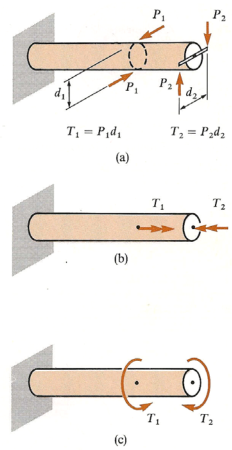
Torsion

Torsion:

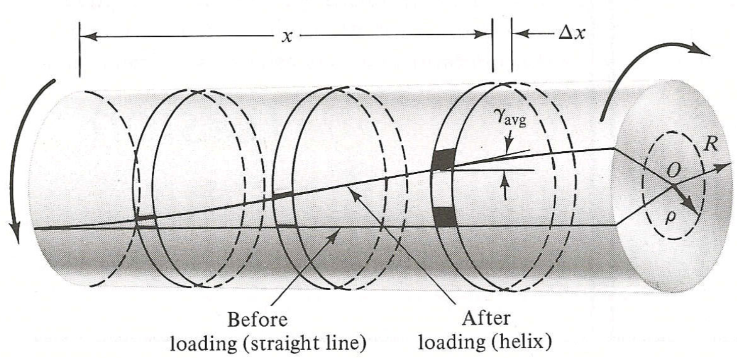
Torsion refers to the twisting of a structural member that is loaded by couples (torque) that produce rotation about the member's longitudinal axis. In other words, the member is loaded in such a way that the stress resultant is a couple about the longitudinal axis and the response is a twisting motion about that axis.

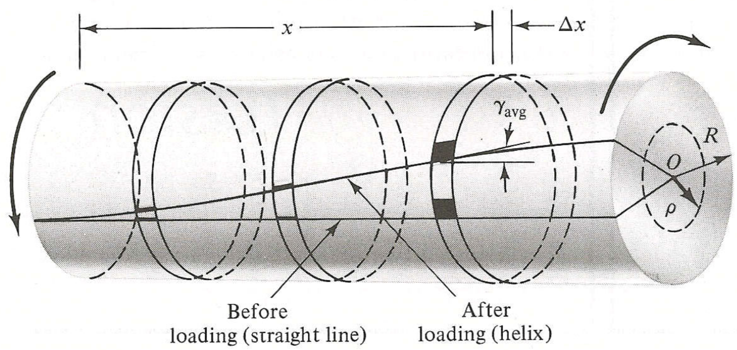
Couples that produce twisting of a bar are called **torques**, **twisting couples** or **twisting moments** and may be represented in several ways.



Deformations in a Circular Shaft:

Experiments have shown that when a circular shaft made from a homogeneous and isotropic material is twisted by couples it will deforms in such a way that planes perpendicular to its axis before loading remain plane and perpendicular to the axis after loading. In addition, radial lines in the cross section remain radial and the length does not change appreciably. Another analogy is to think of the shaft as a series of thin disks that rotate slightly with respect to each other as the shaft is twisted.

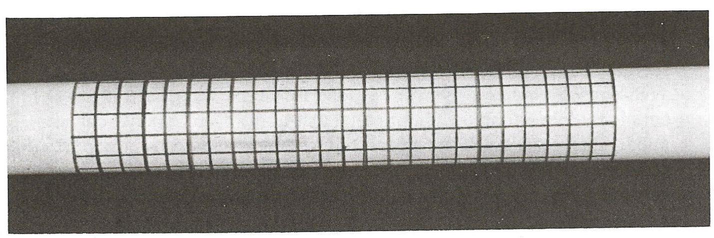




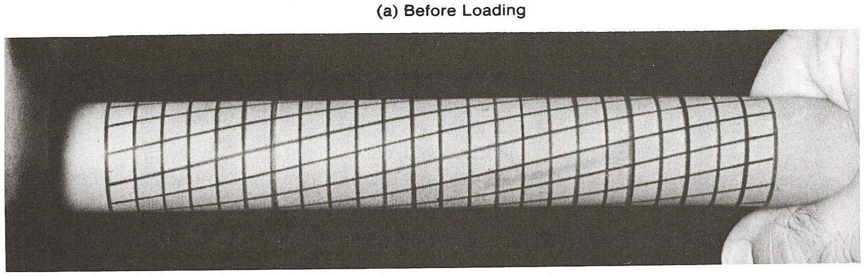
In the above example, each disk rotates slightly with respect to the previous disk and plane sections remain plane. The rotation of each disk with respect to the previous disk causes the shaded element that was originally square to change shape.

When a bar or shaft of circular cross section is loaded in this manner (*twisted by couples*) the bar is said to be in **pure torsion** and the deformed element, shown above, are said to be in a state of **pure shear**.

As shown below, lines parallel with the axis of the shaft distort into helixes. Lines perpendicular to the axis of the shaft remain perpendicular. Therefore, plane sections remain plane and undistorted.



