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The kinetic center of the cantilever beam

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ABSTRACT

Elastic mechanisms are almost always designed using the linearized elastic equations which are true for infinitesimal deflections, only. As the deflections become finite in size they change the geometry of the elastic medium and the distribution of stresses and strains within it. This effect gives rise to a "geometric" nonlinearity which requires a "large displacement" solution to the equations of elasticity, a solution that is generally very difficult, if possible at all.

In the application of elastic mechanisms (flexures) to precision instruments it is often important to determine or bound some of the "large displacement" characteristics of the mechanisms in order to quantify parasitic, or undesirable motions. The foreshortening of a beam in bending is one of the "large displacement" characteristics. This paper show how to estimate the foreshortening of a beam in the domain of small deflections (near zero) and applies it to the task of determining the radius of curvature of the locus of the free end of a double cantilever beam. The center of curvature of the locus is the "kinetic center" of the beam.

Key Words: flexure, large displacement, foreshortening, parasitic

1.0 THE EQUIVALENT RIGID LENGTH OF A BEAM

Figure 1 shows a cantilever beam being bent under the influence of a transverse load, P. As the free end of the beam

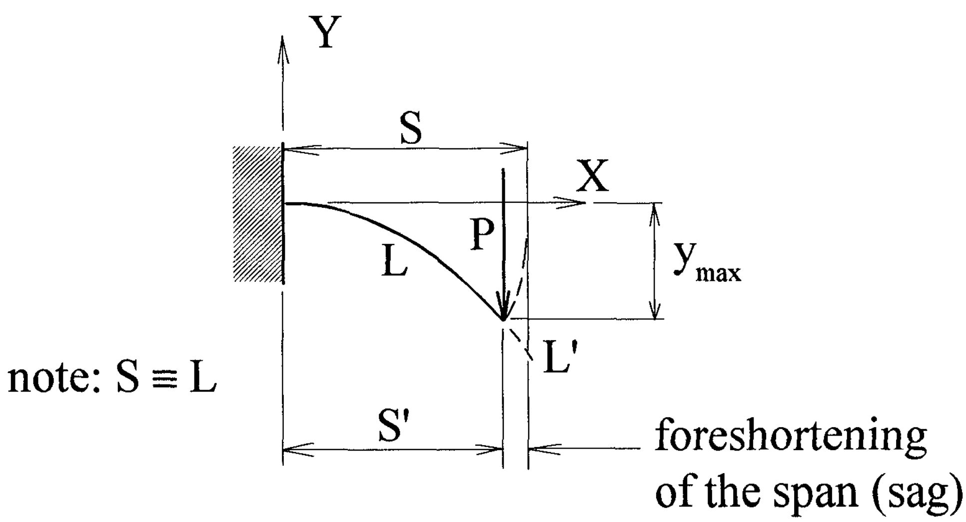

Figure 1. A cantilever beam in transverse bending.

deflects downward, the length of the beam, L, remains constant. As a result the span, S', of the beam must be shorter than when the beam is undeflected, S. This effect is called foreshortening of the span of the beam. Conversely, the length of the elastic curve, L', projected over the original span, S, must be longer than the undeflected physical length, L, of the beam. As the beam deflects and foreshortens the free end of the beam describes a curve called the locus of the free end of the beam. We shall estimate the magnitude of the foreshortening effect for small displacements and we shall use it to calculate the radius of curvature of the locus of the free end at zero deflection (and, thereby, locate the kinetic center of the beam).

The radius of curvature of the locus is also the "equivalent rigid length," R, that will produce the same foreshortening of the free end as the elastic beam under a transverse load, Figure 2.

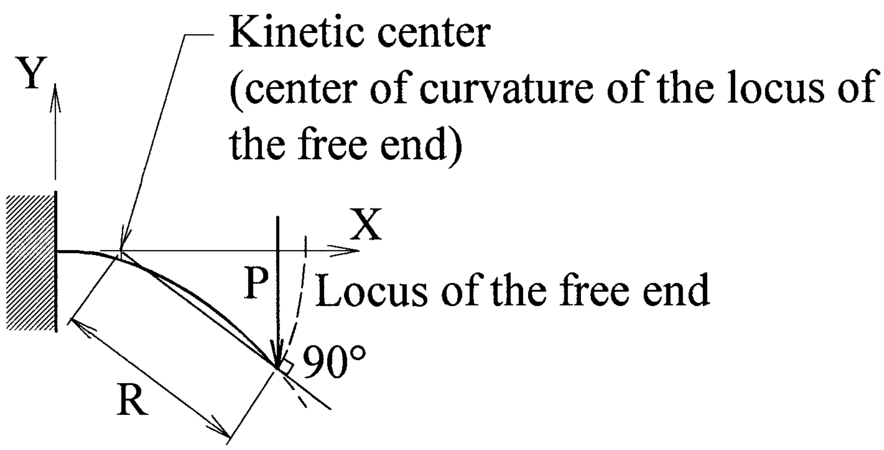

Figure 2. The radius of curvature of the locus of the free end of the beam.

