigma_y = 0$$
$$\sigma_n = 2\,\tau_{xy}\ Sin\theta\, Cos\theta$$
$$\tau_{n\tau} = \tau_{xy}\left(Cos^2\theta - Sin^2\theta\right)$$

or

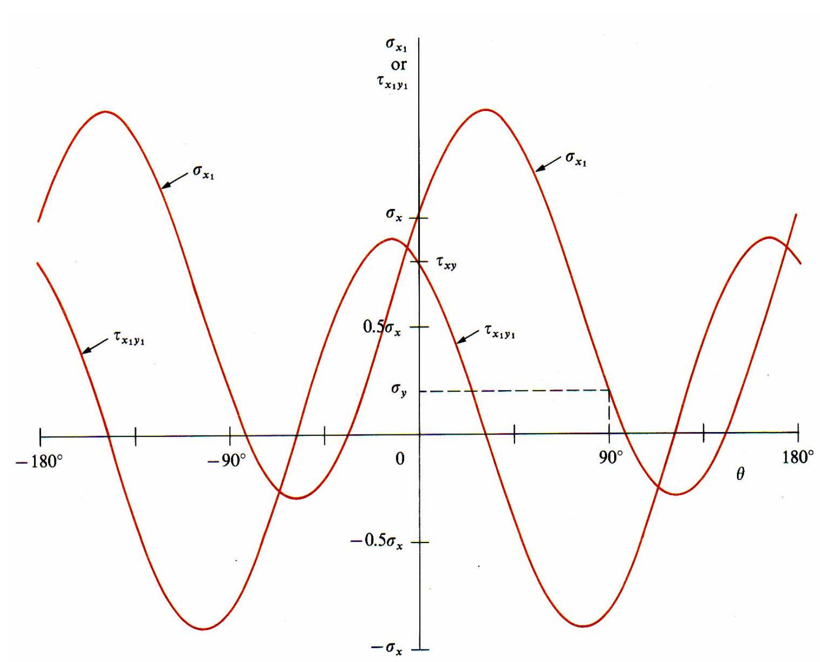
$$\sigma_n = \tau_{xy}\, Sin2\theta$$
$$\tau_{nt} = \tau_{xy}\ Cos2\theta$$

Biaxial Stress

$$\tau_{xy} = 0$$

$$\sigma_n = \sigma_x Cos^2\theta + \sigma_y\ Sin^2\theta$$
$$\tau_{nt} = -\left(\sigma_x = \sigma_y\right) Sin\theta\, Cos\theta$$

Graph of Normal Stress and Shear Stress Vs Theta



$\sigma_y = 0.2\ \sigma_x$

$\tau_{xy} = 0.8\ \sigma_x$