Image rotation in plane-mirror optical systems

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ABSTRACT

Image rotation is an inherent part of plane-mirror optical systems. The amount of rotation caused by any one mirror is a function of its relative orientation within the system. Fixed mirrors introduce a fixed amount of rotation, while gimbaled and flexured mirrors introduce a variable amount. Further, in vehicle-mounted systems, additional image rotation can be introduced by changes in vehicle orientation. Derotation devices are added to optical systems when a stable image orientation is required. A digital controller with a feed-forward algorithm can be used advantageously to control these devices. Control of a derotation device requires knowledge of the image rotation angle as a function of the mirror and vehicle orientation angles. Three methods of calculating image rotation for any plane-mirror optical system are presented. These methods are based on a new optical kinematic construction, the line-of-sight (LOS) reference frame. Using LOS reference frames, image rotation due to fixed mirrors, movable mirrors, derotation prisms, and vehicle orientation is calculated exactly. The first two methods give the total image rotation angle without regard to the source of the rotation. The last method gives the amount of rotation imparted into the system by each component. Total image rotation is the sum of these individual rotation angles. All methods produce equations for the real-time calculation of the derotation prism command signal. Image rotation within an aerial photography system is calculated as an example.

1. INTRODUCTION

The existence of image rotation within plane-mirror optical systems is well known and generally undesired. For human observers, non-upright images are confusing at best and disconcerting or even nauseating at worst. While not prone to vestibular disruptions, computer-imaged systems can none the less be adversely affected by the image processing complications of image rotation.

Image rotation is introduced in plane-mirror systems from four possible sources: fixed components, movable components (such as gimbaled or flexured mirrors), vehicle orientation, and derotation devices. The last item, a derotation device, is usually added to the system to remove the unwanted rotation imparted by the first three items. The amount of image rotation due to the fixed components of a system is constant and a function only of their relative orientation. If movable optics are present, the amount of image rotation in the system will be a function of the state variables (such as gimbal angles) that describe the orientations of the movable components. If the system is vehicle mounted, then an additional amount of rotation is imparted due to the general motions of the vehicle. Note that this last effect is not unique to plane-mirror systems.

As mentioned, derotation devices are added to systems when a controlled-orientation image is desired. The vast majority of these are prisms, and the term "derotation prism" is ubiquitous. There are many different derotation prism configurations, but most 1) do not change the direction of the optical axis, 2) are mounted to physically rotate about the optical axis, and 3) impart an amount of image rotation that is twice that of prism rotation. In systems with variable image rotation, the derotation prism is servo-driven to maintain correct image orientation. In the past, very complex analog circuits were often required for control of these devices. Today's microprocessors allow greatly simplified digital controllers and feed-forward algorithms. To accurately control a derotation prism, the exact amount of unwanted image rotation associated with the vehicle and mirror orientation states must be calculated. This image rotation angle, correctly scaled, is used as the command signal to the derotation prism. Three general methods of deriving the equations relating vehicle and mirror orientations to image rotation are presented herein. The first two methods (*back projection* and *outgoing LOS reference frame*) both yield a single equation for the total system rotation. The third method (*multi-angle formulation*) can be used to calculate the image rotation associated with the individual system components.

The methodology presented in this paper makes extensive use of line-of-sight (LOS) reference frames. An LOS reference frame is defined by a vector triad. One of the three vectors is parallel to the optical axis, while the other two lie in the perpendicular image plane. An LOS reference frame, which is transformed through an optical system by its components, provides orientation information for the optical axis as well as the optical image. The general properties of LOS reference frames are described in detail in a companion paper by the authors[1]. Application of LOS reference frames to LOS stabilization is detailed by DeBruin[2].

2. HISTORICAL PERSPECTIVE

Much has been written about image derotators and their application. Hopkins[3,4] summarizes the properties of several common prisms, as does Swift[5], who also covers other, non-prismatic, derotation devices. Durie[6] and Walker[7] each discuss in detail one particular prism configuration. A typical application example is provided by Boot[8], while Stetson and Elkins[9] and Fagan and Waddell[10] both outline the use of image rotators in the observation of rotating machinery. The image processing complications of *not* derotating an image is detailed by Freitag and MacLeod[11].

