Synopsis: Optimal design techniques for kinematic couplings

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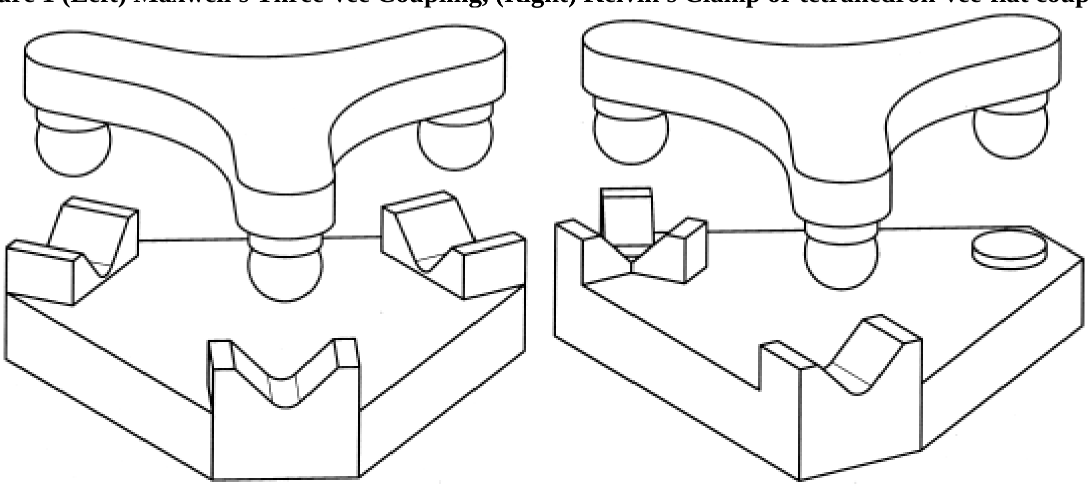
Introduction

A literature search found that kinematic couplings have been discussed prior to 1849. In 1876 James Clerk Maxwell described a three-vee coupling method and William Thomson's (Lord Kelvin's') tetrahedron-vee-flat coupling method for accurate positioning. However they both learned about kinematic coupling from a shared professor, Professor Robert Willis around 1849.

Figure 1 (Left) Maxwell's Three-vee Coupling, (Right) Kelvin's Clamp or tetrahedron-vee-flat coupling



Typically kinematic couplings are used in instrument design with light and static loads. But they can be used for high load and high stiffness with well designed contact areas or ceramic materials.

This paper discusses the process of finding the best configuration of a kinematic coupling given certain specifications. The first optimization method comes from Maxwell. He stated that each constraint should be in the direction of motion allowed when the other five constraints are engaged. In this way each constraint will carry a minimum load and supply its full stiffness to the assembly. The other three methods of optimizing the coupling involve mathematical modeling of the systems geometry and behavior. Two of these mathematical methods take into consideration the friction at the contact surfaces and the other the modal frequencies of the system.

Modeling the System

Prior to being able to perform the last three optimization techniques a mathematical model needs to be constructed for the system. The system can be modeled as a system of six springs, one for each constraint. To simplify this, the springs can be modeled as a parallel combination. Or additional springs may be added to represent flexures and the bodies being coupled. The springs' behaviors are normally linearized but a complete nonlinear vector function may be used as well.

Using the linear spring model will give a system model that is represented by a [6 X 6] stiffness matrix which defines a relationship between deflection and load. Within the stiffness matrix are four [3 X 3] that are similar, as can be seen in equations 1 and 2.

