Line-of-sight reference frames: a unified approach to plane-mirror optical kinematics

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ABSTRACT

Plane mirrors are commonly used to steer, rotate, and stabilize the image and line-of-sight (LOS) of optical systems. Plane-mirror optical kinematics is the study of fixed, flexured, and gimbaled mirrors in this application. LOS pointing, stabilization, image mapping, image derotation, boresight coefficient determination and mechanical tolerance analysis are all areas of plane-mirror optical kinematics. Specific problems in these areas have been addressed in the literature by a wide variety of analytical techniques. None of these techniques, however, have been generalized for application across the field. A unified analytical framework for plane-mirror optical kinematics is presented in this paper. This methodology is based on a new optical kinematic construction, the line-of-sight reference frame. An LOS reference frame is a unit vector triad that defines the LOS and the associated image plane. The use of optical and basis transformations is central to LOS reference frame analysis. These transformations often look similar, but are conceptually unrelated. A thorough understanding of each is required. Both are discussed in detail, and a direct comparison is made. Use of LOS reference frames as a general optical kinematics tool is outlined. The pertinent LOS reference frames of an aerial photography system are constructed as an example.

1. INTRODUCTION

Vector methods have been used in the analysis of gimbaled plane mirror systems for over twenty-five years. The optical line-of-sight (LOS) is represented as a vector, and the reflection properties of the mirror are represented as a matrix. The reflected, outgoing LOS is found by operating on the incoming LOS with the mirror matrix. Such an operation represents an optical transformation. Vector methods can also be applied to kinematic analysis of gimbals. Unit vector sets that define kinematic reference frames are attached to the gimbals. The relative orientations of the gimbals are then described by the basis transformations between the unit vector sets. Optical transformations and vector basis transformations must be combined to determine the relation between the direction of the LOS and orientations of the gimbals.

An analytical method for the analysis of gimbaled plane-mirror systems is presented that makes use of a new optical kinematic construction, the line-of-sight reference frame. LOS reference frames can be used in image mapping, rotation and derotation analysis, boresight coefficient determination, stabilization analysis, and mechanical tolerance analysis. Basis and optical transformations are both used with LOS reference frames. A thorough discussion of each is presented, with an eye toward their application to LOS reference frames. Their similarities and differences are demonstrated by a description of an example system.

2. HISTORICAL PERSPECTIVE

To the optical designer, the term "matrix method" generally implies the use of matrices, as opposed to the use of geometric "ynv" ray-tracing, to design and analyze paraxial lens systems. Sinclair provides a summary of the use of paraxial transfer matrices[1], which is also discussed in most introductory optics textbooks[2]. Matrix analysis of plane-mirror systems has often been considered a subset of this field. Brouwer devotes a brief first chapter of his book to plane-mirror systems[3]. However, a substantial body of literature exists that refers specifically to problems associated with plane-mirror systems. Many of the papers that treat plane-mirror problems were written, not by optical designers, but by controls, design, system, and structural engineers. Their publications address certain areas of the system-level performance of optical instruments that are generally outside the realm of the optical designer, such as pointing control, line-of-sight stabilization, and structural interaction. The methodology discussed herein is of general application to these areas, and, as such, this paper is directed to the engineering side of the optics community.

Analytical, as opposed to geometric, treatment of plane-mirror kinematics first appeared in the literature during the years following World War I, with Smith first outlining the matrix approach in 1928[4-6]. The dramatic increase in instrument complexity that occurred after World War II spawned further progress in analytical development[7-14]. Much of this work discussed the mirror-equivalent, internal-reflection prism. Levi's summary is typical[15]. Polasek directly addressed gimbaled mirror systems in 1967[16], with Royalty adding to this work as late as 1990[17]. Redding and Breckenridges's work last year is in response to the growing interest in wavefront control, segmented optics, deformable optical elements, and active structural control[18].

Most of the references above discuss the transformation of light rays, modeled as vectors, by plane-mirror elements. The foundation of image transformation, as opposed to light ray, or line-of-sight transformation, is found in the works of Hopkins, who uses "rotated coordinate axes" to discuss image orientation[19,20]. Sitsov uses a vector triad throughout his series of papers on the properties and synthesis of certain fixed plane-mirror systems[21-25]. These techniques have not been extended to movable-component, vehicle-mounted systems.

While the literature is rich in specific analytical techniques for particular plane-mirror systems, this paper provides a new methodology that is applicable to a broad class of problems in plane-mirror optical kinematics. Further, the technique directly produces the algebraic equations relating the system states. These equations are easily programmed if numerical data is desired. In addition, the method is based on modern vector analysis techniques. Thus, the analyst need not be familiar with classical optical constructions such as image and object space to solve problems. Finally, the clear distinction described herein between optical and basis transformations allows the technique to be discussed and applied algorithmically. That is, formulas and procedures can be derived for general application to the field. It is the authors' hope that the methodology of this paper goes a long way in changing the analysis of plane-mirror systems from an art to a science.

Much of kinematic construction of this paper is based on the work of Kane in his landmark textbook on rigid-body dynamics[26]. Specifically, the concept of a vector triad, especially a *non-physical* vector triad, as a reference frame for the purposes of modeling and analysis is core to the method. While the reader is certainly encouraged to read the first chapter of the reference, familiarity with "Kane's Method," which is usually taught in a one-year graduate course, is certainly *not* required for application of the method.

3. LINE-OF-SIGHT VECTORS

Most pointed optical systems contain an imaging system. The imaging components are generally radially symmetric and coaxially mounted with respect to a central optical axis. This axis forms the line-of-sight of the system. The system reticle (the crosshairs) is aligned with, and thus designates, the line-of-sight. A unit vector is used to model the direction of the optical axis. Though light energy enters the primary aperture and travels through the system to the imaging focal plane, we choose for modeling purposes the opposite convention. Thus, the LOS vector $\underline{r}_1$ is pointed outward, as in the simple system shown in Figure 1.

The direction of the optical axis, and thus its LOS vector, is unaffected by purely imaging elements. Only those elements that change the direction of the optical axis affect the LOS vector. A new LOS vector is designated every time the optical axis changes direction. The LOS vector $\underline{r}_1$ leaving the focal plane in Figure 1 is unchanged by the lens. The optical axis is reflected (or *folded)* by the mirror, however, and a new LOS vector $\underline{s}_1$ is required to describe the new orientation. By this convention, a system with one optical axis could have several LOS vectors.

Nearly all modern optical systems have mirrors, prisms, or other elements to fold the optical axis. Imaging elements that are mounted symmetrically to a folded optical axis are considered optically coaxial for the purposes of imaging system design. Since only two dimensions are required to design a radially-symmetric imaging system, a system with folds in the optical axis can be unfolded to simplify element layout and design. Note, however, that optical kinematics is generally a three-dimensional problem. All the folds in a system must be considered during optical kinematic analysis.

4. IMAGE VECTORS

Imaging systems are designed to create an image of the optical scene on a curved or flat surface. The image is converted to useful information on this surface, either by exposing film or by stimulating an electronic detector. Whether this surface be curved or flat, the tangent plane at the optical axis is always perpendicular to the axis. For modeling purposes then, we assume the imaging surface to be planar and will refer to it generically as a focal plane.

