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# Controlling non-linearities in elastic actuators

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## 1. ABSTRACT

A new line of very high precision actuators has been developed by the author (Hatheway, 1988). One of the features of these actuators is a very high degree of linearity over the full range of motion. The actuators are based upon an elastic transducer design which should make both the linear and non-linear characteristics of the actuator predictable from the basic theory of elasticity. The author has measured the characteristics of a typical actuator based upon these elastic principles. In this report he shows that the non-linearities are predictable from the elastic theory for large displacements. The actuators may be calibrated for "open-loop" operation in applications requiring accuracies to 1.8 parts in 10,000, which is the limit of accuracy of this study.

Key words: elastic, angstrom, precision, linear, actuator, transducer, calibration, elasticity.

## 2. INTRODUCTION

There are two sources of non-linearity in elastic transducers: non-linearities in the Young's modulus of the parent material and large displacements which violate the "small displacement" assumption of linear elastic theory. This research examines the influence of large displacements on the deviations from linearity of elastic transducers.

One form of the elastic actuator is the model 1000-1 elastic actuator. It is driven by a micrometer deforming a cantilever beam as shown in Figure 1. The deformations of the cantilever beam create small strains and displacements which may be used

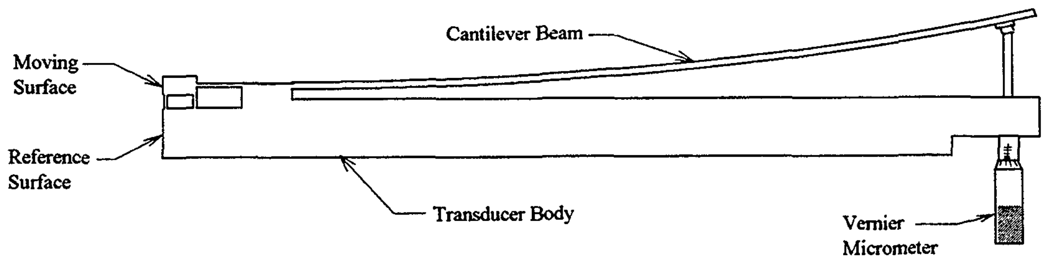


Figure 1. Model 1000-1 Elastic Actuator.

to displace an attached component through small motions. The strains and displacements are dependent upon the magnitude of the load created by deforming the cantilever beam. The principal load in the actuator is the bending moment created at the root of the beam. Traditional analysis of the deflection of the beam's force displacement characteristic assumes that the displacement of the beam is small, or more exactly that the slope of the deflected beam is negligibly small (compared to unity). This assumption is never strictly true but it is usually adequate for engineering work. The 1000-1 actuator has a nominal range of one micrometer (10,000 angstroms) and a repeatability of one angstrom.

Testing of elastic actuators of the type shown in Figure 1 has been completed. The results are apparently quite linear as shown in Figure 2. Over the full range of motion for the specific transducer body tested (9,337 angstroms) the actuator was repeatable to about 1.1 angstroms rms. However, when a best fit straight line is removed from the position data to calculate the residuals, they show a systematic curved behavior indicating a residual non-linear characteristic in the actuator (Figure 3). This research

