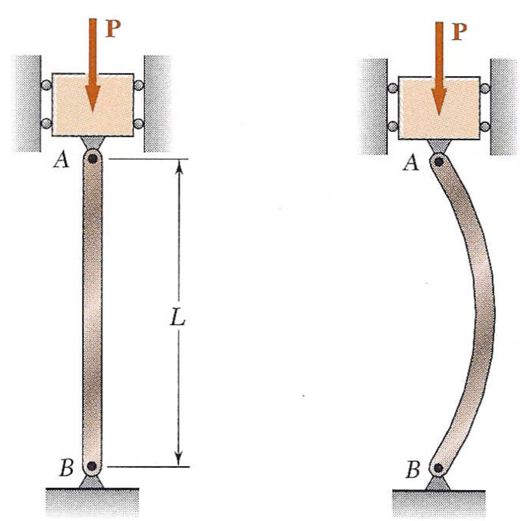
Buckling

Buckling and Stability:

As we learned in the previous lectures, structures may fail in a variety of ways, depending on the materials, load and support conditions. We had two primary concerns: (1) the **strength** of structure, (*i.e. its ability to support a given load without excessive stress*) or (2) the **stiffness** of a structure, (*i.e. its ability to support a given load without excessive deformation*). We will now consider the **stability** of a structure – its ability to support a given load without experiencing a sudden change in its configuration or shape. Our discussion will be primarily related to the analysis and design of columns. A **column** is a straight, slender member subjected to an axial **compressive** load. Such members are commonly encountered in trusses and in the framework of buildings, but may also be found in machine linkages, machine elements and optical systems.

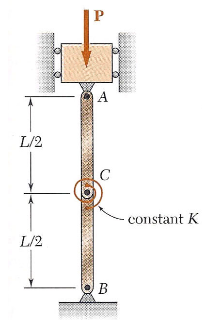
If a compression member is relatively short, it will remain straight when loaded and the load-deformation relations previously presented will apply. However, if the member is long and slender, **buckling** will be the principle mode of failure. Instead of failing by direct compression, the member bends and deflects laterally and we say the member has **buckled**. In other words, when the compressive loads reach a certain critical value, the column undergoes a bending action in which the lateral deflection becomes very large with little increase in load. Failures due to buckling are frequently catastrophic because they occur suddenly with little warning when the critical load is reached.

To illustrate the phenomenon of buckling in an elementary manner consider the following idealized structure. The member is sized such that the design stress, $\sigma = P/A$, is less than the allowable stress for the material and the deformation, $\delta = PL/AE$ is within the given specifications. We may conclude that the column has been properly designed. However, before the design load is reached, the column may buckle and become sharply curved – clearly an indication that the column has not been properly designed.

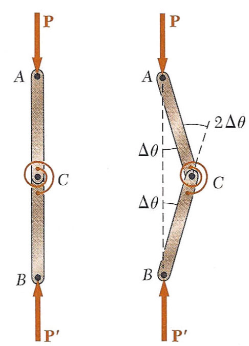


Stability of Structures:

We may gain insight into this problem by considering a simplified model consisting of two rigid rods AB and BC connected at C by a pin and a torsional spring of constant K.



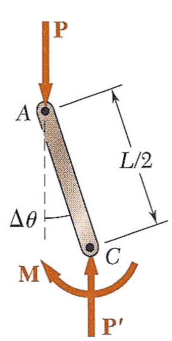
As shown below, if the two rods and the two forces **P** and **P'** are perfectly aligned, the system will remain in the equilibrium position, (*a*). However, if joint C is displaced slightly to the right so that each rod now forms a small angle $\Delta\theta$ with the vertical, (*b*), will the system be **stable** or **unstable**?



(*a*) (*b*)

If joint C returns to the equilibrium position, the system is considered to be stable. Otherwise, if joint C continues to move away from the equilibrium position, the system is unstable.

We can determine if the system is stable or unstable by considering the forces acting on rod AC. The forces consist of two couples, namely the couple formed by P and P' and the moment M.



The moment, which tends to move the rod away from the vertical, is:

$$P\,(L/2)\sin\Delta\theta$$

Couple M, exerted by the spring, tends to bring the rod back to the vertical position. Since the angle of deflection of the spring is $2\Delta\theta$, the moment of couple M is:

$$M = K\,(2\Delta\theta)$$

If the moment of the second couple is larger than the moment of the first couple, the system tends to return to its equilibrium position and is stable. However, it the moment of the first couple is larger than the moment of the second couple, the system tends to move away from the equilibrium position and is unstable. The value of the load for which the two couples balance each other is called the **critical load** and is denoted by P_{cr}. Equating the two we have:

$$P_{cr}\,(L/2)\sin\Delta\theta = K\,(2\Delta\theta)$$

Applying small angle theory:

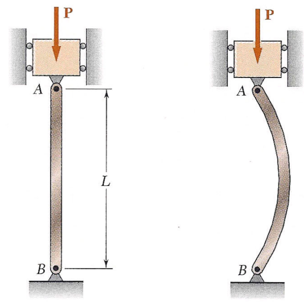
$$P_{cr} = 4K/L$$

The system is stable for $P < P_{cr}$ (*values of load less than the critical value*) and unstable for $P > P_{cr}$.