The orientation of the focal plane is defined by its normal, the LOS vector. However, optical kinematics is also concerned with the orientation of the image as projected onto the focal plane. To aid in the analysis of image orientation, image vectors are introduced. Image vectors are defined to be perpendicular to the LOS and of unit magnitude. Note that one image vector is sufficient to determine the rotational orientation of an image. However, two-nonparallel image vectors are required to track the image inversion of folded optical systems. For convenience, we select these two image vectors to be at right angles to each other. Further, we align the image vectors with the natural orientation of the focal plane, which is usually rectangular. Thus, the image vectors $\underline{r}_2$ and $\underline{r}_3$ shown in Figure 1 are aligned respectively with the horizontal and vertical directions of the focal plane. Though image vectors do not represent outgoing rays of the system, they none the less are transformed by the optical system. This process, as described in the next section, is fundamental to the analytical methodology of this paper.

Unlike the LOS vector, image vectors are affected by optical lenses. For instance, the simple lens system shown in Figure 1 rotates the image 180° about the optical axis. The transformations of lens systems are not covered in this paper. The focusing lenses and focal planes of most systems are housed together as a camera. Since the orientation of the focal plane relative to the camera housing is usually known and fixed, the LOS and image vectors $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ can be fixed to the camera LOS as shown in Figure 2 without loss of generality.

5. LINE-OF-SIGHT REFERENCE FRAMES

The combination of a line-of-sight vector and two mutually-normal image plane vectors collectively forms a line-of-sight vector set. An LOS vector set kinematically defines a line-of-sight reference frame, and the two terms will be considered interchangeable. For example, the line-of-sight vector $\underline{r}_1$ and the horizontal and vertical image vectors $\underline{r}_2$ and $\underline{r}_3$ of the camera in Figure 2 together define LOS reference frame *R*. LOS vector set $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ forms an orthonormal, right-handed triad.

If the incoming LOS vector $\underline{r}_1$ and the outgoing LOS vector $\underline{s}_1$ are both expressed in terms of a set of mutually normal unit vectors $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$ as

$$\underline{r}_1 = r_{11}\underline{u}_1 + r_{12}\underline{u}_2 + r_{13}\underline{u}_3 \qquad\qquad \underline{s}_1 = s_{11}\underline{u}_1 + s_{12}\underline{u}_2 + s_{13}\underline{u}_3$$

then the optical transformation of $\underline{r}_1$ into $\underline{s}_1$ by the mirror can be represented by the matrix equation

$$[\underline{s}_1]^T = [\mathbf{M}][\underline{r}_1]^T \tag{1}$$

where [M] is the mirror transformation matrix and

$$[\underline{s}_1] = [s_{11} \; s_{12} \; s_{13}] \qquad [\underline{r}_1] = [r_{11} \; r_{12} \; r_{13}] \qquad [M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

The incoming image vectors $\underline{r}_2$ and $\underline{r}_3$ are also optically transformed by the mirror into outgoing image vectors $\underline{s}_2$ and $\underline{s}_3$. Consequently, the incoming line-of-sight vector set $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ which defines LOS reference frame R is optically transformed by the mirror into the outgoing LOS vector set $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ which defines LOS reference frame S. Eq. 1 applies to the image vectors as well as the LOS vectors, so that the optical transformation between LOS vector sets $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ and $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ can be expressed as

$$\begin{bmatrix} s_{11} & s_{21} & s_{31} \\ s_{12} & s_{22} & s_{32} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \tag{2}$$

where

$$\underline{s}_i = s_{i1}\underline{u}_1 + s_{i2}\underline{u}_2 + s_{i3}\underline{u}_3 \qquad \underline{r}_i = r_{i1}\underline{u}_1 + r_{i2}\underline{u}_2 + r_{i3}\underline{u}_3 \qquad i=1,2,3$$

That vectors $\underline{s}_2$ and $\underline{s}_3$ actually represent the transformation of the image can be shown by geometric ray-tracing.

Equation 2 can be alternately expressed using two different compact notations:

$$[\underline{s}_1 \ \underline{s}_2 \ \underline{s}_3] = [\mathbf{M}][\underline{r}_1 \ \underline{r}_2 \ \underline{r}_3] \tag{3}$$

$$[\mathbf{S}]^T = [\mathbf{M}][\mathbf{R}]^T \tag{4}$$

Comparison of Eqs. 2, 3, and 4 yield three equivalent forms for the LOS reference frame matrices (LOS matrices) [**S**] and [**R**]:

$$[\mathbf{S}] = \begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \underline{s}_3 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad [\mathbf{R}] = \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The fully expanded notation of Eq. 2 emphasizes the scalar components of the vectors and is used when operational details are important. The notation of the Eq. 3 is used when the vectors within the LOS matrices are to be considered as single entities, irrespective of their scalar components. Finally, the notation of Eq. 4 is used to express the vector triad as a single quantity. Equations are expressed throughout the remainder of this paper in more than one notation if no one form is better or worse than another.

The mirror transformation matrix is discussed in detail in the next section. For now, it suffices to state that [**M**] is *orthogonal*. This property guarantees that orthogonality between vectors is preserved by the matrix transformation. Further, it ensures that vector length is unchanged. Thus, if the orthonormal LOS reference frame $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ is transformed through the mirror as in Figure 2, the resulting LOS reference frame $S(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ will also be orthonormal. The vectors $\underline{s}_2$ and $\underline{s}_3$ are taken as the model of the transformed optical image. Thus, an LOS vector set or LOS reference frame provides, in one kinematic quantity, complete information on the pointing direction of the optical axis and on the rotational and inversional orientation of the optical image. Note that the location of the origin of the LOS reference frame *S* along the optical axis between optical elements has no importance, and in this sense LOS vector sets can be thought of as free vectors. LOS reference frames provide directional and orientational, but not positional, information.

6. THE MIRROR TRANSFORMATION MATRIX

The mirror transformation matrix $[\mathbf{M}]$ is assembled from the scalar components (n_1,n_2,n_3) of the mirror normal vector $\underline{n}$ as follows:

$$[\mathbf{M}] = \begin{bmatrix} 1-2n_1n_1 & -2n_1n_2 & -2n_1n_3 \\ -2n_2n_1 & 1-2n_2n_2 & -2n_2n_3 \\ -2n_3n_1 & -2n_3n_2 & 1-2n_3n_3 \end{bmatrix} \tag{5}$$

Matrix $[\mathbf{M}]$, as expressed by Eq. 5, is well known. Details of the development of Eq. 5 can be found in references 15 and 16. Matrix $[\mathbf{M}]$ can be shown to be orthogonal by noting that the transpose of $[\mathbf{M}]$ is identical to its inverse. Since $[\mathbf{M}]$ is also symmetric, the following relations apply:

$$[\mathbf{M}] = [\mathbf{M}]^T = [\mathbf{M}]^{-1} \tag{6}$$

Though orthogonality is preserved by $[\mathbf{M}]$, handedness is not. As seen in Figure 2, the outgoing LOS vector set $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ is left-handed. Dextrality can be restored by transforming *S* through another mirror. The change in handedness of the LOS reference frame is indicative of the *inversion/reversion* property of mirrors.

