Calculation of Image Rotation for a Scanning Optical System

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Summary

The scanning device in Figure 1 can scan an area in the object space that has been collimated by a lens and have all the data collected by a single sensor. However as the mirrors rotate the image on the sensor rotates as well.

Older scanning designs have been updated to correct for this. To correct this problem it is necessary to quantify the image rotation as the optical system scans. The Rank Organisation Ltd. holds a US patent, 4,202,597, that corrects for image rotation of an older design, US patent 4,106,845. A simplified version of this older scanner design is used in this tutorial.

The simplified design replaces the rotating polygon mirror with a single pivoting flat mirror. The other non-rotating optics are also removed from the system.

Figure 1 The Rank Organisation Ltd. Scanner Top View (above) and Side View (below)

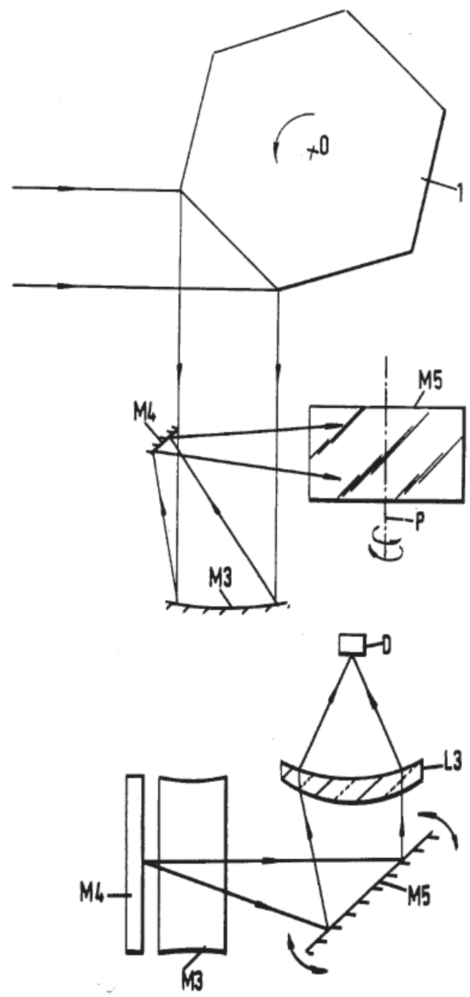


Figure 2 Simplified Scanner Top View

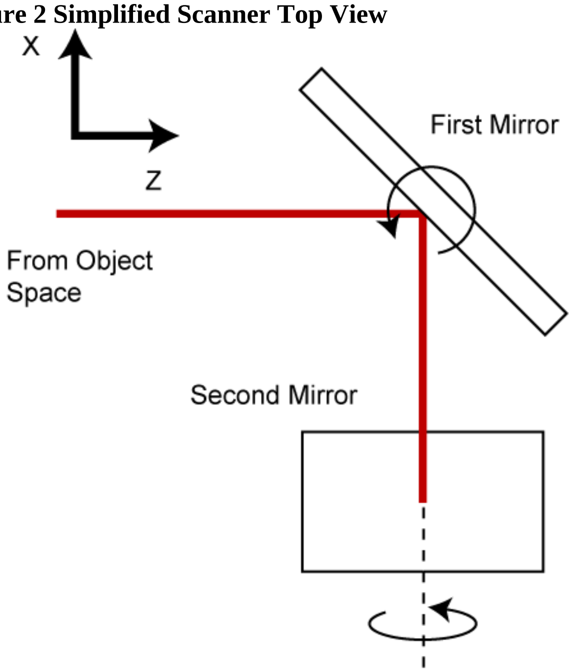
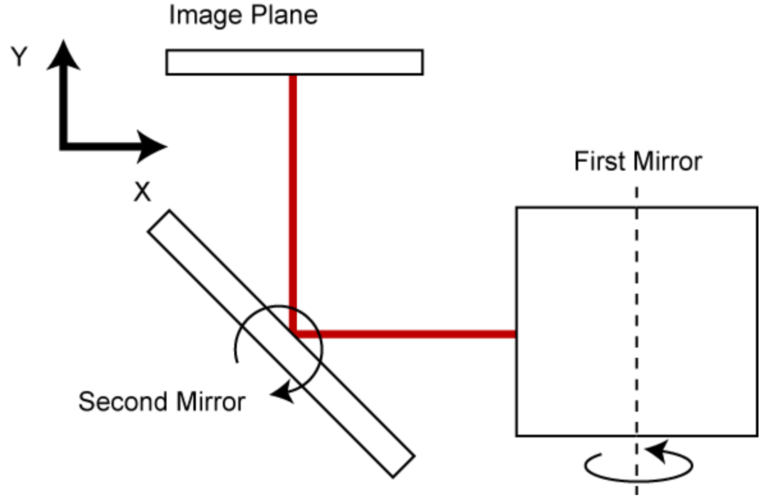


Figure 3 Simplified Scanner Side View



The first thing to do is to sketch the system, taking into account the location and orientations of all the components: the mirrors, object, and image plane. Also it is important to know the axis of rotations of the rotating mirrors.