Image rotation was analyzed in early optical instruments with a manual, graphical ray trace of the optical system. Computers have long supplanted pencil-and-paper in this regard, and are essential for application of the ray-trace technique to modern, movable-element systems. Dolan[12] lists image rotation analysis among the features of a particular ray-tracing computer program. It should be noted, however, that this method provides only numerical information on image rotation. The algebraic equations relating image rotation to the system state variables (gimble angles, etc) are not provided by ray tracing.

The method described herein is based on the matrix-vector approach to plane-mirror analysis. Hopkins[3,4] and Walles and Hopkins[13] describe the orientation of an image in terms of the coordinate transformation properties of the system mirror matrices. Sitsov[14-18] uses vector triad transformations throughout his series on plane-mirror system design. Royalty[19] discusses transforming the gravity vector through the optical system to provide an error signal for derotation prism control. The authors extend these previous works by providing 1) a definition and general equation for image rotation, 2) a method of directly calculating the derotation servo command signal from the system states, and 3) a technique for possibly increasing the efficiency of such calculations when implemented in real time. Further, these methods are based on an analytical framework that is of general applicability to kinematic analysis of vehicle-mounted, plane-mirror optical systems.

3. ANALYTICAL DEFINITION OF IMAGE ROTATION

The subject of image rotation is biased by the experience of the human observer. Most "viewing" optical systems are designed to provide an upright image to the observer. As such, image rotation is defined as the amount an image is skewed from that seen by a human standing erect and looking along the same line-of-sight. This definition, which implies natural "up" and "level" directions, becomes insufficient in circumstances where up and level aren't defined. For example, in aerial mapping, a camera looking directly down images a scene with no "up" direction. In this circumstance, the image derotation requirement may be to keep the "north" direction at the top of the image.

The methods described in this paper require that the desired image orientation be analytically defined in vector terms. To do this, an "orientation vector" must be specified by its vector components in one of the vector bases of the system model. Image rotation is then defined by the orientation of the projection of this vector onto the image plane of the optical system. Care should be taken in defining this vector. For instance, if the local north vector is specified as the orientation vector, one must take into account that the direction of this vector changes relative to the earth as the earth is traversed.

4. METHOD ONE: CALCULATING IMAGE ROTATION BY BACK PROJECTION

The "human standing erect" definition of image rotation is the most common and will be used herein. One, but not the only, way of specifying this in vector terms is shown in Figure 1. A generic optical system is represented as a simple lens and imaging focal plane. The system is viewing a scene in which up is defined by a local gravity vector $\mathbf{g}$, which is designated as the orientation vector. Vector $\mathbf{g}$ images onto the focal plane as vector $\underline{f}$. Note that the system outgoing line-of-sight will not, in general, be perpendicular to $\mathbf{g}$. As such, vector $\underline{f}$ will be foreshortened.

A line v is shown on the focal plane to designate the vertical direction as seen by a human observer. Note that the observer can be looking directly, viewing on a remote video monitor, or reviewing a later-developed photograph. The image rotation angle ϕ is defined as the angle between $\underline{f}$ and v. A suitable means of defining a positive direction of rotation is required, and to this end an LOS vector set $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ is fixed in the optical system and aligned with the focal plane as shown. Vector $\underline{r}_1$ is parallel to the optical axis and

$\underline{r}_2$ is aligned with line v. Vector $\underline{r}_3$ completes the right-handed triad. Following the right-hand rule, a positive ϕ is defined as taking $\underline{f}$ from $\underline{r}_2$ toward $\underline{r}_3$.

Vector $\underline{f}$ can be found by use of g and $[O_S]$, the optical transformation of the system. This transformation is defined from the focal plane to the outgoing optical axis. The inverse of $[O_S]$ is used to project g back through the system. Note that the vector $\underline{h}$, the projection of g through $[O_S]$, will not, in general, lie in the focal plane. Making use of the orthogonality of $[O_S]$:

$$[\underline{h}] = [O_S]^{-1}[g] = [O_S]^T[g] \tag{1}$$

Vector $\underline{f}$ is found as the projection of $\underline{h}$ onto the focal plane:

$$\underline{f} = (\underline{h}\cdot\underline{r}_2)\underline{r}_2 + (\underline{h}\cdot\underline{r}_3)\underline{r}_3 \tag{2}$$