Note from Eq. 5 that $[\mathbf{M}]$ is insensitive to a sign reversal of the mirror normal. Therefore, both sides of a double-sided mirror are represented by the same mirror matrix. This fact and Eq. 6 can be used with the concepts presented in the next section to show that a LOS vector set passing through a parallel mirror pair is unchanged. This is another way of saying that a mirror is self-inverting.

7. THE OPTICAL TRANSFORMATION MATRIX

The overall optical transformation matrix associated with multiple mirrors is obtained by post-multiplying the individual mirror transformation matrices in sequence. Thus, the matrix product $[\mathbf{M}_2][\mathbf{M}_1]$ represents the optical transformation matrix for the parallel mirror pair of Figure 3. Optical transformation matrices model the transformation of an LOS reference frame through an optical system or instrument. Optical transformation matrices, which are orthogonal but not generally symmetric, will be designated by the letter $[\mathbf{O}]$.

Consider the two-mirror optical instrument shown in Figure 3. The incoming camera LOS vector set $(\underline{q}_1,\underline{q}_2,\underline{q}_3)$ is transformed by the mirror M_1 into LOS vector set $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$, which is then transformed by mirror M_2 into the outgoing LOS vector set $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$. Operationally, the optical transformation matrix gives the outgoing LOS set $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ in terms of the incoming set $(\underline{q}_1,\underline{q}_2,\underline{q}_3)$ as

$$[\underline{s}_1\ \underline{s}_2\ \underline{s}_3] = [O][\underline{q}_1\ \underline{q}_2\ \underline{q}_3] \qquad [S]^T = [O][Q]^T \tag{7}$$

where

$$[O] = [\mathbf{M}_2][\mathbf{M}_1] \tag{8}$$

and $[\mathbf{M}_1]$ and $[\mathbf{M}_2]$ are the transformation matrices of the first and second mirrors respectively. For instance, the optical transformation matrix [O] for the mirror pair shown in Figure 3 is equal to the identity matrix (a general result only for *parallel* mirror pairs). For systems with *n* elements, Eq. 8 expands to

$$[O] = [\mathbf{M}_n][\mathbf{M}_{n-1}]\ldots[\mathbf{M}_2][\mathbf{M}_1] \tag{9}$$

Note that the mirror matrix [**M**] of Eq. 5 is the optical transformation matrix for a simple optical instrument that consists of a single plane mirror.

8. VECTOR BASES

Eq. 5 gives the mirror matrix in terms of the scalar components (n_1,n_2,n_3) of the mirror normal vector $\underline{n}$. Implicit in this statement is the existence of a vector basis to which the components are referred. A vector basis is an orthonormal triad that can be used to define any three dimensional vector. For example, if the components of $\underline{n}$ are referred to the vector basis $(\underline{a}_1,\underline{a}_2,\underline{a}_3)$, then $\underline{n}$ can be expressed as the sum of vector components

$$\underline{n} = n_1\underline{a}_1 + n_2\underline{a}_2 + n_3\underline{a}_3 \tag{10}$$

For the matrix in Eq. 5 to be valid in Eqs. 1 and 3, the mirror normal $\underline{n}$ must be a unit vector and must be expressed in the same unit vector basis as the vector to be transformed. This rule can be extended to include all optical transformations. Basis consistency is a fundamental requirement of the methodology of this paper.

Vector triads are also used to analytically describe the orientation of one rigid body or reference frame relative to another. For instance, let the vector set $(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ be attached to the base of the instrument shown in Figure 4. Let set $(\underline{b}_1,\underline{b}_2,\underline{b}_3)$ be attached to the mirror, which can rotate with respect to the instrument base. The orientation of the mirror with respect to the base is defined by the relative orientation of the two vector sets $(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ and $(\underline{b}_1,\underline{b}_2,\underline{b}_3)$. Orthonormal triads that are used to define the orientations of the kinematic reference frames corresponding to the various rigid bodies within a system are also useful as vector bases for the expression of LOS vectors, image vectors, and optical transformations.

A vector, such as the mirror normal $\underline{n}$, can be expressed in any vector basis defined in the system. To avoid ambiguity, a superscript is added to a vector if a particular vector basis is intended. Thus $\underline{n}^A$ and $\underline{n}^B$ indicate vector $\underline{n}$ expressed in the *A* and *B* bases as follows:

$$\underline{n}^A = n_1^A\underline{a}_1 + n_2^A\underline{a}_2 + n_3^A\underline{a}_3$$

$$\underline{n}^B = n_1^B\underline{b}_1 + n_2^B\underline{b}_2 + n_3^B\underline{b}_3 \qquad (11)$$

Note that while $\underline{n}^A$ and $\underline{n}^B$ represent the same vector and can be considered equivalent ($\underline{n}^A=\underline{n}^B$), their components ($n_1^A,n_2^A,n_3^A$) and ($n_1^B,n_2^B,n_3^B$) will not in general be equal ($n_i^A \neq n_i^B$, i=1,2,3).

9. THE BASIS TRANSFORMATION MATRIX

A basis transformation matrix is used to change the representation of a vector from one basis to another. For instance, the vector components of $\underline{n}$ in basis *B* can be found by a transformation of the components of $\underline{n}$ in basis *A* using the basis transformation $[{}^B T^A]$ according to

$$\begin{bmatrix} n_1^B \\ n_2^B \\ n_3^B \end{bmatrix} = [{}^B T^A] \begin{bmatrix} n_1^A \\ n_2^A \\ n_3^A \end{bmatrix} \qquad (12)$$

The same transformation used to transfer the components of a vector from one basis to another can also be used to transfer the basis vectors themselves:

$$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} = [{}^B T^A] \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \qquad (13)$$

Eq. 13, rather than Eq. 12, is generally regarded as the defining equation for a basis transformation. In fact, basis transformation matrices are usually developed from observations of the relative orientations of the basis vectors. That is, the vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are expressed in terms of $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ as

$$\underline{b}_i^A = b_{i1}^A\underline{a}_1 + b_{i2}^A\underline{a}_2 + b_{i3}^A\underline{a}_3 \qquad i=1,3 \qquad (14)$$

Inspection of Eqs. 13 and 14 reveals that the components of $[{}^B T^A]$ can be identified as

$$[{}^B T^A] = \begin{bmatrix} {}^B T_{11}^A & {}^B T_{21}^A & {}^B T_{31}^A \\ {}^B T_{12}^A & {}^B T_{22}^A & {}^B T_{32}^A \\ {}^B T_{13}^A & {}^B T_{23}^A & {}^B T_{33}^A \end{bmatrix} = \begin{bmatrix} b_{11}^A & b_{12}^A & b_{13}^A \\ b_{21}^A & b_{22}^A & b_{23}^A \\ b_{31}^A & b_{32}^A & b_{33}^A \end{bmatrix} \qquad (15)$$

LOS vector sets, while not necessarily attached to rigid bodies, none the less define reference frames, and serve as suitable vector bases for expressions of system vectors. Basis transformations should not be confused with optical transformations. A basis transformation of a vector does not change the magnitude or direction of that vector, but merely redescribes it in a different reference frame. An optical transformation of a vector, however, produces a new vector. A more detailed discussion of basis and optical transformations is provided in the next section.