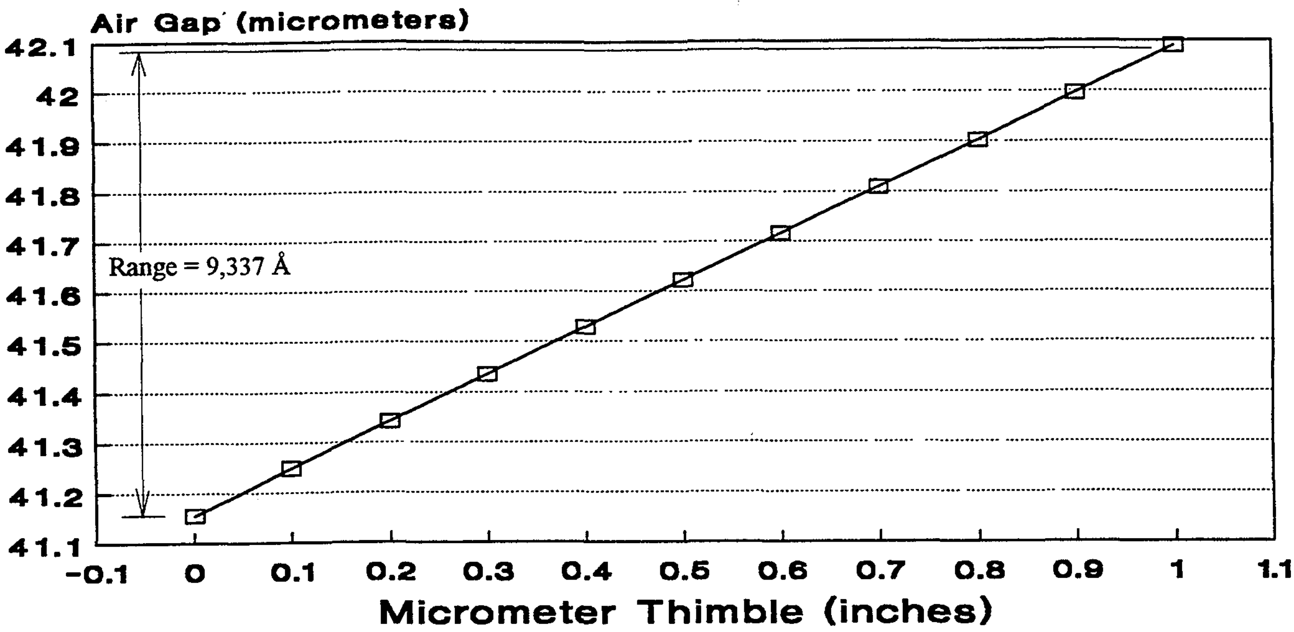


Figure 2. Characteristic curve for a Model 1000-1 actuator, as tested.

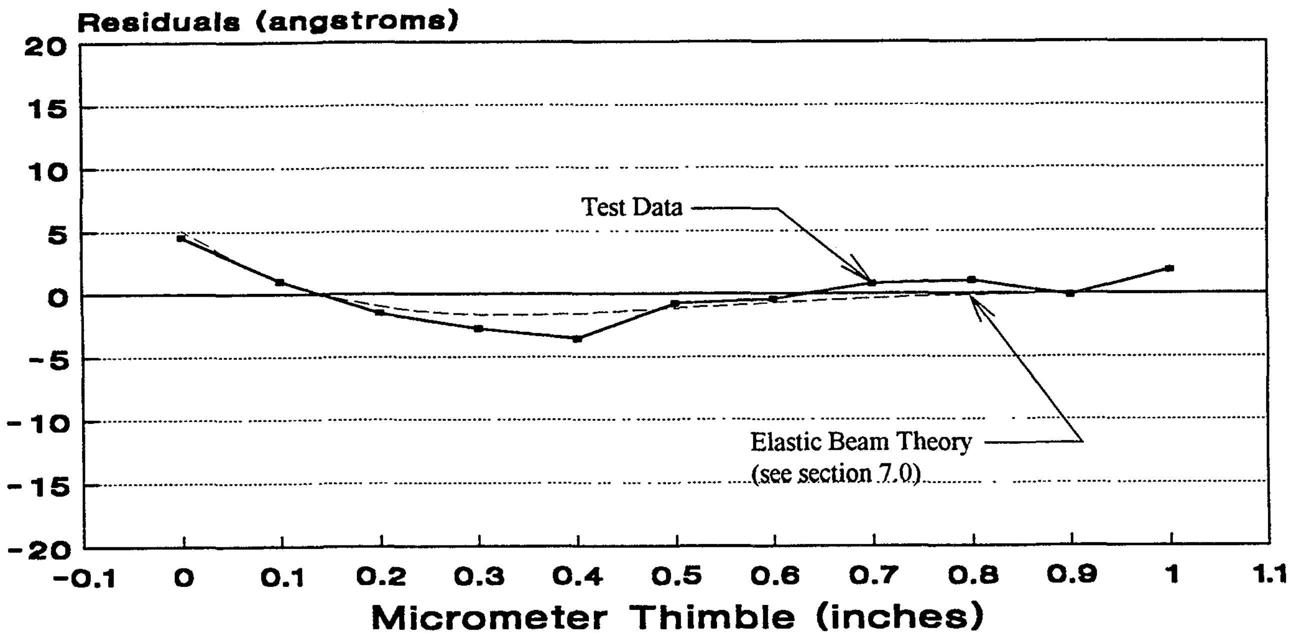


Figure 3. Non-linear residuals from a linear fit to the data of Figure 2.

attempts to determine the extent to which this non-linearity is related to the relatively large deflections of the cantilever beam as opposed to possible non-linearities in the stress-strain characteristics of the material of construction.