2.0 THE SIMPLE CANTILEVER

We may define the moment loading on the simple cantilever beam by making a free body diagram of the end of the beam, Figure 3.

Using the small displacement equations for the bending of a beam,

$$d^2y/dx^2 = M/IE = P(L-x)/EI, \qquad (1)$$

and integrating twice,

$$dy/dx = (PLx - Px^2/2)/EI \qquad (2)$$

and

$$y = (PLx^2/2 - Px^3/6)/EI, \qquad (3)$$

we have the small displacement equation for the elastic curve.

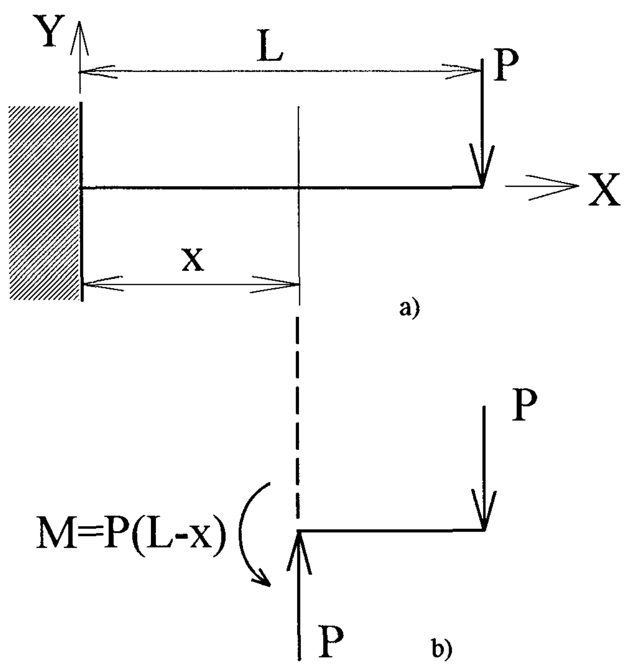

Figure 3. The moment loading on a cantilever beam.
a) the beam's loading b) the free body diagram.

By substituting x=L into equation (3) we may see that the deflection at the free end of the beam is,

$$y_{max} = PL^3/3EI, \tag{4}$$

as one would find in any handbook on structural mechanics (Roark, 1965).

The elastic curve is described by L' in Figure 1. An infinitesimal length, dL', may be described by (Figure 3.),

$$dL'^2 = dx^2 + dy^2, \tag{5}$$

$$dL' = [1 + (dy/dx)^2]^{1/2}dx. \tag{6}$$

The length of the elastic curve, L', may be calculated by integrating dL' over the span, S,

$$L' = \int_{x=0}^{x=S} dL' = \int_0^S [1 + (dy/dx)^2]^{1/2}dx. \tag{7}$$

The radical may be removed by expanding the binomial and the integral becomes,

$$L' = \int_0^S [1 + (1/2)(dy/dx)^2 - (1/8)(dy/dx)^4 + ...]dx, \tag{8}$$

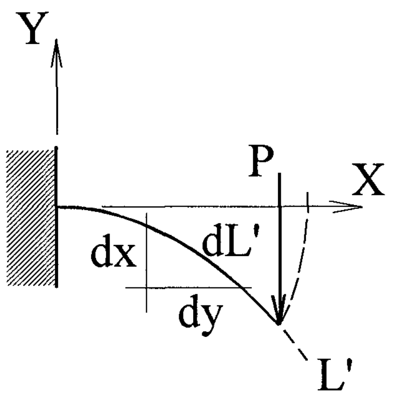

Figure 3. The elastic curve, L', of the beam.

in which the terms above second degree may be ignored for small values of dy/dx. Substituting dy/dx as derived earlier, equation (2),

$$L' = \int_0^S [1 + (1/2)(PLx - Px^2/2)^2/E^2I^2]dx, \tag{9}$$

which integrates (after expansion) to become

$$L' = S + (P/EI)^2S^5/15. \tag{10}$$

The second term on the right hand side represents the foreshortening of the span, S, when the slope, dy/dx, is small. It may also be considered to be the sag of the locus of the free end at the deflection y_{max},

$$sag = (P/EI)^2S^5/15 = y_{max}^2/2R. \tag{11}$$

where R is the radius of curvature of the locus of the free end. By rearranging equation (11) and using the expression for y_{max} derived above, equation (4), we may calculate the radius of curvature of the locus and the location of the kinetic center,

$$R = y_{max}^2/2(sag) = [PL^3/3EI]^2/2[(P/EI)^2S^5/15] = (5/6)L \sim (.8333....)L, \tag{12}$$

noting that

$L \equiv S$.