The angle ϕ follows directly:

$$\phi = \arctan\left(\frac{\underline{f}\cdot\underline{r}_3}{\underline{f}\cdot\underline{r}_2}\right) \tag{3}$$

Vector $\underline{f}$ can be eliminated using Eq. 2:

$$\phi = \arctan\left(\frac{\underline{h}\cdot\underline{r}_3}{\underline{h}\cdot\underline{r}_2}\right) \tag{4}$$

Vector $\underline{f}$ can likewise be eliminated using Eq. 1. Note that, following the notation of Reference 1, Eq. 1 is a matrix equation. Whenever the matrix form of vectors are substituted into dot (inner) product equations, the matrix form of the inner product will be used (*e.g.*, $\underline{h}\cdot\underline{r}_3 = [\underline{h}]^T[\underline{r}_3]$).

$$\phi = \arctan\left(\frac{[[O_S]^T[g]]^T[\underline{r}_3]}{[[O_S]^T[g]]^T[\underline{r}_2]}\right) \tag{5}$$

Eq. 5 provides a nice compact formula for calculation of image rotation. It should be noted, however, that for Eq. 5 to be operational the components must be expressed in a consistent vector basis. For example, say that $[O_S]$ is expressed in the vehicle basis P (and thus written as $[O_S^P]$), g is expressed in the local-level basis $\boldsymbol{N}$, and $\underline{r}_2$ and $\underline{r}_3$ are expressed in their own basis R. The basis transformations $[{}^P\mathbf{T}^N]$ and $[{}^P\mathbf{T}^R]$ would then be required to perform the calculations of Eq. 5:

$$\phi = \arctan\left(\frac{[[O_S^P]^T[{}^P\mathbf{T}^N][g^N]]^T[{}^P\mathbf{T}^R][\underline{r}_3^R]}{[[O_S^P]^T[{}^P\mathbf{T}^N][g^N]]^T[{}^P\mathbf{T}^R][\underline{r}_2^R]}\right) \tag{6}$$

5. METHOD TWO: USE OF THE OUTGOING LOS REFERENCE FRAME

Method two is presented as an alternative to calculating image rotation by back projection. This second method makes use of the system outgoing LOS reference frame. As shown for the simple one-mirror system of Figure 2, outgoing LOS reference frame $S(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ is the projection of camera set $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ through the optical system (The camera set is substituted for the focal plane set without loss of generality as explained in Reference 1):

$$[\underline{s}_1 \ \underline{s}_2 \ \underline{s}_3] = [O_S][\underline{r}_1 \ \underline{r}_2 \ \underline{r}_3] \qquad [S]^T = [O_S][R]^T \tag{7}$$

Set S can be calculated early in the analysis of an optical system and then used for the calculation of image rotation. The following equation follows from Eq. 4 based on the orthogonality of $[O_S]$ and the preservation of inner products in orthogonal transformations:

$$\phi = \arctan\left(\frac{[[O_S][\underline{h}]]^T[O_S][\underline{r}_3]}{[[O_S][\underline{h}]]^T[O_S][\underline{r}_2]}\right) \tag{8}$$

Eq. 1 can be used to eliminate $[O_S][\underline{h}]$. Likewise Eq. 7 can be used to eliminate $[O_S][\underline{r}_2]$ and $[O_S][\underline{r}_3]$:

$$\phi = \arctan\left(\frac{\underline{g}\cdot\underline{s}_3}{\underline{g}\cdot\underline{s}_2}\right) \tag{9}$$

Eq. 9 states mathematically that the orientation of scene in the outgoing LOS reference frame is identical to the orientation of the scene image in the camera LOS reference frame. This fact is important not only in image rotation calculations but also in the determination of structural dynamics boresight coefficients and error and alignment analyses. Again, note that Eq. 9 must be expressed in a consistent vector basis to be operational.

6. THE DEROTATION COMMAND SIGNAL

As a component of the optical system, the optical transformation $[O_D]$ of a derotation prism is included in the system transformation $[O_S]$. As such, the image rotation ϕ given by Eqs. 3 through 9 includes the image rotation λ imparted by the derotation prism as well as the unwanted system-imparted image rotation ν:

$$\phi = \lambda + \nu \tag{10}$$

The validity of Eq. 10 follows the arguments for the multi-angle formulation developed later in Section 7. The derotation prism is added to the system to eliminate image rotation. Inspection of Eq. 9 reveals that image rotation ϕ is equal to zero provided that:

$$\lambda = -\nu \tag{11}$$

Eq. 11 represents the root form of the derotation control equation. The variables λ and ν, however, are optical kinematic variables, which generally cannot be directly measured or controlled. It is true, however, that both variables are related to physical kinematic states of the system which *can* be directly measured and controlled.