The test data shown in Figures 2 and 3 were taken with an Andeen-Hagerling precision capacitance bridge measuring a custom designed, high stability, air-gap capacitor. The precision of these measurements has been shown to be on the order of 1.0 angstroms, rms, the limit of resolution of the micrometer used to drive the actuator (Hatheway, 1994).

The construction material for the transcucer body is aluminum alloy 6061-T6 which is well known and well documented . All available data for this alloy indicate that it's stress-strain curve is linear over a wide range, approaching the elastic limit of about 247 MPa. (36,000 psi.). However, the accuracy of the instrumentation traditionally used for measurements of the Young's modulus is about one percent (one part in one hundred). Since the actuator is repeatable to one part in ten thousand we need to understand the non-linearities more precisely than the data on material properties allows.

Since there are only two sources of non-linearity in the actuator the analysis of the large deflection non-linearities may allow us to assess the relative influence of the two sources.

### 3. THE CANTILEVER BEAM WITH A FORCE AT ITS FREE END NORMAL TO ITS NEUTRAL AXIS

We may consider that the displacements produced by the actuator are controlled by deflections of the cantilever beam. Simple calculations for the small displacements produced by the transducer may be estimated from the conventional equations for a cantilever beam available in ay text on structural mechanics (Popov, 1952). These equations are (ref. Figure 4):

$$y = PL^3/3EI \qquad (1)$$
$$x = L$$
$$\theta = PL^2/2EI$$

where,

$y$ is the transverse location of the free end,
$x$ is the axial location of the free end,
$\theta$ is the angular rotation of the free end,
$P$ is the transverse force normal to the neutral axis,
$L$ is the length of the cantilever beam,
$E$ is the Young's modulus of the material and
$I$ is the moment of inertia of the beam's cross section.

These simple linear equations assume small (approaching zero) displacement and no axial force. It will be convenient to expand these equations (1) into the form of infinite series,

$$y = (2L/3)(PL^2/2EI) + a_2L(PL^2/2EI)^2 + a_3L(PL^2/2EI)^3 + \cdots \qquad (1a)$$
$$x = L + b_1L(PL^2/2EI) + b_2L(PL^2/2EI)^2 + b_3L(PL^2/2EI)^3 + \cdots$$
$$\theta = (PL^2/2EI) + c_2(PL^2/2EI)^2 + c_3(PL^2/2EI)^3 + \cdots$$

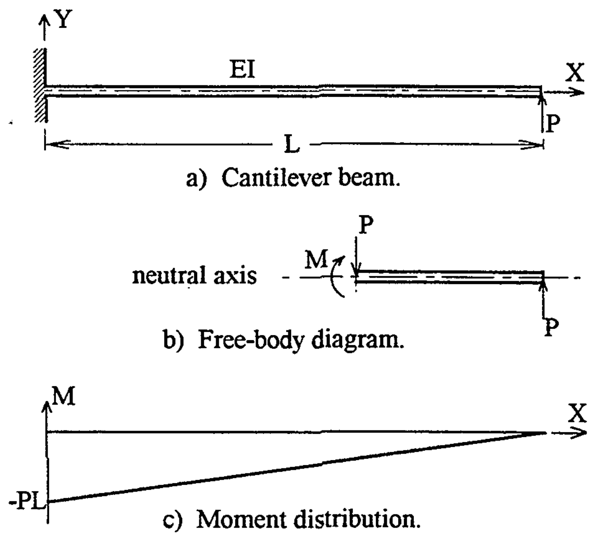


Figure 4. A cantilever beam with a transverse force.

in which the first terms define the linear behavior, as in (1) and the remainder of the terms define the non-linear components of location of the free end (note the coordinate system). The magnitudes of these non-linearities depends upon the coefficients $a_i$, $b_i$ and $c_i$ that accompanies each term. Since the magnitudes of the constant and linear terms are known these coefficients are also known,

$a_0 = 0.0$, $a_1 = 2L/3$
$b_0 = 1.0$
$c_0 = 0.0$, $c_1 = 1.0$ .

