

First Order Optics

Jordan Jur

1. INTRODUCTION

First order optics are the principles and equations which describe the geometrical imaging of any optical system. The foundations of first order optics are derived from the concept of central projection, collinear transformation and the camera obscura. These foundations will be used to demonstrate the fundamental parameters of an optical system such as image position, size and orientation. These fundamental parameters will be calculated from the different cardinal points of an optical system using both Newtonian and Gaussian imaging equations. To extrapolate upon these imaging principles, the derivation of Snell's law and the law of reflection will be covered.

2. FUNDAMENTAL CONCEPTS

Central Projection

The geometry of first order optics is extrapolated from the concept of central projection. The central projection concept states that a given object located in the object plane which is projected through the "projection point" will form an image of the object in the image plane. The central projection concept is an application of the mathematical concept of projection. Projection is defined as a mapping of a set into a subset. In relation to the central projection concept, the set encompasses the points and/or lines which define the object in object space, the mapping function would be the projection point and the subset encompasses the image points and/or lines which define the object in image space. Collinear transformation is a type of projection which has applications to optics, as well.

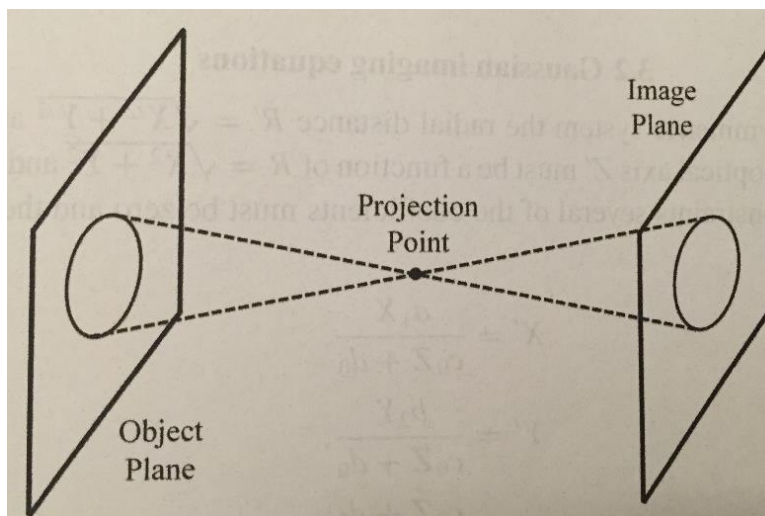


Figure 1. Central Projection

Collinear Transformation

A collinear transformation is a one to one mapping between two spaces. In the world of optics, the spaces are referred to as object space and image space. All of the points within object space pass through the same projection point which maps them to points within the image space. The projection function of a collinear system states that points map to points, lines map to lines and spaces map to spaces. The points, lines and spaces in object space all have corresponding and unique points, lines and spaces in image space. The notion of these corresponding and unique elements describes the basis of conjugate elements.

Next, the camera obscura is described to illustrate the idea of image formation with rays. This is a necessary concept in order to build upon the mathematical principles and equations which govern the geometric behavior of the object and image equations. Further details of these imaging equations and their response to small perturbations can be described by linear shift invariant systems theory.

The Camera Obscura

The camera obscura is the simplest form of an imaging system. It consists of a black box with a pinhole in one side. Light from an object placed outside of the box will propagate through the pinhole and form an inverted image on the wall. Here light can be thought of as a ray which travels from left to right to form the image. Notice that the geometry and concept of the camera obscura is an adaptation of the central projection concept. An important detail of this imaging system lies in the pinhole diameter.