Once all this is known a global coordinate system must be determined for the entire system. In this case the height of the object will be along the Y axis. The ray from the object will travel along the Z axis as it enters the system. The X axis with be perpendicular to the other two axis. The first mirror will rotate about the Y axis and will be nominally located at an angle of 45° from the Z axis in the X-Z plane. This steers the light to the second mirror that will rotate about the Z and is nominally located at 45° from the Y axis in the X-Y plane. The image plane is located in the X-Z plane.

The global coordinate system should be chosen to simplify calculations. It can be transformed at a later time to another system by using rotation matrices. If it is not feasible to create a simple global coordinate system it may be necessary to determine the mirrors axis of rotations in a local coordinate system and then transform them to the global coordinate system.

Equation 1 X-axis unit vector: $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

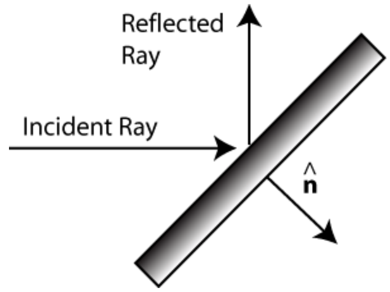
Equation 2 Y-axis unit vector: $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Equation 3 Z-axis unit vector: $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In order to begin the calculations the mirror matrices are needed. They are defined by the mirrors surface normal vector. The surface normal vector of a mirror is defined by the surface normal ray projecting opposite of the reflective surface as shown in Figure 4. These vectors should be scaled to unit vectors. In this example the mirrors' normal vectors are:

Equation 4 Mirror 1normal vector: $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **Equation 5 Mirror 2 normal vector:** $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

Figure 4 Reflection and Surface Normal Vector



For this example the mirror matrices are found with the mirrors in their initial state. The mirror matrices are defined from their surface normal vectors by:

Equation 6 Mirror Matrix: $\mathbf{M} = \mathbf{I} - 2\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}^{\mathbf{T}}$.

Using the equation above and previously determined normal vectors the mirror matrices for the two mirrors are:

Equation 7 Mirror 1 Mirror Matrix: $\mathbf{M}_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Equation 8 Mirror 2 Mirror Matrix: $\mathbf{M}_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In order to perturb the mirrors about their axis a matrix transformation is computed using a rotation matrix. The rotated mirror matrix is equal to:

Equation 9 Mirror Rotation Transformation: $\mathbf{M}_{\mathbf{R}} = \mathbf{R} \cdot \mathbf{M} \cdot \mathbf{R}^{\mathbf{T}}$
where M_R is the rotated mirror matrix, R is the rotation matrix, and M is the initial mirror matrix.

The rotation matrices for the mirrors are:

Equation 10 Mirror 1 Rotation about Y-axis $R_1 = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}$

Equation 11 Mirror 2 Rotation about X-axis $R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix}$.

There are several methods to determine rotation matrices. Euler Parameters may be used to determine the rotation matrix for cases where the axis of rotation is not along one of the axis. If the axis of rotation is a unit vector $\hat{\mathbf{n}}$ and the angle of rotation about that axis is ϕ, then the rotation matrix elements (here in Einstein summation) are:

Equation 12 Rotation Matrix Elements from Euler Parameters:
$r_{ij} = \delta_{ij}\left(e_0^2 - e_k e_k\right) + 2e_i e_j + 2\varepsilon_{ijk} e_0 e_k$
where δ_{ij} is the Kronecker delta, and ε_{ijk} is the permutation symbol.

The Euler parameters are defined by:

Equation 13 First Euler Parameter: $e_0 \equiv \cos\left(\frac{\phi}{2}\right)$

Equation 14 Remaining Euler Parameters: $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{\mathbf{n}} \sin\left(\frac{\phi}{2}\right)$

With all the matrices found the system equation can be computed. The system equation is built going from right to left and from object to the image.

Equation 15 System Equation: $\mathbf{img} = \mathbf{R_2 M_2 R_2^T R_1 M_1 R_1^T obj}$
where img is the image vector, R_n is the respective mirror rotation matrix, M_n is the respective initial mirror matrix, and obj is the object vector.