10. RELATION BETWEEN OPTICAL AND BASIS TRANSFORMATIONS

As previously mentioned, the vector set associated with an LOS reference frame can be used a vector basis. Therefore, basis transformation matrices that involve LOS vector sets are of interest. How these basis transformations are related to the optical transformations that produce the LOS vector sets is the subject of this section. The optical transformation given by Eq. 7 and the basis transformation given by Eq. 13 are repeated below for convenience:

Optical Transformation: $$[\underline{s}_1\ \underline{s}_2\ \underline{s}_3] = [\mathbf{O}][\underline{q}_1\ \underline{q}_2\ \underline{q}_3]$$

$$[\mathbf{S}]^T = [\mathbf{O}][\mathbf{Q}]^T$$

Basis Transformation: $$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} = [{}^B\mathbf{T}^A] \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix}$$

Note that the optical transformation of Eq. 7 is different in form than the basis transformation of Eq. 13. In Eq. 7, each of the vectors on the left-hand side is a function of only one vector on the right. For example, $\underline{s}_1$ is dependent on $\underline{q}_1$ but independent of $\underline{q}_2$ and $\underline{q}_3$. In Eq. 13, however, each of the left-hand vectors is expressed as a combination of all three vectors on the right. Further, note that both equations are notationally compact. The transformations represented as a single letter are actually 3-by-3 matrices. Eq. 13 is made explicit by simply expanding the matrix $[{}^B\mathbf{T}^A]$:

$$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} = \begin{bmatrix} {}^BT^A_{11} & {}^BT^A_{21} & {}^BT^A_{31} \\ {}^BT^A_{12} & {}^BT^A_{22} & {}^BT^A_{32} \\ {}^BT^A_{13} & {}^BT^A_{23} & {}^BT^A_{33} \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \tag{16}$$

For Eq. 7 to be explicit, however, the matrix *and* the vectors must be expanded, the latter as columns of their components referred to an appropriate vector basis. For instance, expanding with repect to basis $(\underline{a}_1, \underline{a}_2, \underline{a}_3)$ yields

$$\begin{bmatrix} s^A_{11} & s^A_{21} & s^A_{31} \\ s^A_{12} & s^A_{22} & s^A_{32} \\ s^A_{13} & s^A_{23} & s^A_{33} \end{bmatrix} = \begin{bmatrix} O^A_{11} & O^A_{21} & O^A_{31} \\ O^A_{12} & O^A_{22} & O^A_{32} \\ O^A_{13} & O^A_{23} & O^A_{33} \end{bmatrix} \begin{bmatrix} q^A_{11} & q^A_{21} & q^A_{31} \\ q^A_{12} & q^A_{22} & q^A_{32} \\ q^A_{13} & q^A_{23} & q^A_{33} \end{bmatrix} \tag{17}$$

where

$$\underline{s}^A_i = s^A_{i1}\underline{u}_1 + s^A_{i2}\underline{u}_2 + s^A_{i3}\underline{u}_3 \qquad \underline{q}^A_i = q^A_{i1}\underline{u}_1 + q^A_{i2}\underline{u}_2 + q^A_{i3}\underline{u}_3 \qquad i=1,2,3$$

To further stress the difference between basis and optical transformations, we reiterate that Eq. 12 is used to change the components of a single vector from one vector basis to another. Eq. 16 describes one set of vectors in the basis of a second set. Finally, Eq. 17 represents the optical transformation of a set of vectors into a different set of vectors, both sets of which are described in the same vector basis.

We now seek the basis transformation matrix between two reference frames related by an optical transformation. In other words, given the optical transformation matrix [O] and the two LOS vector sets $(\underline{q}_1, \underline{q}_2, \underline{q}_3)$ and $(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ of Eqs. 7 and 15, what is the basis transformation matrix $[{}^S\mathbf{T}^Q]$?

We begin by transposing Eq. 7, using both forms of compact notation:

$$\begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \underline{s}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \end{bmatrix} [O]^T \qquad [S] = [Q][O]^T \tag{18}$$

We seek a transformation of the form of Eq. 13:

$$\begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \underline{s}_3 \end{bmatrix} = [{}^S\mathbf{T}^Q] \begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \end{bmatrix} \qquad [S] = [{}^S\mathbf{T}^Q][Q] \tag{19}$$

Equating the right-hand sides of Eqs. 18 and 19 and solving for $[{}^S\mathbf{T}^Q]$ leads to

$$[{}^S\mathbf{T}^Q] = \begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \end{bmatrix} [O]^T [\underline{q}_1 \; \underline{q}_2 \; \underline{q}_3] \qquad [{}^S\mathbf{T}^Q] = [Q][O]^T[Q]^T \tag{20}$$

As mentioned, Eq. 20 is compact. To expand the equation, a consistent basis must be chosen for [O] and $\underline{q}_i$. For instance, if [O] is expressed in the A-basis, then the vector $\underline{q}_1$ would have components r^A_{11}, r^A_{12}, and r^A_{13}. When the A-basis is used, the expanded, or explicit , form of Eq. 20 is

$$\begin{bmatrix} {}^S T^Q_{11} & {}^S T^Q_{21} & {}^S T^Q_{31} \\ {}^S T^Q_{12} & {}^S T^Q_{22} & {}^S T^Q_{32} \\ {}^S T^Q_{13} & {}^S T^Q_{23} & {}^S T^Q_{33} \end{bmatrix} = \begin{bmatrix} q^A_{11} & q^A_{12} & q^A_{13} \\ q^A_{21} & q^A_{22} & q^A_{23} \\ q^A_{31} & q^A_{32} & q^A_{33} \end{bmatrix} \begin{bmatrix} O^A_{11} & O^A_{21} & O^A_{31} \\ O^A_{12} & O^A_{22} & O^A_{32} \\ O^A_{13} & O^A_{23} & O^A_{33} \end{bmatrix} \begin{bmatrix} q^A_{11} & q^A_{21} & q^A_{31} \\ q^A_{12} & q^A_{22} & q^A_{32} \\ q^A_{13} & q^A_{23} & q^A_{33} \end{bmatrix} \tag{21}$$

The first matrix on the right is the basis transformation matrix $[{}^Q\mathbf{T}^A]$ between the vector bases $(\underline{q}_1, \underline{q}_2, \underline{q}_3)$ and $(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ written in the form of Eq. 15. The last matrix is the transpose (and, since the matrix is orthogonal, the inverse) of the first. Thus, returning to compact form, Eq. 21, which represents the desired basis transformation $[{}^S\mathbf{T}^Q]$, can be written as

$$[{}^S\mathbf{T}^Q] = [{}^Q\mathbf{T}^A][\mathbf{O}^A]^T[{}^A\mathbf{T}^Q] \tag{22}$$

Transformation matrix [O] is superscripted in Eq. 22 to indicate that this matrix must be expressed in the *A*-basis for the equation to be valid. Note, however, that reference frame *A* serves only as an intermediate reference frame. As the equation is valid for *any* intermediate reference frame, the *A*-basis can be regarded as a dummy parameter.