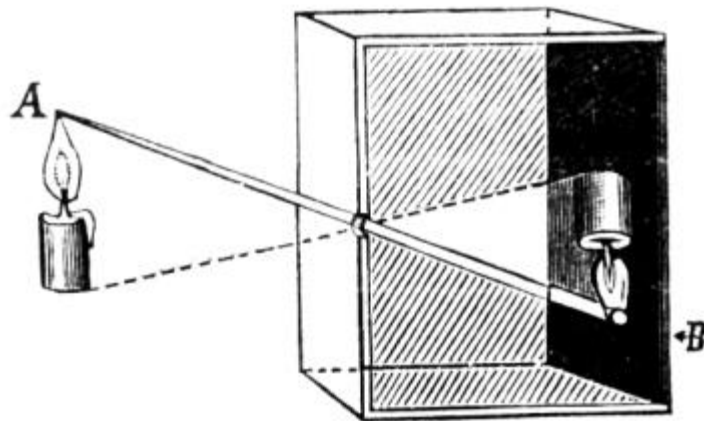


Figure 2. Camera Obscura

As the pinhole diameter increases so does the image brightness. This is because a larger aperture will enable more light to pass through the optical system. Though, if the aperture is too large then the image will become blurry. This, of course, describes the preliminary concept of resolution and the point spread function (PSF). Conversely, as the pinhole diameter decreases, thereby improving image resolution and PSF, so does the image brightness. Before expanding upon the complexities of resolution and PSF, the geometric relationship between the object and image distances must be understood.

The distance from the object to the pinhole is known as the object distance while the distance from the pinhole to the image is known as the image distance. The idea of object and image distances leads to the mathematical concepts of imaging as independently explain by Sir Isaac Newton and Karl Gauss.

With the concepts of central projection, collinear transformation and the camera obscura laid out, a mathematical and physical geometric basis can now be constructed to help derive and illustrate, respectively, the first order properties of an optical system.

3. IMAGING EQUATIONS

The imaging equations can be derived by tracing the chief and marginal rays of an optical system. The marginal ray is an on-axis ray which travels from the center of an object to the edge of the stop and to the center of the image. The chief ray is an off-axis ray which travels from the edge of the object through the center of the stop to the edge of the image. In other words, the marginal and chief rays define the extrema of rays which define an image to the first order. That is to say, all other rays which can be traced through an optical system lie between the marginal (image center) and chief (image edge) rays. The image location is known as the focal plane and is our first introduction into the cardinal points.

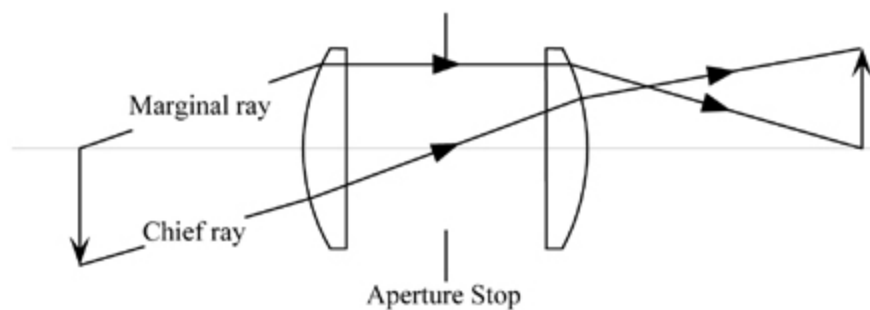


Figure 3. Marginal and Chief Rays

The cardinal points define points of angular and spatial significance within an optical system. Any optical system can be decomposed in terms of the six cardinal planes. The six cardinal planes are the front and rear focal planes, nodal planes and the principle planes. The focal planes are the planes at which all rays parallel to the optical axis in object space will come to focus in image space. The same is true for rays traveling from image space towards object space. The nodal planes are the planes of unit angular magnification. A ray which passes through the front nodal point will pass through the rear nodal point at the same angle with respect to the optical axis. The principle planes are the planes of unit lateral magnification. A ray which passes through the front principal point will pass through the rear principal point at the same vertical distance from the optical axis. The front and rear cardinal points are conjugate to one another.

For reference, the symbolic interpretation of the Newtonian and Gaussian imaging systems is explain in table 1.