It may be further reasoned from the geometry of the cantilever beam that the first derivative of x with respect to P will be 0.0 and therefore

$b_1 = 0.0$

and our complete equations for the large deflection of the cantilever beam become

$$y = (2L/3)(PL^2/2EI) + a_2L(PL^2/2EI)^2 + a_3L(PL^2/2EI)^3 + \cdots \quad (1b)$$
$$x = L + b_2L(PL^2/2EI)^2 + b_3L(PL^2/2EI)^3 + \cdots$$
$$\theta = (PL^2/2EI) + c_2(PL^2/2EI)^2 + c_3(PL^2/2EI)^3 + \cdots .$$

It should be noted that the modulus in these equations has been selected to be the linear component of rotation of the free end of the beam, $\theta$. In the research reported here the maximum angular deflection has been about 1/10 radian. As a result the magnitudes of the non-linear terms tend to fall off about an order of magnitude for each increase in the degree of the modulus. Only the first few terms are required to evaluate the non-linearities in the action of the actuator's cantilever beam.

In the elastic transducer the cantilever beam is deflected by an accurately measured displacement from the tip of a micrometer. Several features of the mechanics of the micrometer drive should be noted.

First, since the micrometer is mounted to the structure of the transducer's relatively stiff body, the micrometer operates over a fixed span. Subsequently, as the micrometer deflects the cantilever beam the micrometer tip slides longitudinally along the beam to accommodate the difference between the beam's undeformed length, L, and its deformed length, $L+\Delta L$, over the fixed span which is also L. The conventional equations for the deflection of the cantilever beam assume small deflections, i.e., $\Delta L=0$.

Second, the tip of the micrometer is spherical so the measured deflection imparts a force to the free end of the beam that is perpendicular to the neutral axis of the beam. The fact that the beam takes on a finite slope as it deflects produces an axial force component as well as the transverse component. The conventional equations for the cantilever beam assume small deflections and therefore ignore the effects of the axial component of force on the beam.

Finally, the conventional equations for the cantilever beam assumes that the forces and deflections are measured at the neutral axis of the beam. In practice the micrometer operates through a track-pad and anvil combination which produce an effective contact surface which is offset some distance from the neutral axis.

All of these influences must be considered to quantify the non-linear components of motion of the transducer.

Since the beam is nominally free to rotate at the micrometer the beam's deflection is better described as being due to a transverse force at the free end. The bending moment causing deflections is variable over the length of the beam, from a maximum at the fixed end to early zero at the micrometer. Also, in the transducer the force is normal to the neutral axis (not parallel to the "y" axis as assumed in the linear case) of the beam. Although the force will initially, at zero displacement, be parallel to the "y" axis, at all finite displacements the force will have a component in the "x," or axial, direction as well. This "x" component of force contributes its own bending moment to the beam, increasing the deflection and making it impossible to determine the exact moment distribution in the beam.

Another complicating feature of the transducer is that the span is fixed and the beam's length is a variable because the tip of the micrometer may slide longitudinally on the surface of the beam to maintain the span. This latter effect has the influence of softening the beam because the load is applied at a slightly greater distance from the fixed end than assumed in equations (1b).

## 4. THE CONSTANT-MOMENT BEAM

Consider the case of a cantilever beam loaded by a bending moment, $M_e$, at its free end (Figure 5). If a free body diagram is prepared by cutting the beam at any section between the free end and the fixed end it may be determined from considerations of static equilibrium that the net moment at the cut section must be equal to and opposite in sense to the moment applied at the free end. Since the cut section may be at any longitudinal location in the beam it is clear that the bending moments at all cross sections of the beam are equal to each other and also equal to (and opposed to) the moment applied to the free end. Elementary structural mechanics shows (Timoshenko, 1955, page 138) that if a beam is composed of a uniform isotropic elastic material and if the beam's cross sectional shape is constant over the length of the beam, then the radius of curvature of the beam at any cross section is proportional to the bending moment at that cross section. It follows that any uniform cantilever beam deformed only by a moment at its free end will have its length bent into a circular arc (of constant radius). This knowledge allows us to determine the exact, non-linear, shape of a cantilever beam loaded only by a moment at its free end.