An alternative to Eqs. 20, 21, and 22 can be found by solving Eq. 19 for $[{}^{S}T^{Q}]$:

$$[{}^{S}T^{Q}] = \begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \underline{s}_3 \end{bmatrix} [\underline{q}_1 \; \underline{q}_2 \; \underline{q}_3] \qquad [{}^{S}T^{Q}] = [S][Q]^{T} \tag{23}$$

The following two equations can be derived from Eq. 23 in the same manner that Eqs. 21 and 22 were derived from Eq. 20:

$$\begin{bmatrix} {}^{S}T^{Q}_{11} & {}^{S}T^{Q}_{21} & {}^{S}T^{Q}_{31} \\ {}^{S}T^{Q}_{12} & {}^{S}T^{Q}_{22} & {}^{S}T^{Q}_{32} \\ {}^{S}T^{Q}_{13} & {}^{S}T^{Q}_{23} & {}^{S}T^{Q}_{33} \end{bmatrix} = \begin{bmatrix} s^{A}_{11} & s^{A}_{12} & s^{A}_{13} \\ s^{A}_{21} & s^{A}_{22} & s^{A}_{23} \\ s^{A}_{31} & s^{A}_{32} & s^{A}_{33} \end{bmatrix} \begin{bmatrix} q^{A}_{11} & q^{A}_{21} & q^{A}_{31} \\ q^{A}_{12} & q^{A}_{22} & q^{A}_{32} \\ q^{A}_{13} & q^{A}_{23} & q^{A}_{33} \end{bmatrix} \tag{24}$$

$$[{}^{S}T^{Q}] = [{}^{S}T^{A}][{}^{A}T^{Q}] \tag{25}$$

If the *Q*-basis is used in Eq. 22, the result is

$$[{}^{S}T^{Q}] = [{}^{Q}T^{Q}][O^{Q}]^{T}[{}^{Q}T^{Q}] = [O^{Q}]^{T} \tag{26}$$

since $[{}^{Q}T^{Q}]$ is simply the identity matrix. Note that Eq. 26 is valid only if optical transformation is expressed in the basis of the incoming LOS reference frame.

Now, the right-hand sides of Eqs. 22 and 26 can be equated and then transposed to provide a formula to transfer an optical transformation matrix from one basis to another (in this case, from basis *A* to basis *Q*):

$$[O^{Q}] = [{}^{Q}T^{A}][O^{A}]^{T}[{}^{A}T^{Q}] \tag{27}$$

Eq. 26 indicates that for certain cases an optical transformation matrix can be related to a basis transformation matrix by a simple transpose. It is important, then, to clearly delineate which function is intended when using these transformations. This is especially true if an optical transformation matrix is symmetric, as is always the case for single mirrors, double-mirror pairs, and any prism used as an image rotator. For symmetric transformations, Eq. 26 and its inverse can be used to show that

$$[{}^{S}T^{Q}] = [{}^{Q}T^{S}] = [O^{Q}] = [O^{S}] \tag{28}$$

Eq. 28 indicates that, if the optical transformation matrix from *Q* to *S* is symmetric, then the forward and backward optical and basis transformation matrices are all equal, and that they are valid for vectors written in either the *Q* or *S* basis. Eq. 22 must be used, however, if the optical transformation matrix is not symmetric.

11. APPLICATION OF LOS REFERENCE FRAMES

LOS reference frames are an analytical tool for use in solving a wide range of optical kinematic problems associated with plane-mirror systems. They are especially useful in analysis of gimbaled-mirror, vehicle-mounted imaging systems. For example, control equations for stabilizing the optical axis can be derived by calculating the angular velocity of the outgoing LOS reference frame[27]. Further, the amount of image rotation and the resulting command signal to a derotation device can be ascertained by analysis of the LOS reference frame orientation[28]. The methodology introduced in this paper is also of use in mechanical tolerance and alignment analysis, and in determining the boresight coefficients associated with structural dynamics. Accompanying work on these latter topics is in progress by the authors. Finally, many of the published solution techniques for specific LOS pointing problems are readily adapted to the analytical framework described in this paper.

Successful application of LOS reference frames to problem solving requires manipulation of vector triads using both optical and basis transformations. The basic techniques of this paper are used to assemble the LOS reference frames of the example in the next section.

12. EXAMPLE: AN AERIAL PHOTOGRAPHY SYSTEM

A gimbaled-mirror aerial photography system is shown in Figure 5. The camera C is mounted internally to the floor of aircraft P. The forward, right-wing, and down directions of P are indicated. The optical axis of the camera, shown as LOS vector $\underline{q}_1$, reflects first as $\underline{r}_1$ off the fold mirror M_1 and then as $\underline{s}_1$ off the gimbaled mirror M_2 before leaving the aircraft through a hole in the floor (not shown). The optical axis lies in a nominally horizontal plane until reflected off of M_2, which is brought into the desired orientation with respect to the aircraft by rotations ψ of the outer gimbal A and θ of the inner gimbal B, to which M_2 is rigidly attached. The following unit vector sets are introduced to define the geometry and aid in the analysis of the system. Each forms a right-handed orthonormal triad and is shown in Figure 6 unless otherwise indicated:

- Airframe basis set $P(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ - Fixed in the aircraft and aligned in the forward, right, and down directions respectively.

- Camera LOS set $Q(\underline{q}_1,\underline{q}_2,\underline{q}_3)$ - Vector $\underline{q}_1$ is aligned with the optical axis from the focal plane to mirror M_1. Vectors $\underline{q}_2$ and $\underline{q}_3$ are aligned with the horizontal and vertical direction of the focal plane. Vector $\underline{q}_3$ is parallel to vector $\underline{p}_3$.

- Intermediate LOS set $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ - (not shown) - The transformation of the camera set Q through mirror M_1, which is fixed such that $\underline{r}_1$ is aligned with $-\underline{p}_2$. Set R is left-handed.

- Outgoing LOS set $S(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ - (not shown) - The transformation of the intermediate set R through mirror M_2.

- Outer gimbal basis set $A(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ - Attached to the outer gimbal A and aligned with the set $(\underline{p}_2,-\underline{p}_1,\underline{p}_3)$ when angle ψ is zero. Vector $\underline{a}_2$ is parallel to the inner gimbal axis.

- Inner gimbal basis set $B(\underline{b}_1,\underline{b}_2,\underline{b}_3)$ - Fixed to the inner gimbal (and thus to mirror M_2) and aligned with $A(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ when θ is zero. Vector $\underline{b}_1$ is coincident with $\underline{n}_2$, the vector normal to the surface of mirror M_2.

The following basis transformations can be assembled from observation of the system configuration. Note that C_ι and S_ι stand for $\cos(\iota)$ and $\sin(\iota)$ respectively, where ι is either a fixed angle (as with 135°, the angle between $\underline{p}_1$ and $\underline{q}_1$), or a variable angle (such as the gimbal angles ψ or θ):

$$\begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \\ \underline{q}_3 \end{bmatrix} = [{}^Q\mathbf{T}^P] \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} = \begin{bmatrix} C_{135} & S_{135} & 0 \\ -S_{135} & C_{135} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} = [{}^A\mathbf{T}^P] \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -C_\psi & 0 & S_\psi \\ S_\psi & 0 & C_\psi \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} = [{}^B\mathbf{T}^A] \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \tag{31}$$

Expressions in the *P*-basis for the intermediate and outgoing LOS reference frame matrices $[\mathbf{R}^P]$ and $[\mathbf{S}^P]$ and the system optical transformation matrix $[\mathbf{O}^P]$ are desired.