Symbol	Meaning
F	Front Focal Plane
P	Front Principal Plane
z	Object Distance
f_F	Front Focal Distance
h	Object Height
n	Object Space Refractive Index
F'	Rear Focal Plane
P'	Rear Principal Plane
z'	Image Distance
f'_R	Rear Focal Distance
h'	Image Height
n'	Image Space Refractive Index
m	Magnification
f_E	

Table 1. Symbol Meaning

Newtonian Imaging Equations

The imaging equations were first derived by Sir Isaac Newton in 1666 using similar triangles. In relation to the conjugate object and image planes, the Newtonian equations are referenced to the focal planes.

$$\begin{array}{l}
 z = -\frac{f_F}{m} \\
 z' = -mf'_R \\
 zz' = f_F f'_R
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{l}
 \frac{z}{n} = \frac{f_E}{m} \\
 \frac{z'}{n'} = -mf_E \\
 \left(\frac{z}{n}\right)\left(\frac{z'}{n'}\right) = -f_E^2
 \end{array}$$

Figure 4. Newtonian Imaging Equations

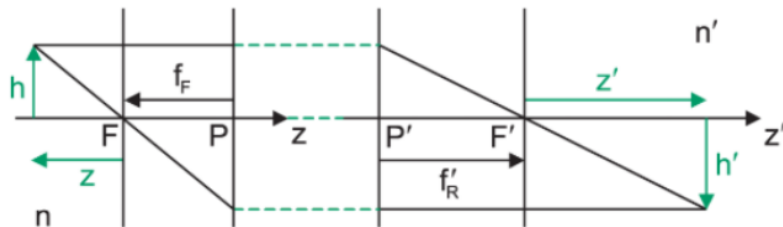


Figure 5. Newtonian Imaging Geometry

Gaussian Imaging Equations

Gauss later derived similar imaging equations where the conjugate object and image planes are referenced to the principal planes. Both sets of imaging equations assume that the lenses are thin and therefore the principal and nodal planes lie at the center of the lens.

$$\begin{array}{l|l}
 z = -\frac{(1-m)}{m}f_F & \frac{z}{n} = \frac{(1-m)}{m}f_E \\
 z' = (1-m)f'_R & \frac{z'}{n'} = (1-m)f'_E \\
 m = -\left(\frac{z'}{z}\right)\left(\frac{f_F}{f'_R}\right) & m = \frac{z'/n'}{z/n} \\
 \frac{f'_R}{z'} + \frac{f_F}{z} = 1 & \frac{n'}{z'} = \frac{n}{z} + \frac{1}{f_E}
 \end{array}$$

Figure 6. Gaussian Imaging Equations

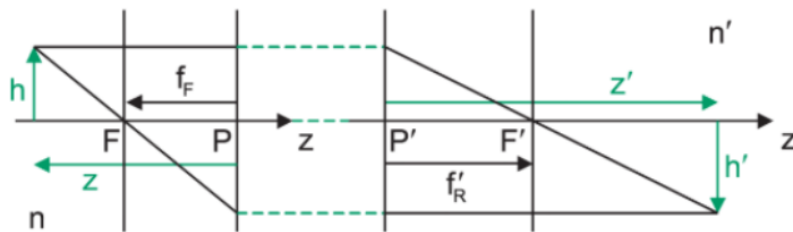


Figure 7. Gaussian Imaging Geometry

4. OPTICS LAWS

Law of Reflection

The simplest form of ray propagating can be understood by the law of reflection. The law of reflection states that for a ray incident upon a flat surface at an angle of α with respect to the surface normal reflect away from the flat surface at an angle of $-\alpha$ with respect to the surface normal. Figure 8 provides a visualization of the reflection which takes place at a flat surface.