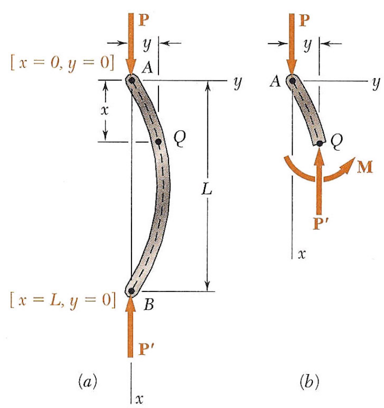
<u>Euler's Formula for Beams with Pin-Pin Connections:</u>

To investigate the stability behavior of columns, we will begin by considering a long slender column with pinned joints at each end. The column is loaded with a vertical force P that is applied through the centroid of the cross section and aligned with the longitudinal axis of the column. In addition, the loading is conservative *(i.e. the force is guided and remains vertical*). The column itself is perfectly straight and is made of a linearly elastic material that follows Hooke's law. Thus, it is considered to be an **ideal column**.

When the axial load P has a small value, the column remains straight and undergoes only axial compression, $\sigma = P/A$. This straight form of equilibrium is **stable** and if disturbed, the column will return to the straight position. As the axial load increases, we will reach a condition of **neutral equilibrium** in which the column will have a bent shape. The corresponding load is the **critical load**, P_{cr}. At this load, the ideal column may undergo small lateral deflections with no change in axial force and a small lateral force will produce a bent shape that does not disappear when the lateral load is removed. Thus, the critical load can maintain the column in static equilibrium in either the straight position or in a slightly bent condition. At higher values of load, the column is **unstable** and will collapse by bending.



Now lets examine a free body diagram of the bend column. Recalling that a beam is a structural member whose length is large compared to its cross sectional area, a column can be considered as a vertical beam subjected to an axial load. As shown below, *x* denotes the distance from the end of the column to a given point *Q* of its elastic curve and y denotes the defection of that point. This is identical to our beam of previous lectures, however the *X* axis in now vertical and directed downward and the *Y*-axis is horizontal and directed to the right.



Summing moments about point Q, we have M = -Py. Therefore:

$$EI\frac{d^2y}{dx^2} = M = -Py$$

Rearranging we have:

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

Which is a linear, homogeneous differential equation of second order with constant coefficients. Setting

$$k^2 = \frac{P}{EI}$$

and introducing prime notation, we obtain:

$$y'' + k^2 y = 0$$

The general solution of this equation is

$$y = C_1 \sin kx + C_2 \cos kx$$

To evaluate the constants of integration we use the boundary conditions at the ends of the beam.

$$y(0) = 0 \text{ and } y(L) = 0$$

The first yields $C_2 = 0$ and the second yields $C_1 \sin kL = 0$. From this we conclude:

$$C_1 = 0 \ (\textit{trivial solution})$$

or

$$\sin kL = 0 \ (\textit{nontrivial solution})$$

Therefore:

$$kL = n\pi$$

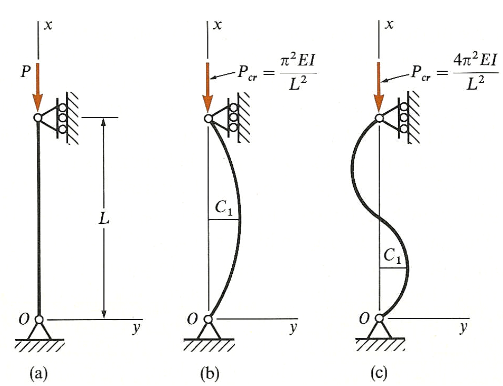
Substituting for k we obtain:

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad n = 1,2,3\ldots$$

The smallest critical load for the column is obtained when n=1:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

The above expression is known as **Euler's formula** and the critical load is also known as the **Euler load**. The corresponding buckled shape is also called the **mode shape**. Buckling of a pinned-end column in the first mode (*n = 1*) is called the fundamental case of column buckling.



Returning to the differential and substituting for P we obtain:

$$y = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L} \quad \text{n=1,2,3...}$$

For the first mode of buckling (*n = 1*):

$$y = C_1 \sin \frac{\pi x}{L}$$

As shown of the previous page, the column buckles into a sine wave and the constant C_1 represents the maximum deflection of the midpoint of the column. The value of C_1 is indeterminate and can be obtained from a nonlinear analysis. This is typically of little concern because we are typically interested only in the value of the critical load.

The value of stress corresponding to the critical load is called the critical stress and is denoted by σ_{cr}. Therefore:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2}$$

Recalling from a previous lecture that the moment of inertia I = r^2A, where r is the radius of gyration. Substituting for I we obtain:

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$$

The quantity L/r is called the slenderness ratio of the column.

As shown on the previous page, the critical stress is proportional to E and inversely proportional to $(L/r)^2$. For a constant value of E (*E = 250 GPa*), we can plot stress vs. L/r.