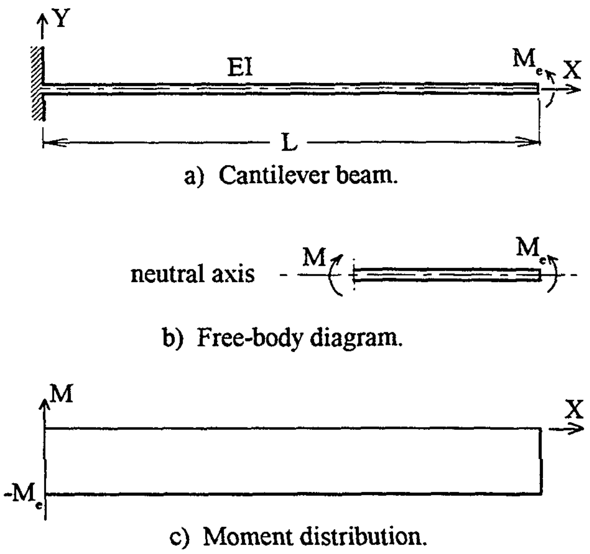

Figure 5. A cantilever beam with an end moment.

It has been shown (Hatheway, 1989) that the exact equation for the deflected free end of a cantilever beam with a uniform moment, M, applied at the free end may be expressed in parametric form by the equations,

$$y = (L/2)(ML/EI) - (L/24)(ML/EI)^3 + (L/720)(ML/EI)^5 - \cdots \quad (2)$$
$$x = L - (L/6)(ML/EI)^2 + (L/120)(ML/EI)^4 - \cdots$$
$$\theta = ML/EI + (1/3)(ML/EI)^3 + (2/15)(ML/EI)^5 + \cdots .$$

These expressions are based upon the Maclaurin's series for the sine and cosine trigonometric functions. The modulus in the expressions is again the linear component of rotation of the free end. Only the first three terms are shown because the influence of the higher order terms becomes increasing small as their order increases. Using just the first terms of these equations yields the traditional linear, small deflection, equations for the deflection of a cantilever beam deflected by an end moment, M,

$$y = ML^2/2EI \quad (3)$$
$$x = L$$
$$\theta = ML/EI$$

The subsequent terms in the expressions (2) describe the deviations from linear behavior. In the "y" deflection the terms indicate a stiffening of the beam as the load, M, is increased. The "x" deflection shows a complimentary foreshortening of the span of the beam's length under the same conditions.

These equations have been developed for a cantilever beam with a moment applied at the free end and it is based upon the assumption that he moment will bend the neutral axis into a circular arc (true only for beams in which the bending moment is a constant over their entire length).

This knowledge will be used to estimate the non-linearities in the bending stiffness of the cantilever beam in the transducer. It is proposed that for any cantilever beam with bending moments varying throughout its length the magnitude of the non-linearities in the stiffness (proportional to the non-linearities in the deflection, y) and the non-linearities in foreshortening, x, will be of about the same magnitude *with respect to the corresponding non-linearities in the beam with a constant bending moment throughout its length.* This proposition is important because it is difficult to accurately measure the non-linearities in the cantilever beam's spring constant whereas it is reasonably easy to measure the beam's foreshortening from which, via similarity, the non-linearities in the beam's stiffness may be estimated, even for very small effects.