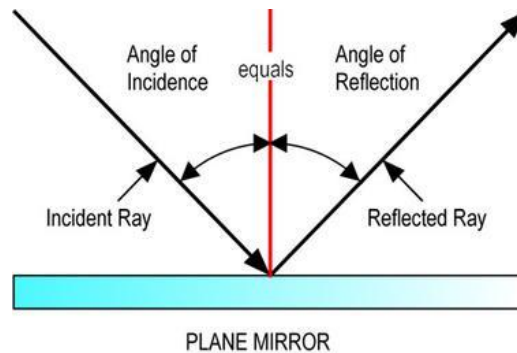


Figure 8. Law of Reflection

Law of Refraction: Snell's Law

In any optical system, light travels through different types of media. As light travels from one media to another the direction of propagation will alter depending upon the incident angle and the properties of both media. The simplest model which describes the change in direction is known as Snell's Law. Snell's Law states that a ray in media 1 which is incident upon a surface at an angle of α with respect to surface normal will alter its direction into media 2 at angle β with respect to the surface normal. This altering of direction is known as refraction. The equation which mathematically describes Snell's Law is shown in equation 1.

$$n_1 \sin \alpha = n_2 \sin \beta \quad (1)$$

Where n_1 and n_2 are the indices of refraction for the incident and transmitted surface. One can image that any time a ray of light interacts with different media that Snell's Law can be applied to determine the change in the direction of propagation. Figure 9 provides a visualization of the refraction which takes place at an interface between two different media.

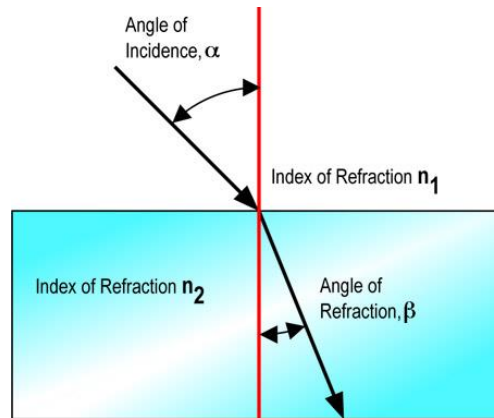


Figure 9. Law of Refraction

When the transmitted medium has a higher refractive index than the incident medium the ray will bend towards the surface normal. When the transmitted medium has a lower refractive index than the incident medium the ray will bend away from the surface normal. The principles of reflection and refraction can be applied to optical systems to understand how object and image motion are affected.

5. OBJECT AND IMAGE MOTION

Object and image motion is described in relation to a three dimensional Cartesian coordinate system. This coordinate system has six effective degrees of freedom: x, y & z motion and θ_x , θ_y , & θ_z rotation, as described in figure 10. There exist reflective and refractive optical elements which will alter the ray propagation through an optical system and thus effect the object and image motion. Common reflective elements include mirrors and prisms and common refractive elements include plane parallel plates (PPP) and lenses.

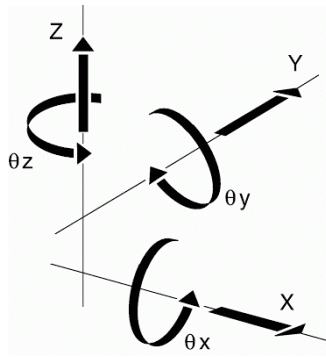


Figure 10. The Six Degrees of Freedom

Reflective Elements

Prisms

Prisms are useful tools which make it possible to compact an optical system into a smaller form factor. The process of compacting an optical system is known as folding. There exist five main categories of prisms which can be used to fold an optical system into different geometries. They are the deviation, dispersion, displacement, rotation and expansion prisms.

In addition to the 3D coordinate system, image motion due to a prism can also be described by the image handedness and parity. Image handedness refers to the number of reflections. There are two types of image handedness- right and left. Right handedness refers to an image which has been reflected an even number of times. Conversely, left handedness refers to an image which has been reflected an odd number of times. Figure 11 demonstrates right and left handedness.