3.0 A VALIDATION MEASUREMENT

A quick check of the accuracy of the above derivation was made on a Series 5001 parallel blade flexure stage. The set up is shown in Figure 4. The blades of the stage are 2.0 inches long and undergo bending as "double" cantilevers when the table is displaced by the vernier micrometer. Table I presents the measured data on the change in the table height, sag, when the flexures are deflected, y_{max}.

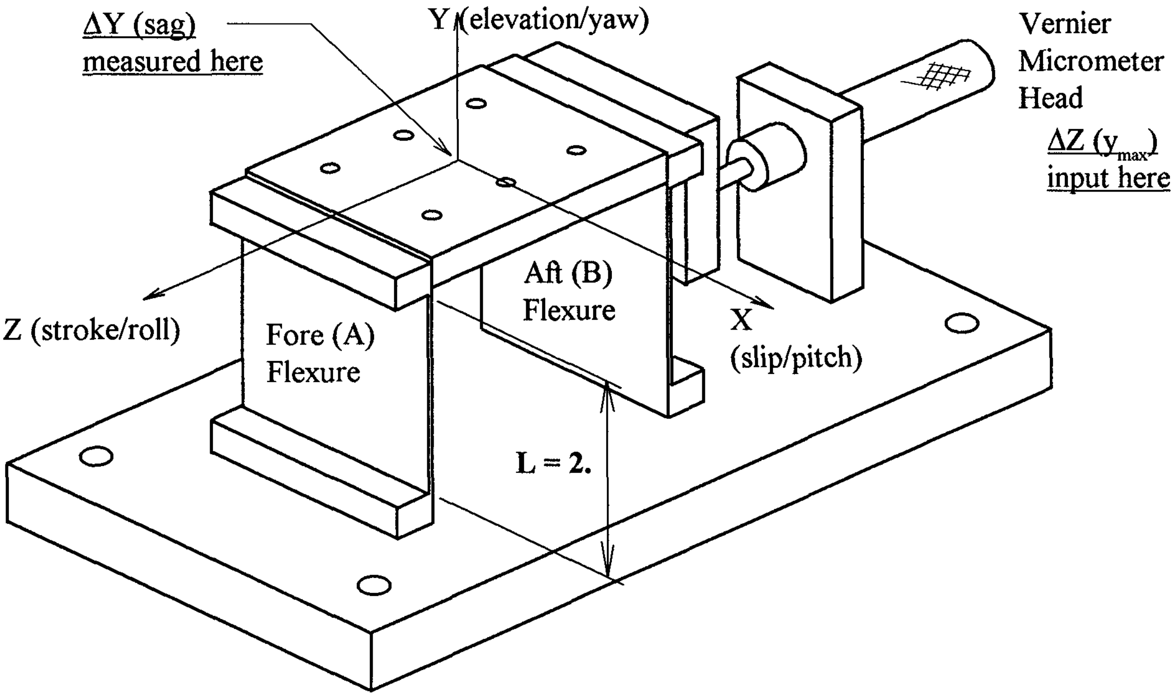

Figure 4. A Series 5001 parallel blade flexure stage with 2.0 inch long flexures

Table I
Measured Radius of Curvature, R, of the Locus of the Free End

$\Delta Z(y_{max})$ in.	ΔY(sag) in.	R in.	R/L -
.1913	-.0108	1.694	.847

The measured R/L, .847, is about 1.7% larger than the theoretical derivation, .8333.... This is consistent with the accuracy of the metrology method but may also include some additional "large displacement" effects. The ΔZ motion (y_{max}) was about ten percent of the span of the flexures and in very careful measurements at displacements of this magnitude one should be able to see some higher order elastic effects; the locus has been assumed to be (only) a second degree function of the deflection (y_{max}), the higher degree terms having been discarded in the formulation of the theory.

4.0 CONCLUSION

Integrating the "small displacement" elastic curve of a beam provides an effective and accurate method for estimating the foreshortening (sag) of the end of an elastic beam in bending at small deflections. The resulting curvature of the locus of the free end of the beam is true only at zero deflection but provides useful estimates at finite deflections. It is seen to be no more than two percent in error when the deflection is ten percent of the span of the beam.

5.0 ACKNOWLEDGMENT

The author is indebted to Malcolm R. Howels of the Lawrence Berkeley Laboratory for pointing out the significance of the above derivation.

6.0 ERRATUM

The author previously published an article (Hatheway, 1995) which used an erroneous derivation for the foreshortening of cantilever beams in bending. An updated and corrected issue of the paper (AEH Doc. No. 93435B, 1995) is available from the author on request.

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