If it is assumed that the non-linearities will be dominated by the low-order terms, we may truncate the equations in (2) to two terms on the right hand side. If we identify the first term as the linear term and the second term as the non-linear term we may establish the relative magnitudes of the non-linearities in y and x as the ratios of the second terms to the first terms,

$$y_{non}^{large\ deflection}/y = [(L/24)(ML/EI)^3]/[(L/2)(ML/EI)] = (ML/EI)^2/12$$

and

$x_{non}^{large\ deflection}/L = [(L/6)(ML/EI)^2]/L = (ML/EI)^2/6$

from which it may be seen that

$y_{non}^{large\ deflection}/y = (1/2)x_{non}/L$ ; (4)

> *The relative magnitude of the non-linear component of transverse defection is 1/2 the relative magnitude of the foreshortening.*

## 5. NON-LINEARITIES IN THE CANTILEVER BEAM

The above development will allow estimation of the non-linear behavior of the cantilever beam in the transducer being studied. The cantilever beam of the transducer has been deflected through its maximum range of 25.4 mm (1.0 inch) and the corresponding foreshortening of the beam has been measured:

foreshortening = $x_{non}^{large\ deflection}$ = 1.034 mm. (.0407 inches).

The span, L, of the cantilever beam from the root to the micrometer tip is 355.6 mm. (14.000 inches). Therefore the foreshortening as a fraction of the length of the beam is,

$x_{non}^{large\ deflection}/L = 1.034/355.6 = .002908.$

The non-linearity in the stiffness may now be estimated from equation (4) above,

$y_{non}^{large\ deflection}/y = (1/2)x_{non}/L = (1/2)(.002908) = .001454$ .

The degree of similarity between the simple cantilever beam and the constant moment beam may also be estimated from the measured data. From equations (2) the foreshortening of the constant moment beam may be calculated to be,

$x_{non}^{constant\ moment}$ = 1.209 mm. (.0476 inches).

As was pointed out above the foreshortening of the simple cantilever beam was measured to be,

$x_{non}^{simple\ beam}$ = 1.034 mm. (.0407 inches).

The similarity, S, of these two beams may be quantified to be the ratio of their non-linear terms at the same deflections,

$S = 1.034/1.209 = .855$ .

Since

y = 25.4 mm. (1.0 inches)

$y_{non}^{large\ deflection} = Sy(y_{non}^{large\ deflection}/y) = (.855)(25.4)(.001454) = .03158$ mm. (.001243 inches.)

which is a stiffening effect, based upon the signs on the terms in equations (2), and assumes that the neutral axis of the beam has a fixed length equal to L. In reality, the micrometer tip slides axially along the beam as the beam bends while maintaining a constant span equal to L. Consequently, the beam's active length increases as the free end deflects, maintaining a constant span. The amount of this axial sliding is determined by the beam's foreshortening in the "x" direction. Since the deflected end of the beam has rotated through an angle, θ, the axial sliding tends to reduce the stiffness of the cantilever beam.

The slope at the deflected free end may be expressed as (derived from equations (1)),

$\theta = 3y/2L = (3 \times 25.4)/(2 \times 355.6) = .1071$ rad.

The foreshortening of the beam has been measured as 1.034 mm. (.0407 inches) so the reduction in deflection is,

$y_{non}^{fixed\ span} = 1.034 \tan(.1071) = .1111$ mm. (.004375 inches).

The net non-linearity in the stiffness of the cantilever beam based upon a constant span is,

$y_{non}^{large\ deflection,\ fixed\ span} = .03158 - .1111 = -.0795$ mm. (-.00313 inches).

This combined non-linearity represents the sum of 1) relaxation of the beam's original end point when the micrometer tip slides axially to maintain a constant span during the deflection of the beam and 2) the stiffening of the beam under the influence of finite displacements. If one plots the deflection of the beam's original end point versus the force required at a fixed span necessary to produce that deflection one would observe a gentle curve. At full deflection the deviation of the curve from the slope at the origin is .0795 mm., or about 31 parts per 10,000. This also represents the proportion by which the micrometer's force is reduced by the non-linearities. Since the transducer's motion is proportional to the stresses and strains produced by the force, its motion at full deflection will deviate about 29 angstroms, over its full range of 9,337 angstroms, from its slope at the origin.