Figure 11. Right and Left Handedness (respectively)

Image parity refers to an even or odd number of reflections. Even parity refers to an image which has been reflected an even number of times and odd parity refers to an image which has been reflected an odd number of times. Each handedness is capable of producing four different image orientations depending upon the number of reflections, as shown in figure 12.

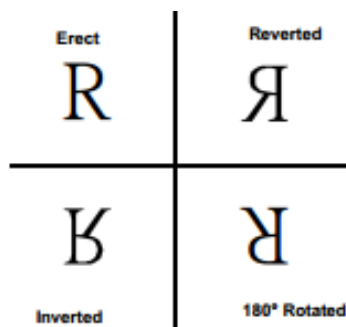


Figure 12. Image Orientations

90° Deviation & Displacement Prisms

Common 90° deviation and displacement prisms include the right angle prism, the Amici prism, and the penta prism. The right angle prism is the simplest form of a prism. It consists of a single reflecting surface oriented at 45° to the incident beam. The right angle prism has right handedness and odd parity; it will keep an erect image. The Amici prism is identical to the right angle prism with the exception that it has a roof. The Amici prism has right handedness and even parity; it will revert the image. The Penta prism contains two reflecting surfaces oriented at 22.5°. Each reflection will cause a 45° change in the beam pointing resulting in a total of 90° of image deviation. The Penta prism has right handedness and even parity; it will invert an image.

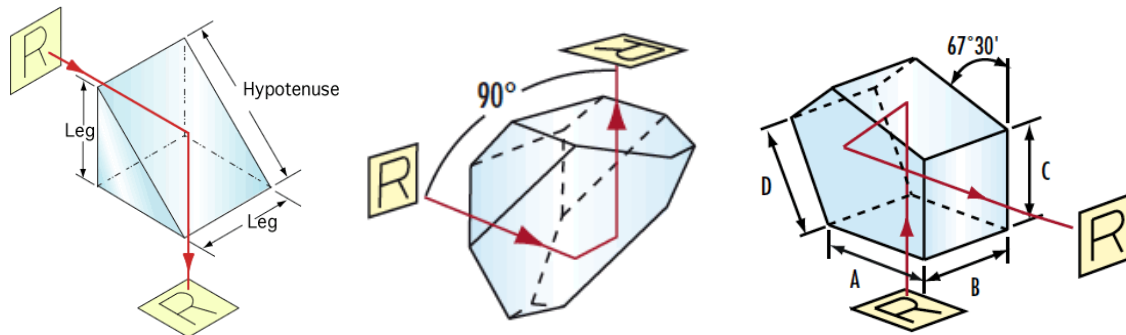


Figure 13. Right Angle, Amici, & Penta Prisms (respectively)

180° Deviation & Displacement Prisms

The most common 180° deviation and displacement prism is the Porro prism. It consists of two reflective surfaces both oriented at 45° to the incident beam. Each reflecting surface causes a 90° deviation for a total of 180° of deviation. The Porro prism has right handedness and even parity; it will invert an image.

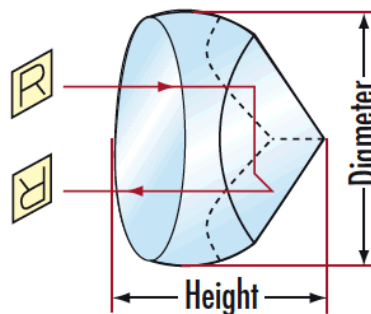


Figure 14. Porro Prism

45° Deviation & Displacement Prisms

Common 45° deviation and displacement prisms include the half penta and Schmidt prisms. The half penta prism consists of two reflecting surfaces oriented at 11.25° to the incident beam causing a total deviation of 45°. The half-penta prism has right handedness and even parity; it will keep the image erect. The Schmidt prism consists of four reflecting surfaces which deviate the beam 45°. The Schmidt prism has right handedness and even parity; it will rotate the image by 180°.