To study the image found by the system an object that is defined by the unit vector along the Y-axis is used.

Equation 16 Object Vector $\mathbf{obj} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

This object vector is then placed through the system with both mirrors at zero rotation. This will give an image vector to compare to later on when the system's mirrors are rotated. In the case of no rotation the rotation matrices become the identity matrix so the system equation may be simplified as:

Equation 17 System Equation with No Rotation: $\mathbf{img_{nr}} = \mathbf{M_2 M_1 obj}$

The example system has a non-rotated image equal to:

Equation 18 Example Systems Non-rotated Image: $\mathbf{img}_{\mathbf{nr}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

This can be verified with the pencil bounce test.

To find the image rotation it is necessary to define an image plane. In this case the easiest image plane to define would be perpendicular to the non-rotated image. The plane will be defined by a unit vector normal to its surface.

Equation 19 Image Plane Normal Unit Vector: $\hat{\mathbf{n}}_{\mathbf{img}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Next, find an image for a given mirror rotation. For example, the first mirror is rotated 2° and the second mirror is rotated 5°. This gives a rotated image of:

Equation 20 Rotated Image Example: $\mathbf{img}_{2,5} = \begin{bmatrix} -0.99619 \\ 0.007596 \\ -0.08682 \end{bmatrix}$

To find how much the image is rotated both the initial and rotated images need to be projected onto the image plane. This can be done by finding the projection of the image vector onto the normal unit vector of the image plane. This vector is then subtracted from the image vector to give the projection of the image vector onto the image plane.

Equation 21 Projected Image on Image Plane: $\mathbf{pimg} = \mathbf{proj}_{\mathbf{image\_plane}}(\mathbf{img}) = \mathbf{img} - (\mathbf{img} \cdot \hat{\mathbf{n}}_{\mathbf{img}})$

The cross product of the two projected image vectors is then taken. The length of the cross product vector is then divided by the product of the lengths of the two original image vectors. The inverse sine of this value gives the angle between the two vectors. The cross product vector also gives the axis of rotation

Equation 22 Cross Product Definition: $|\mathbf{pimg}_{\mathbf{nr}} \times \mathbf{pimg}_{\mathbf{rot}}| = |\mathbf{pimg}_{\mathbf{nr}}||\mathbf{pimg}_{\mathbf{rot}}|\sin(\theta)$

Equation 23 Angle of Image Rotation: $\theta = \sin^{-1}\left(\frac{|\mathbf{pimg}_{\mathbf{nr}} \times \mathbf{pimg}_{\mathbf{rot}}|}{|\mathbf{pimg}_{\mathbf{nr}}||\mathbf{pimg}_{\mathbf{rot}}|}\right)$

For the example the projected images are shown below with their calculated angle of rotation. In this case the first mirror was rotated by 2° and the second mirror is rotated 5°.

Equation 24 Projected Non-rotated Image: $\mathbf{pimg}_{\mathbf{nr}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

Equation 25 Projected Rotated Image: $\mathbf{pimg}_{2,5} = \begin{bmatrix} -0.99619 \\ 0.007596 \\ 0 \end{bmatrix}$

Equation 26 Angle of Rotated Image $\theta = 0.437^{\circ}$

In summary the following are the steps to perform to calculate the angle of the rotated image.

1. Sketch the system including:
    a. Mirror locations
    b. Mirrors' axis of rotation.
    c. Input beam.
    d. Image plane.
2. Choose a global coordinate system.
3. Define all objects in the global coordinate system.
    a. Define normal vectors of mirrors in global coordinate system.
    b. Define mirror matrices for mirrors in global coordinate system.
    c. Define axis of rotation for mirrors in global coordinate system.
4. Find image for non-rotated case.
5. Add rotation to mirrors.
6. Find image in rotated case.
7. Project rotated and non-rotated images onto image plane.
8. Compare the projected images of the rotated case with the non-rotated case to determine rotation of image.

References:

For information on how to perform mirror rotations in Zemax see:
http://www.zemax.com/kb/articles/25/1/How-To-Model-a-Scanning-Mirror/Page1.html

For more information on Euler Parameters see:
http://mathworld.wolfram.com/EulerParameters.html.

For further information on projection see:
http://www.euclideanspace.com/maths/geometry/elements/plane/lineOnPlane/index.htm and
http://www.euclideanspace.com/maths/geometry/elements/line/projections/index.htm.

Attached is a spreadsheet that implements the process used in this tutorial:

Scanning
Rotation.xls