(a) Before Loading

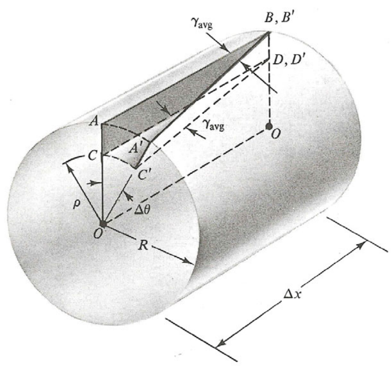


(b) After Loading

The above assumptions are correct for circular bars or shaft whether solid or hollow (i.e. round tubes) but are incorrect for other shapes. Other shapes, such as a rectangular bar, distort when twisted (*i.e. plane sections do not remain plane*).

Strains in a Circular Shaft:

Deformations of a circular shaft due to pure torsion can be related to the strains by considering a short segment of the shaft with length Δx.



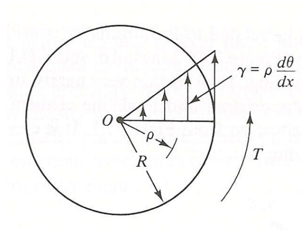
$$\gamma_{avg} \approx \tan\gamma_{avg} = \frac{CC'}{\Delta x} = \frac{\rho\Delta\theta}{\Delta x}$$

The shear strain at a point is obtained by taking the limit as $\Delta x \to 0$:

$$\gamma = \lim_{\Delta x \to 0} \frac{\rho\Delta\theta}{\Delta x} = \frac{\rho d\theta}{dx}$$

$d\theta/dx$ represents the rate of change of the angle of twist θ

The previous equation shows that shear strain varies linearly with the radius and reaches a maximum at the outer surface of the shaft ($\rho = R$).



If the shaft is uniform and the torque does not vary along its length, the previous equation can be integrated directly to obtain:

$$\theta = \int_0^L \frac{\gamma}{\rho} dx = \frac{\gamma L}{\rho}$$

Rearranging and letting ρ= R

$$\gamma = \frac{R\theta}{L}$$

Where:

R = Outer radius of shaft

θ = Angle of twist (radians)

γ = Shear strain

L = Total length of shaft

Note that the preceding equations are based only geometric concepts and are valid for a circular bar of any homogeneous and isotropic material. In other words, the strain distribution is independent of the resistive properties of the material. There are, however, several conditions that must be satisfied for shear strain equation to be valid.

1. The member must be straight or nearly so and circular.
2. The member must be loaded with a couple about the longitudinal axis.
3. The cross-sectional area must be constant or nearly so.
4. The section of interest must be away from connections, supports and the load application points (St. Venant's Principle).
5. The material must be homogeneous and isotropic.

Shear Stresses in a Circular Shaft:

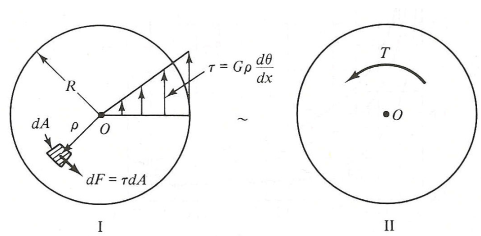
Recall from our discussion on material properties, for a homogeneous and isotropic material with stresses below the elastic limit, the following stress-strain relationship applies:

$$G = \frac{\tau}{\gamma}$$

Therefore:

$$\tau = G\gamma = G\rho\frac{d\theta}{dx} \tag{1}$$

Now, the shear stresses can be related to the stress resultant (torque).



$$\int \rho\tau dA = T \tag{2}$$

Substituting Equation (1) into Equation (2) we obtain:

$$T = \int \rho\left(G\rho\frac{d\theta}{dx}\right)dA = G\frac{d\theta}{dx}\int \rho^2 dA$$

From previous lectures we determined that:

$$J = \int \rho^2 dA$$

Where J is the polar moment of inertia. Substituting we obtain:

$$T = GJ\frac{d\theta}{dx} \tag{3}$$

Finally, combining Equations (1) and (3) we obtain the torque-stress relationship for a circular bar in pure torsion.

$$\tau = \frac{T\rho}{J}$$

In summary we have:

$$\gamma = \frac{\rho\theta}{L} \tag{4}$$

$$\tau = \frac{T\rho}{J} \tag{5}$$

$$G = \frac{\tau}{\gamma} \tag{6}$$

Frequently in many structural problems, including optics, the angle of twist, θ, is of paramount importance.

Substituting Equations 4 and 5 into Equation 6 we obtain:

$$\theta = \frac{TL}{GJ}$$

Where:

θ = Angle of twist (radians)
T = Applied torque (moment)
L = length of member
G = Shear modulus of material
J = Polar moment of inertia

As seen above the angle of twist is proportional to the torque applied to the shaft.