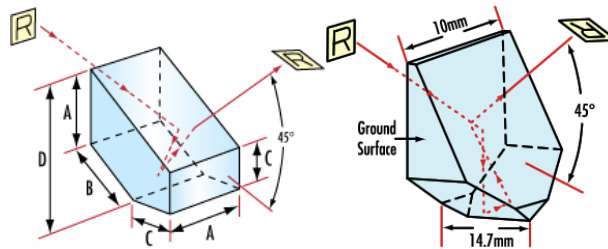


Figure 15. Half Penta and Schmidt Prisms (respectively)

Image Rotation and Erection Prisms

Common image rotation and erection prisms include the Dove, Schmidt-Pechan, and Abbe-Koenig prisms. The Dove prism consists of one reflecting surface. The Dove prism has left handedness and odd parity; it will invert an image. The Schmidt-Pechan prism consists of two elements separated by an air gap, six reflecting surfaces and a roof. The Schmidt-Pechan prism has righthandedness and even parity; it will rotate an image by 180°. The Abbe-Koenig prism consists of four reflecting surfaces including a roof. The abbe-Koenig prism has righthandedness and even parity; it will rotate an image by 180°.

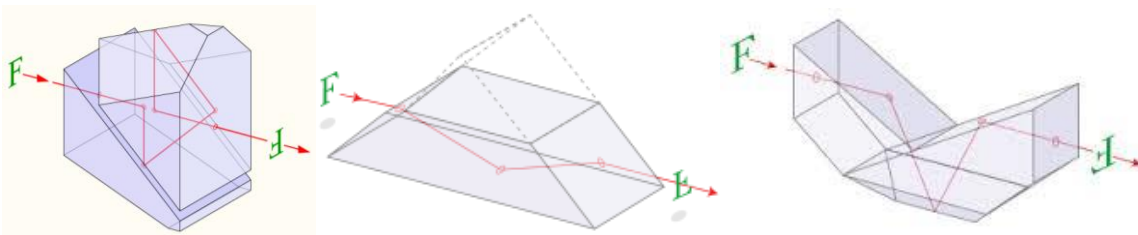


Figure 16. Dove, Schmidt-Pechan, Abbe-Koenig Prisms (respectively)

Prism	Type	Reflections	Image Orientation	Application
Right Angle	Deviation & Displacement	1	Erect	Endoscopy
Amici	Deviation	2	Revert	Eye Pieces
Penta	Deviation	2	Invert	Projection
Porro	Deviation & Displacement	2	Invert	Binoculars
Schmidt	Deviation	4	Erect	Stereo Microscope
Half-Penta	Deviation	2	Erect	Pechan Erector Assembly
Dove	Rotation	1	Invert	Astronomy
Pechan	Rotation	6	180° rotation	Binoculars
Abbe-Koenig	Rotation	4	180° rotation	Binoculars
Equilateral	Dispersion	0	Erect	Spectroscopy
Anamorphic Pair	Expansion	0	Erect	Laser Diode Beam Expander

Table 2. Prism Summary

For the special case of a thin prism, the beam will deviate according to index of the prism and the wedge angle of the prism, as shown in figure 17.

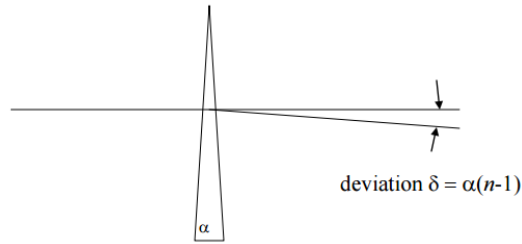


Figure 17. Thin Prism

Mirrors

Prisms are effectively mirrors which have been compacted into smaller form factors. It is important to understand how the movement of a mirror effects image motion. The general rule of thumb for image rotation is that any rotation of a mirror will cause a image rotation of twice the mirror rotation.