Equation 1 [3 X 3] Block from Stiffness Matrix

$$\mathbf{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix} \cdot \begin{bmatrix} d\delta_x \\ d\delta_y \\ d\delta_z \end{bmatrix} = \mathbf{K}_{\mathbf{f}/\delta} \cdot \mathrm{d}\delta$$

Equation 2 Stiffness Matrix

$$\begin{bmatrix} \mathbf{f} \\ \hline \mathbf{m} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}_{\mathbf{f}/\delta} & \mathbf{K}_{\mathbf{f}/\theta} \\ \hline \mathbf{K}_{\mathbf{m}/\delta} & \mathbf{K}_{\mathbf{m}/\theta} \end{array}\right] \cdot \begin{bmatrix} d\delta \\ \hline \mathrm{d}\theta \end{bmatrix} \qquad \mathbf{K}_{\mathbf{m}/\delta} = \mathbf{K}_{\mathbf{f}/\theta}^{\mathrm{T}}$$

Equation 3 Compliance Matrix

$$\begin{bmatrix} d\delta \\ \hline \mathrm{d}\theta \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{C}_{\delta/\mathbf{f}} & \mathbf{C}_{\delta/\mathbf{m}} \\ \hline \mathbf{C}_{\theta/\mathbf{f}} & \mathbf{C}_{\theta/\mathbf{m}} \end{array}\right] \cdot \begin{bmatrix} \mathbf{f} \\ \hline \mathbf{m} \end{bmatrix} \qquad \mathbf{C}_{\theta/\mathbf{f}} = \mathbf{C}_{\delta/\mathbf{m}}^{\mathrm{T}}$$

The most convenient method of finding the stiffness or compliance matrix is to setup a local coordinate system (CS) where the z-axis is normal to the contact surface of the constraint. This way the stiffness or compliance matrix will have off-diagonal terms equal to zero. Also if the contact is friction free there will only be a k_{zz} term. If there is no slip k_{xx} and k_{yy} will have some value. The compliance matrix is the easier of the two to find and is the resulting translation-rotation vector found by applying a unit force or moment in the axis matching that column.

After the stiffness or compliance matrix is found it must be transformed to the base coordinate system so that each compliance matrix for the entire system may be formed. This is done by using a [6 X 6] transformation matrix. This transformation matrix is formed from a [3 X 3] rotation matrix and a [3 X 3] cross product matrix. The details on how the rotation matrix and cross product are formed are discussed further in the appendix of the paper.

Equation 4 Transformation Matrix for Stiffness Matrix, 2 matrices in center

$$\begin{bmatrix} \mathbf{f}_0 \\ \hline \mathbf{m}_0 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline & & & 1 & 0 & 0 \\ & \mathbf{C}(r_0) & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & \mathbf{R}_{0/1} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{R}_{0/1} & \\ 0 & 0 & 0 & & & \end{array}\right] \cdot \begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{m}_1 \end{bmatrix}$$

Equation 5 Transformation Matrix for Compliance Matrix, 2 matrices in center

$$\begin{bmatrix} d\boldsymbol{\delta}_1 \\ \hline d\boldsymbol{\theta}_1 \end{bmatrix} = \left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & \mathbf{R}_{0/1}{}^{T} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{R}_{0/1}{}^{T} & \\ 0 & 0 & 0 & & & \end{array}\right] \cdot \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & \mathbf{C}(\mathbf{r}_0)^{T} & \\ 0 & 0 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} d\boldsymbol{\delta}_0 \\ \hline d\boldsymbol{\theta}_0 \end{bmatrix}$$

These are the tools the analyst will need to use when performing optimization. Some of the parameters in these equations will be open to optimization. Usually the position and orientation of the contact surfaces will be optimized but other pieces like the stiffness or compliance may be taken into account by the analyst.

Optimizing the System

Four methods of optimization were looked at in the paper. It is not necessary that all following methods be used for optimizing a system but instead to use only those that will help meet the requirements of the system. In this case it may be only one of the methods. In any case thought it will be necessary to combine the values from the six constraints. The author suggests using a generalized mean, equation 6, where p is positive gives greater weight to larger values, p is negative will place greater weight on smaller values.