As mentioned, the image rotation λ imparted by the derotation prism is twice that of the physical rotation ρ of the derotation prism:

$$\lambda = 2\rho \tag{12}$$

Eq. 12 is derived from the optical transformation of a derotation prism, which is discussed in detail in the Appendix. Using Eq. 11 in Eq. 12, the desired derotation prism angle, and thus the derotation servo-control command signal, can be solved in terms of the unwanted system image rotation:

$$\rho_{cmd} = -\frac{1}{2}\nu \tag{13}$$

It can be seen from Eq. 10 that if λ is zero that ν is equal to ϕ, that is, that all of the system rotation is unwanted. Angle ν can be calculated then by positioning the derotation prism in its home position (ρ=0), calculating the system transformation, and then using any of Eqs. 3 through 9 as appropriate. The derotation command signal ρ_{cmd} follows by substituting ν into Eq. 13.

Signal ρ_{cmd} as calculated by use of Eq. 13 is a function of the geometry constants, movable mirrors states, and vehicle orientation angles. Attempting to write Eq. 13 directly in terms of these variables can be unwieldy for complex systems. Of course, numerically there is no reason to do this. The components of Eqs. 3 through 9 can be calculated separately first, and the intermediate answers subsequently carried forward into the final equation. Even so, the processing load associated with this task can add an undesirable time delay to the control loop (Remember, ρ_{cmd} is calculated from measured system states in real time and subsequently used as the input to the derotation control loop). System performance, then, can be improved by reducing the processing load associated with calculating ρ_{cmd}. The third method of this paper, as outlined in the next section, produces a set of image rotation equations that has been found to reduce the derotation command computational load for some systems.

7. METHOD THREE: MULTI-ANGLE FORMULATION

As mentioned, the system image rotation ϕ is due to four sources: 1) the derotation prism, 2) the fixed system components, 3) the movable system components, and 4) vehicle orientation. In the multi-angle formulation, each of these effects is considered separately. In fact, angle ϕ is explicitly expressed as the sum of four component angles:

$$\phi = \lambda + \gamma + \sigma + \tau \tag{14}$$

Angles λ, γ, σ, and τ represent respectively the image rotation imparted by items 1 through 4 above. Angle λ is the same as in Eq. 10. Comparison of Eqs. 10 and 14 indicates that the unwanted system rotation ν is the sum of γ, σ, and τ. Eq. 13 can be modified appropriately:

$$\rho_{cmd} = -\frac{1}{2}(\gamma + \sigma + \tau) \tag{15}$$

Four concepts are introduced to aid in the construction of the multi-angle formulation: 1) the home-position optical transformation, 2) the home-position LOS reference frame, 3) the orientation reference frame, and 4) the *preservation of relative position* law.

1. Home-Position Optical Transformation: An "H" is added to the subscript of a movable component optical transformation to indicate that the transformation is to be evaluated with the component in its home position, that is, with the component state variable(s) set to zero. For instance, with the derotation prism angle ρ set to zero, the optical transformation matrix of Eq. A1 (Eq. 1 of the appendix) evaluates to:

$$\mathbf{O}^{R}_{DH} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

2. Home-Position LOS Reference Frame: Let LOS reference frame $S(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ be the transformation of $R(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ through optical transformation $[\mathbf{O}_X]$ corresponding to movable-component X. Home-position LOS reference frame $T(\underline{t}_1, \underline{t}_2, \underline{t}_3)$ is the transformation of R through home-position optical transformation $[\mathbf{O}_{XH}]$. Set T is fixed relative to R and is independent of the orientation of X. Note that if the motions of X change the direction of the outgoing LOS, then $\underline{r}_1$ and $\underline{t}_1$ will not stay aligned (that is, T does not follow the LOS).