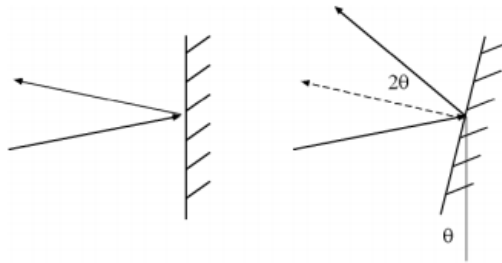


Figure 18. Mirror Rotation

The general rule of thumb for image motion is that any translation of the mirror, which is not orthogonal to the direction of propagation, will increase the OPL by twice the mirror translation.

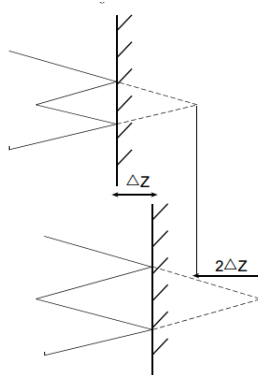


Figure 19. Image Translation

Refractive Elements

Vertical Plane Parallel Plate

Plane parallel plates (PPP) can be used for a number of different applications. In this example, a PPP will be used to demonstrate the concept of the optical path difference (OPD). In this example, the OPD is a function of the index of the PPP glass and the thickness of the glass (assuming the incident medium is air). In figure 19, the dashed lines represent the undeviated beam and the solid lines represent the deviated beam described by Snell's Law. The equation in the figure describes the displacement of the focused beam along the optical axis, Δz .

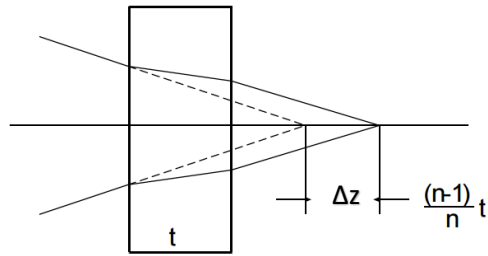


Figure 20. Plane Parallel Plate

Tilted Plane Parallel Plate in a Collimated Beam

A tilted PPP in a collimated beam will cause a vertical deviation of the beam. In this example, the vertical deviation of the beam is a function of the index of the PPP glass, the thickness of the PPP and the angle of the tilted PPP in radians. In figure 20 the dashed line represents the path of the beam for the case when the PPP is vertical and the solid line represents the path of the beam when the PPP is tilted by an angle θ . The equation in the figure describes the displacement of the beam in the vertical axis, Δy .

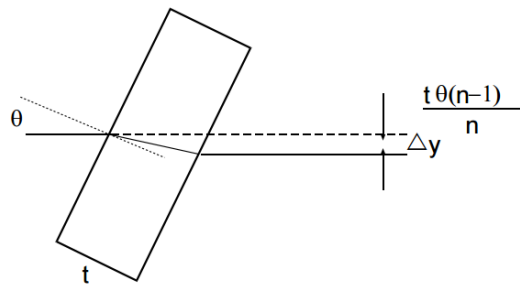


Figure 21. Tilted PPP

Tilted Plane Parallel Plate in a Converging Beam

A tilted PPP in a converging beam will cause an axial and vertical displacement which will also induce several types of fourth order aberrations within the optical system. The axial and vertical displacement of the beam were described in the previous two sections. The aberrations induced by the tilted PPP in the converging beam are a function of the index of the PPP glass, the thickness of the PPP, the angle of the tilted PPP in radians, the $f/\#$ of the system, and the Abbe-Number of the glass. The equations describing these aberrations are in table 3.