## 6. CONTROLLING THE NON-LINEARITIES

It is possible to control the above softening non-linearity by driving the beam's deflection on a surface that is off-set from the beam's neutral axis. If this offset surface is on the micrometer's side of the neutral axis it will tend to stiffen the beam by the amount

$$y_{non}^{offset} = d(\sin\theta)(\tan\theta) \quad (5)$$

where d is the offset between the beams's neutral axis and the drive surface and θ is the local slope of the deflected beam. Evaluating at a full deflection of .1071 radians the offset that is just necessary to correct the .0742 mm. non-linearity above is calculated as follows,

$.0795 = d(\sin .1071)(\tan .1071)$

$d = 6.917$ mm. (.2723 inches).

In the transducer the offset is established by the anvil, tip cushion and spherical micrometer tip. The contributions of each are,

| | |
|---|---|
| anvil | 1.85 mm. (.0725 inches) |
| tip cushion | 1.59 mm. (.0625 inches) |
| mic. tip | 2.245 mm. (.0883 inches) |
| offset (sum) | 5.68 mm. (.2233 inches). |

This is slightly less offset than is necessary to fully compensate for the full 29 parts in 10,000 non-linearity. The uncorrected portion may be estimated to be

$y_{non}^{uncorrected} = 31(6.917-5.68)/6.917 = 5.5$ parts in 10,000 (5.1 angstroms deviation over 9,337 angstroms range).

## 7. DISCUSSION

The full non-linear load vs. deflection curve for a cantilever beam may be expresses as a power series. The author has developed the load vs. deflection series for the case of a constant moment beam. The analytical method for calculating the non-linear behavior of a simply loaded cantilever beam assumes similarity between a simply loaded beam (a transverse force normal to the neutral axis) and a constant moment beam (bending moment applied to the free end). Specifically, it was assumed that the foreshortening is an indication of non-linearity to expect in the bending deflection and the ratio of these two was determined from the established equations for a constant moment beam.

The over-all resolution of the experimental method may be determined from the magnitude of the over-all metrology error of 1.5 parts in 10,000 (Hatheway, 1994) and the limit of resolution of the vernier micrometer head driving the actuator, 1.0 parts in 10,000. Combining these two contributions by the "rss" techinque predicts a combined calibration uncertainty of 1.8 parts in 10,000.

Similarity does not necessarily offer guidance on the distribution of the non-linearities among the terms of the power series, (1a). However, a detailed assessment of the way the axial component of load develops while the beam is being deflected suggests a very strong fifth degree function. A magnitude of 10. for $a_5$ requires a magnitude of -.085 for $a_3$ (reasonable values). If we neglect the even terms and all terms above $a_5$ the result is the dashed curve for elastic beam theory in Figure 3 which duplicates the test data within the accuracy of the experimental set-up. The observed non-linearities appear to be explained by the geometric non-linearities associated with the deflection of the beam. It does not appear that non-linearities in the aluminum alloy are significant at these levels and evaluation of their magnitude will have to await refined experimental accuracy.

## 8. CONCLUSION

Non-linear behavior appears to have three components in the elastic actuators studied in this research. All three are deviations from the usual assumptions about the loading and deflection of beams:

1) large deflections (slopes are not zero),
2) constant span length (beam length increases with deflection) and
3) an offset drive surface (not on the neutral axis).

Considering each component independently accounts for the observed non-linearities in the actuator, within the accuracy of the test data. There is no reason to suspect non-linearities contributed from the material of construction.

The actuator designer and the experiment designer have two transducer features available to them for controlling non-linearities:

1) the offset distance for the drive and
2) the length of the cantilever beam.

Increasing the length of the cantilever beam is effective but often undesirable because it increases the length of the transducer body in the actuator. However, adjusting the offset distance has little effect on the transducer body size. In the actuator tested in this research the offset drive removed about eighty-nine percent of the residual non-linear deviations. Achieving the same linearity by increasing the length of the transducer would require tripling the length of the cantilever beam. It appears that both the range and linearity of the transducer may be predicted very accurately by using the basic elastic theory of materials and accounting for the influence of finite deflections.

This research assures that elastic actuators may be calibrated for "open-loop" operation in applications requiring accuracy as fine as 1.8 parts in 10,000. Nothing in this research suggests this to be a limiting value; but rather that the demonstrated "open-loop" accuracy will likely be refined as the experimental methods are improved. The actuators studied in this research are the subject of U. S. Patents No. 5,187,876 and 5,400,523.

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