3. Orientation Reference Frame: For a system with outgoing LOS reference frame $S(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ and defined orientation vector $\underline{g}$, the orientation reference frame $V(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ is attached to the LOS and aligned as follows: $\underline{v}_1$ is aligned with $\underline{s}_1$; $\underline{v}_2$ is aligned with the projection of $\underline{g}$ onto the image plane of the outgoing LOS (this plane is perpendicular to $\underline{s}_1$); $\underline{v}_3$ completes the vector triad. Reference frame V represents the desired orientation of S, and as such, the derotation prism is driven to align S with V. Note that set V must be defined with the same handedness as S. The orientation reference frame is an extension of the "orientation matrix" defined by Hopkins[4].

4. Preservation of Relative Position: The rotation angle between two LOS reference frames on a common line-of-sight is preserved by any subsequent optical transformation. This is a direct consequence of the *preservation of inner product* rule.

The geometry and construction of the multi-angle formulation is shown in Figure 3. A vehicle-mounted, movable-element (gimbaled-mirror) optical system is shown with its elements symbolically depicted for simplicity. Focal plane (camera) reference frame A passes sequentially through derotation prism D, fixed components F, and gimbaled mirror G. Transformation matrices $[\mathbf{O}_i]$ are defined for i = D, F, and G. The vehicle P moves within reference frame N, in which the gravity vector g is fixed. The outgoing LOS reference frame, which is not shown, is oriented within the vehicle as it leaves $[\mathbf{O}_G]$. The orientation of the vehicle and the outgoing LOS reference frame within N is depicted by the basis transformation $[{}^{N}\mathbf{T}^{P}]$. The vehicle orientation vector $\underline{k}$ is fixed in the vehicle in the "down" direction and aligned with $\underline{g}$ when the vehicle is flat and level. The multi-angle formulation is outlined as follows:

1. Derotation prism D and gimbaled mirror G are placed in their home positions and home-position optical transformations $[\mathbf{O}_{DH}]$ and $[\mathbf{O}_{GH}]$ are calculated.

2. LOS reference frame $B(\underline{b}_1,\underline{b}_2,\underline{b}_3)$ is the transformation of focal-plane LOS reference frame $A(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ through $[\mathbf{O}_D]$. Home-position LOS reference frame $C(\underline{c}_1,\underline{c}_2,\underline{c}_3)$ is the transformation of A through $[\mathbf{O}_{DH}]$. The derotation prism image rotation angle λ is found as follows:

$$\lambda = \arctan\left(\frac{\underline{c}_2\cdot\underline{b}_3}{\underline{c}_2\cdot\underline{b}_2}\right) \tag{17}$$

Eq. 17 is in agreement with Eq. 12, which indicates that a positive prism rotation ρ produces a positive image rotation λ.

3. LOS reference frame $D(\underline{d}_1,\underline{d}_2,\underline{d}_3)$ is the transformation of C through $[\mathbf{o}_F]$. Both C and D are fixed within the optical system.

4. Orientation LOS reference frame $U(\underline{u}_1,\underline{u}_2,\underline{u}_3)$ is constructed as follows: Line-of-sight $\underline{u}_1$ is found by transforming $\underline{d}_1$ through $[\mathbf{O}_G]$. Vector $\underline{u}_2$, the projection of the vehicle orientation vector $\underline{k}$ onto the outgoing image plane, is found as follows:

$$\underline{u}_2 = \frac{\underline{k} - (\underline{k}\cdot\underline{u}_1)\underline{u}_1}{|\underline{k} - (\underline{k}\cdot\underline{u}_1)\underline{u}_1|} \tag{18}$$

Vector set U must match the handedness of the outgoing LOS reference frame. Vector $\underline{u}_3$ is added, therefore, as either plus or minus $\underline{u}_1\mathbf{x}\underline{u}_2$ depending on whether there are an even (+) or odd (-) number of inversions in the optical path. As (arbitrarily) drawn, the system in Figure 3 shows inversion through the derotation prism and the gimbaled-mirror but not through the fixed components.

5. Combination orientation/home-position LOS reference frame $V(\underline{v}_1,\underline{v}_2,\underline{v}_3)$ (not shown) is aligned with U when G is in its home position. Vector set V is fixed in the optical system.

6. Home-position LOS reference frame $E(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ is the transformation of V back through $[\mathbf{O}_{GH}]^T$. The fixed-component image rotation angle γ is found as follows:

$$\gamma = \arctan\left(\frac{\underline{e}_2\cdot\underline{d}_3}{\underline{e}_2\cdot\underline{d}_2}\right) \tag{19}$$

As both D and E are fixed within the optical system, angle γ will be constant.