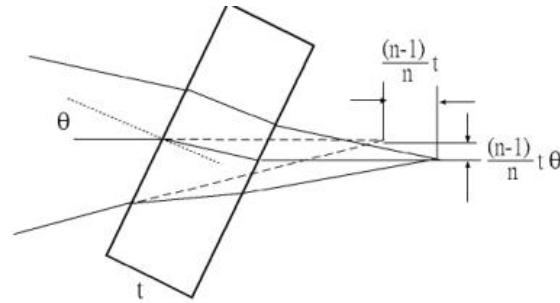


Figure 22. Tilted PPP in Converging Beam

Aberration	Equation
Spherical	$-\frac{t(n^2 - 1)}{f/\#^4 128n^3}$
Coma	$\frac{t\theta(n^2 - 1)}{f/\#^3 16n^3}$
Astigmatism	$\frac{t\theta^2(n^2 - 1)}{f/\#^2 8n^3}$
Transverse Color	$\frac{t\theta(n - 1)}{n^2 v}$
Longitudinal Color	$-\frac{t(n - 1)}{n^2 v}$

Table 3. Fourth Order Aberrations

Lens Movement

When designing optical systems it is important to understand how the motion of a lens will affect the motion of the image. In a 2D system, lens motion can be described in three different ways. The lens can either tilt about the optical axis, move perpendicular to the optical axis (decenter) or move along the optical axis (axially). For the first case, when the lens is thin there is no change in image motion as the lens is tilted about the optical axis.

Lens Decenter

When the lens is decentered by some amount ΔX_L the image moves by some amount ΔX_i . As the lens is decentered the mechanical and optical axes will no longer be collinear. The mechanical axis will remain unchanged but the optical axis will alter by an angle α . The change in the optical axis angle is described by equation 2.

$$\alpha = \frac{\Delta X_L}{o} \quad (2)$$

Where o is the object distance. The effect of lens decenter on image motion can be seen in figure 23.

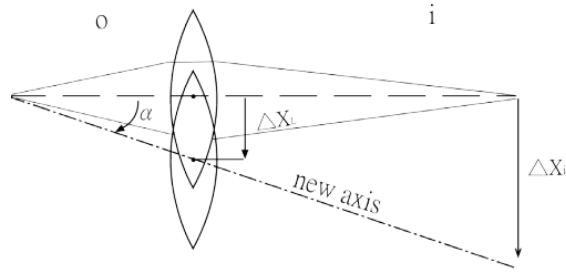


Figure 23. Lens Decenter

The image motion can then be calculated using similar triangles. The image motion is described by equation 3, where i is the image distance.

$$\Delta X_I = \Delta X_L \frac{(o + i)}{o} = \Delta X_L (1 - m) \quad (3)$$

Lens Axial Translation

Axial translation of a lens will result in an image motion along the optical axis. In this example, as the lens is translated axially by a distance Δz_L the resulting image moves $-\Delta z_F$. The object position thus changes from o to o' and the image position changes from i to i' . Given these variables, the image motion can now be derived.

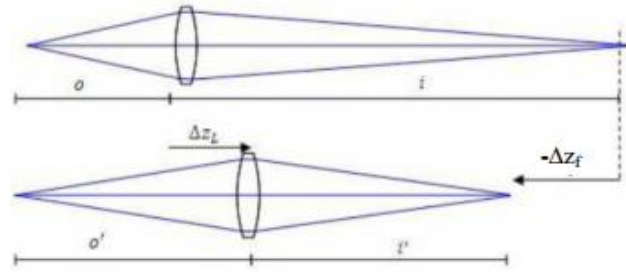


Figure 24. Lens Axial Movement

The change in object and image distance is described by equations 4 and 5, respectively.

$$\Delta o = o - o' = -\Delta z_L \quad (4)$$

$$\Delta i = i - i' = \Delta z_L - \Delta z_F \quad (5)$$

Using the imaging equation (6), the equation can be rearranged to solve for the magnification.

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f} \quad (6)$$

$$\frac{\Delta i}{\Delta o} = -\frac{i^2}{o^2} = -m^2 \quad (7)$$

The resulting image motion can be solved by plugging equations 4 and 5 into 7.

$$\Delta z_F = \Delta z_L (1 - m^2) \quad (8)$$