Vector $\underline{n}_1$, the normal to mirror M_1, bisects the angle between the incoming and outgoing optical axes:

$$\underline{n}_1 = S_{22.5}\underline{p}_1 - C_{22.5}\underline{p}_2 \tag{32}$$

Since $\underline{n}_2 = \underline{b}_1$, the mirror normal $\underline{n}_2$ can be expressed in the *P*-basis by use of Eqs. 30 and 31 as

$$\underline{n}_2 = \underline{b}_1 = -S_\theta S_\psi \underline{p}_1 + C_\theta \underline{p}_2 - S_\theta C_\psi \underline{p}_3 \tag{33}$$

The mirror transformation matrices $[\mathbf{M}_1^P]$ and $[\mathbf{M}_2^P]$ can be constructed by use of Eq. 5, with the scalar coefficients of the mirror normal vectors given in Eqs. 32 and 33. The resulting matrices are

$$[\mathbf{M}_1^P] = \begin{bmatrix} 1-2S_{22.5}^2 & 2S_{22.5}C_{22.5} & 0 \\ 2S_{22.5}C_{22.5} & 1-2C_{22.5}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{45} & S_{45} & 0 \\ S_{45} & -C_{45} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

$$[\mathbf{M}_2^P] = \begin{bmatrix} 1-2S_\theta^2 S_\psi^2 & 2S_\theta C_\theta S_\psi & -2S_\theta^2 S_\psi C_\psi \\ 2S_\theta C_\theta S_\psi & 1-2C_\theta^2 & 2S_\theta C_\theta C_\psi \\ -2S_\theta^2 S_\psi C_\psi & 2S_\theta C_\theta C_\psi & 1-2S_\theta^2 C_\psi^2 \end{bmatrix} \quad (35)$$

The intermediate LOS matrix expressed in the *P*-basis, $[\mathbf{R}^P]$, is constructed by use of Eq. 3 in combination with the expressions for the camera LOS matrix $[\mathbf{Q}^P]$ and the mirror transformation matrix $[\mathbf{M}_1^P]$ found in Eqs. 29 and 34 respectively

$$[\underline{r}_1^P\ \underline{r}_2^P\ \underline{r}_3^P] = [\mathbf{M}_1^P][\underline{q}_1^P\ \underline{q}_2^P\ \underline{q}_3^P] \qquad [\mathbf{R}^P]^T = [\mathbf{M}_1^P][\mathbf{Q}^P]^T$$

$$[\mathbf{R}^P]^T = \begin{bmatrix} C_{45} & S_{45} & 0 \\ S_{45} & -C_{45} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{135} & -S_{135} & 0 \\ S_{135} & C_{135} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad k = \left(\frac{1}{2}\right)^{\frac{1}{2}}$$

$$[\mathbf{R}^P]^T = \begin{bmatrix} k & k & 0 \\ k & -k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -k & -k & 0 \\ k & -k & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (37)$$

Note that the determinant of $[\mathbf{R}^P]$ is equal to -1, verifying that set *R* is left-handed. The outgoing LOS matrix $[\mathbf{S}^P]$ is formed in a like manner by multiplying $[\mathbf{R}^P]$ by the mirror transformation matrix $[\mathbf{M}_2]$:

$$[\ \underline{s}_1^P\ \underline{s}_2^P\ \underline{s}_3^P\ ] = [\mathbf{M}_2^P][\ \underline{r}_1^P\ \underline{r}_2^P\ \underline{r}_3^P\ ] \qquad [\mathbf{S}^P]^T = [\mathbf{M}_2^P][\mathbf{R}^P]^T$$

$$[\mathbf{S}^P]^T = \begin{bmatrix} 1-2S_\theta^2 S_\psi^2 & 2S_\theta C_\theta S_\psi & -2S_\theta^2 S_\psi C_\psi \\ 2S_\theta C_\theta S_\psi & 1-2C_\theta^2 & 2S_\theta C_\theta C_\psi \\ -2S_\theta^2 S_\psi C_\psi & 2S_\theta C_\theta C_\psi & 1-2S_\theta^2 C_\psi^2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2S_\theta C_\theta S_\psi & -1+2S_\theta^2 S_\psi^2 & -2S_\theta^2 S_\psi C_\psi \\ -1+2C_\theta^2 & -2S_\theta C_\theta S_\psi & 2S_\theta C_\theta C_\psi \\ -2S_\theta C_\theta C_\psi & 2S_\theta^2 S_\psi C_\psi & 1-2S_\theta^2 C_\psi^2 \end{bmatrix} \quad (38)$$

The LOS transformation matrix $[\mathbf{S}^P]$ is the result of successive optical transformations of the camera LOS matrix $[\mathbf{R}^P]$ by the mirrors M_1 and M_2, but it can also be regarded as $[{}^S\mathbf{T}^P]$, the basis transformation from *P* to *S*.

The optical transformation of the system from *Q* to *S* is found by post-multiplying the component transformations in sequence:

$$[\mathbf{O}^P] = [\mathbf{M}_2^P][\mathbf{M}_1^P]$$

$$
[O^P] = \begin{bmatrix} 1-2S_\theta^2 S_\psi^2 & 2S_\theta C_\theta S_\psi & -2S_\theta^2 S_\psi C_\psi \\ 2S_\theta C_\theta S_\psi & 1-2C_\theta^2 & 2S_\theta C_\theta C_\psi \\ -2S_\theta^2 S_\psi C_\psi & 2S_\theta C_\theta C_\psi & 1-2S_\theta^2 C_\psi^2 \end{bmatrix} \begin{bmatrix} C_{45} & S_{45} & 0 \\ S_{45} & -C_{45} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
[O^P] = \begin{bmatrix} (1-2S_\theta^2 S_\psi^2 + 2S_\theta C_\theta S_\psi)k & (1-2S_\theta^2 S_\psi^2 - 2S_\theta C_\theta S_\psi)k & -2S_\theta^2 S_\psi C_\psi \\ (2S_\theta C_\theta S_\psi + 1-2C_\theta^2)k & (2S_\theta C_\theta S_\psi - 1-2C_\theta^2)k & 2S_\theta C_\theta C_\psi \\ (-2S_\theta^2 S_\psi C_\psi + 2S_\theta C_\theta C_\psi)k & (-2S_\theta^2 S_\psi C_\psi - 2S_\theta C_\theta C_\psi)k & 1-2S_\theta^2 C_\psi^2 \end{bmatrix} \quad (39)
$$

Multiplying Q by the system transformation matrix $[O^P]$ will produce the outgoing LOS matrix $[S^P]$ of Eq. 38 directly. Note that, unlike the component transformations $[M_1^P]$ and $[M_2^P]$, the system transformation matrix $[O^P]$ is not symmetric. It can be shown that, like $[M_1^P]$ and $[M_2^P]$, $[O^P]$ is orthogonal.