Equation 6 General Mean

$$\text{gen\_mean}(\mathbf{v}, p) = \left\{ \frac{1}{n} \sum_{i=1}^{n} |v_i|^p \right\}^{1/p}$$

Maxwell's Criterion

Maxwell stated that each constraint should be in the direction of motion allowed when the other five constraints are engaged and slide freely. To test a constraint is released and the direction motion is compared to the direction of that particular constraint. The inner product of the two direction unit vectors is taken. The perfect case would produce an inner product equal to one. Each constraint will have an inner product. The object of this optimization is to maximize the inner product of each constraint.

To do this with the stiffness matrix, create the stiffness matrix for the five other constraints besides the one being analyzed. This stiffness matrix will have a single direction with zero stiffness which is the optimal direction of the base for the constraint being tested. This direction is the eigenvector of the stiffness matrix's that corresponds to the zero eigenvalue. This process is repeated for the remaining constraints until the direction of each constraints base is determined. The unit vector of the eigenvector will be the local z-axis of each constraint.

Maximizing the Modal Frequency

Maximizing the modal frequencies can be started off with an approximation. To begin, a [6 X 6] mass matrix similar to the stiffness matrix is created. The mass matrix will have the first three diagonal terms as the mass and the bottom right [3 X 3] matrix as the inertia matrix. The mass matrix or the stiffness matrix must then be transformed to the others coordinate system. Following this the generalized eigenvalue algorithm is applied. Typically the minimum eigenvalue is used to optimize the coupling to allow sliding to a centered position. After this approximation is completed finite element analysis can be used to improve the accuracy.

Minimizing Frictional Nonrepeatability

This optimization is done because friction between the contacting surfaces is the main cause of non-repeatable coupling. In prior work the author found a simple equation to give a value to the nonrepeatability behavior of a coupling. He found that the configuration of a constraint affect the friction and compliance and hence the nonrepeatability of the coupling.

Equation 7 Nonrepeatability

$$\rho \equiv \frac{f}{k} \approx \mu \left(\frac{2}{3R} \right)^{1/3} \left(\frac{P}{E} \right)^{2/3}$$

Similar to Maxwell's optimization technique each constraint must be analyzed separately and thus a five-constraint stiffness matrix for each constraint is needed. The eigenvector for the zero eigenvalue is used to find the frictional force-moment vector. This vector is then used with the full compliance matrix to find a translation-rotation vector. This vector must then be transformed to place the coordinate system to the actual point of coupling. The vector's magnitude returned after the transformation is the nonrepeatability factor. This procedure is repeated for the five remaining constraints. The maximum value is used in design optimization.

Maximizing the Limiting Coefficients of Friction

To provide repeatable coupling the real coefficient of friction must be less than the limiting coefficient of friction when the force needed to center is zero. A typical limit should be 3 times the real coefficient of friction.

Again the six five-constraint stiffness matrices are used. This time however the coefficient of friction is now a variable and must be solved for each constraint. In order for the coupling to slide together the reaction force must be zero. In this way the stiffness matrix equation, equation 2, can be used to solve for the friction coefficient. Other variables are then optimized such as the orientation of each constraint to maximize the solved for friction coefficient. This process is repeated for each constraint to complete the optimization.

An Example

The author provides an example of a using a graphical optimization method using the optimization technique of maximizing the limiting coefficients of friction. Instead of numerically solving all the cases graphs showing the optimization criteria were created to show there respective change on the coefficient of friction.

Conclusion

The article covered four techniques to optimize the design of a kinematic coupling as well as a brief overview of the tools needed to mathematically model the coupling in order to perform those techniques. Throughout the paper the author used the example of the three-vee coupling to show how each of the optimization techniques can be used. Due to the briefness of this synopsis this was not covered here.   Many of the techniques posed in the paper require further knowledge of system modeling matrix methods that are briefly expanded upon in an appendix. However following the paper provides an explanation of most of the tools needed to perform the optimization techniques discussed for the simplest systems.