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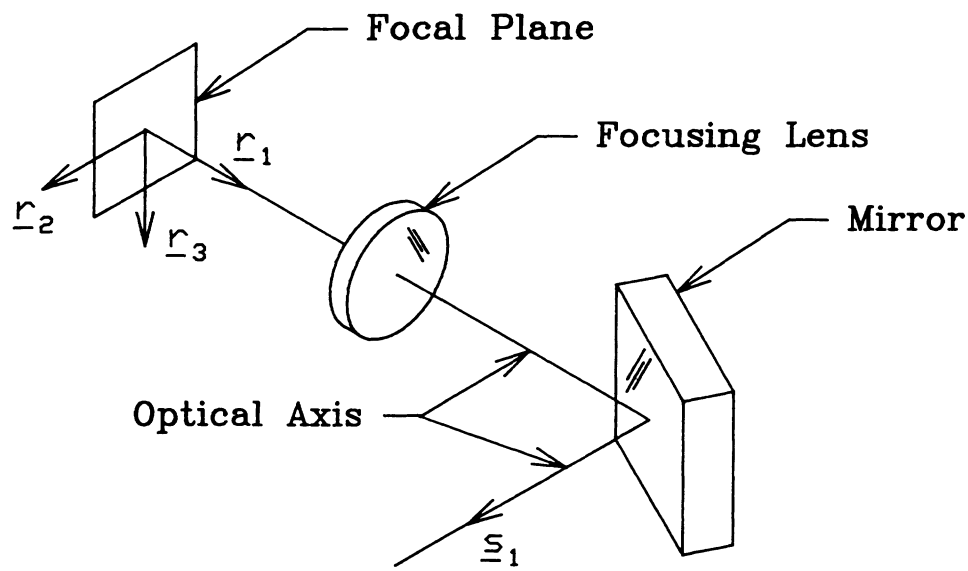

Fig. 1. Line-of-Sight and Image Vectors

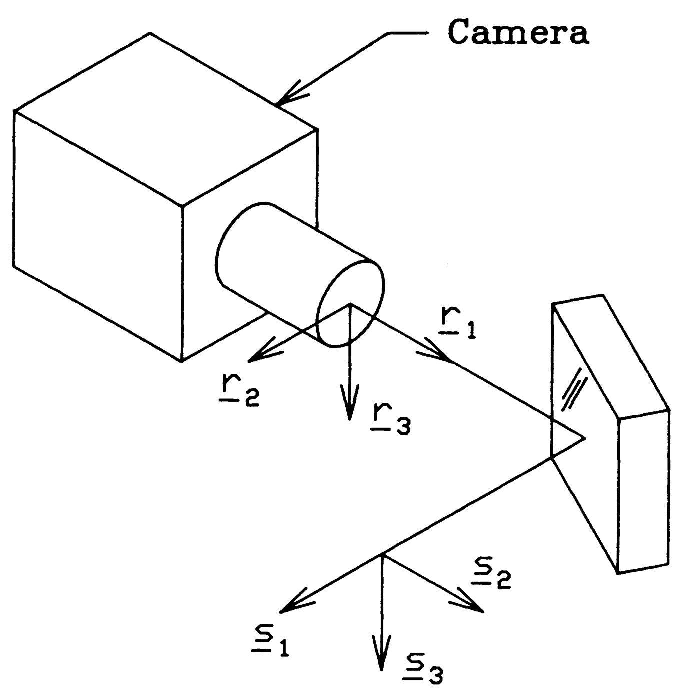

Fig. 2. LOS Reference Frames

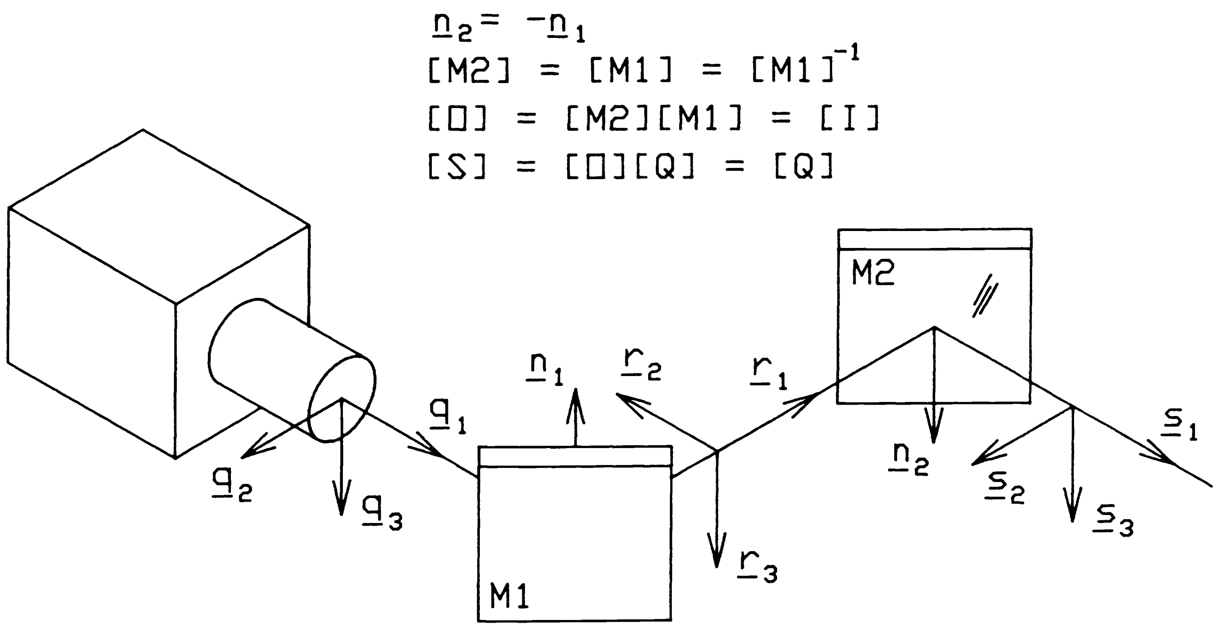

Fig. 3. Parallel Mirror Pair

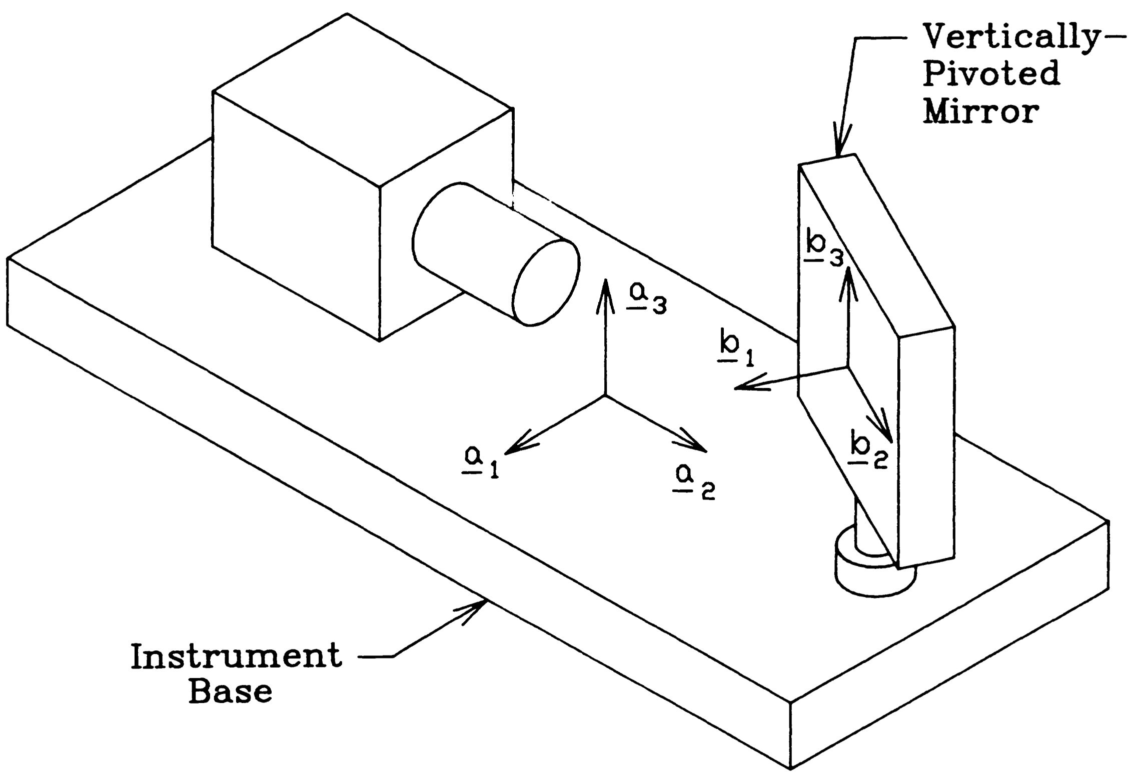

Fig. 4. Vector Bases

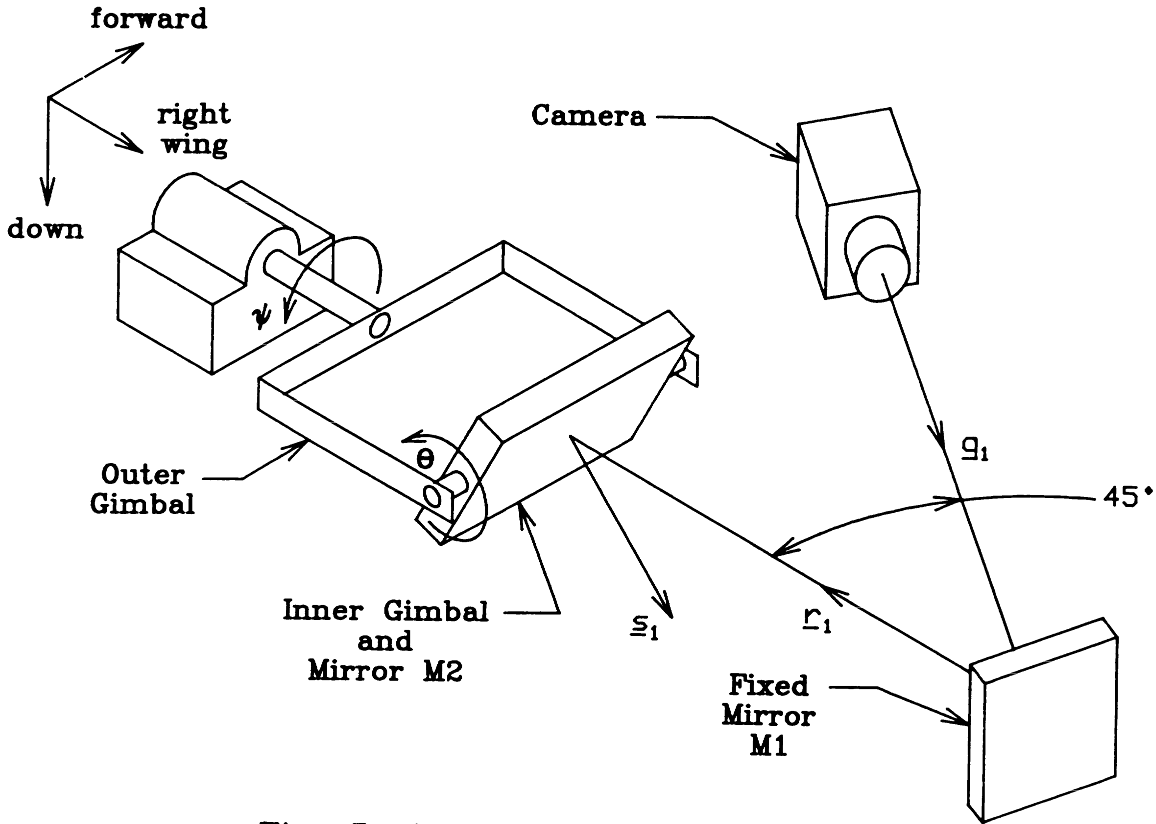

Fig. 5. Aerial Photography System

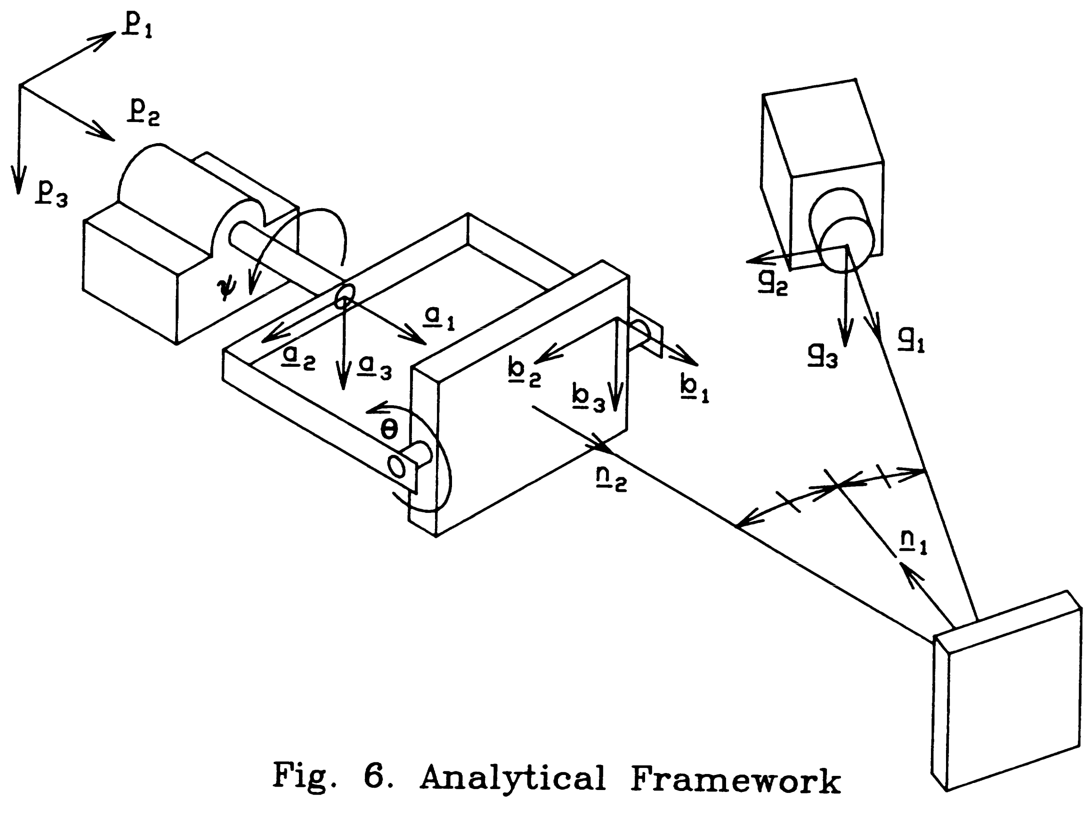

Fig. 6. Analytical Framework