7. LOS reference frame $F(\underline{f}_1,\underline{f}_2,\underline{f}_3)$ is the transformation of E through $[\mathbf{O}_G]$. The gimbaled-mirror image rotation angle σ is found as follows:

$$\sigma = \arctan\left(\frac{\underline{u}_2\cdot\underline{f}_3}{\underline{u}_2\cdot\underline{f}_2}\right) \tag{20}$$

An alternate form of Eq. 20, which can greatly simplify the evaluation of σ, can be found by using Eq. 18 to eliminate $\underline{u}_2$:

$$\sigma = \arctan\left(\frac{\underline{k}\cdot\underline{f}_3}{\underline{k}\cdot\underline{f}_2}\right) \tag{21}$$

8. Orientation LOS reference frame $W(\underline{w}_1,\underline{w}_2,\underline{w}_3)$ is constructed as follows: Line-of-sight $\underline{w}_1$ is aligned with $\underline{u}_1$. Vector $\underline{w}_2$, the projection of the gravity vector $\underline{g}$ onto the outgoing image plane, is found as follows:

$$\underline{w}_2 = \frac{\underline{g} - (\underline{g}\cdot\underline{w}_1)\underline{w}_1}{|\underline{g} - (\underline{g}\cdot\underline{w}_1)\underline{w}_1|} \tag{22}$$

Vector $\underline{w}_3$ is added as $+/-(\underline{w}_1 \mathbf{x} \underline{w}_2)$ to match the handedness of U.

9. The vehicle-orientation rotation angle τ is calculated as follows:

$$\tau = \arctan\left(\frac{\underline{w}_2\cdot\underline{u}_3}{\underline{w}_2\cdot\underline{u}_2}\right) \tag{23}$$

Eq. 22 can be used to eliminate $\underline{w}_2$:

$$\tau = \arctan\left(\frac{\underline{g}\cdot\underline{u}_3}{\underline{g}\cdot\underline{u}_2}\right) \tag{24}$$

10. The derotation command signal is generated by substituting Eqs. 19, 21, and 24 into Eq. 15.

Note that the multi-angle formulation does not always produce a computationally shorter set of equations than the first two methods of this paper. Other factors, however, such as sensor and vehicle bandwidth, the existence of previously calculated terms, and the computationaL characteristics of the microprocessor, must be considered in deciding which set is better to use. Note also that the formulation presented here can be readily adapted to systems in which the components are configured differently from the system shown in Figure 3.

8. EXAMPLE: AN AERIAL PHOTOGRAPHY SYSTEM

A gimbaled-mirror aerial photography system is shown in Figure 4. The system is mounted internally to the floor of aircraft P and consists of camera C, derotation prism D, fixed mirrors M_1 and M_2, and gimbaled mirror G. The outgoing LOS of the system leaves through a hole in the floor of the aircraft (not shown). Prism D is brought into alignment in the system by rotation ρ about the optical axis. Gimbaled mirror G is positioned by rotations ψ of the outer gimbal A and θ of the inner gimbal B, to which G is rigidly attached. The following unit vector sets are introduced to define the geometry and aid in the analysis of the system. Each set forms a right-handed orthonormal triad and is shown in Figure 5 unless otherwise indicated:

- Local-level basis set $N(\underline{n}_1,\underline{n}_2,\underline{n}_3)$ - (not shown) - Aligned with the local north, east and down directions respectively. The aircraft yaw, pitch and roll angles Y,P,R are defined in the traditional manner with respect to this reference frame. The gravity vector $\underline{g}$ is aligned with $\underline{n}_3$

- Airframe basis set $P(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ - Fixed in the aircraft and aligned in the forward, right, and down directions respectively. All optical vectors and transformations will be expressed in this reference frame. Vehicle orientation vector $\underline{k}$ is aligned with $\underline{p}_3$.

- Camera LOS reference frame $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ - Vector $\underline{r}_1$ is aligned with the optical axis from the camera to mirror M_1. Vectors $\underline{r}_2$ and $\underline{r}_3$ are aligned with the horizontal and vertical direction of the camera. Set $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ is aligned with set $(\underline{p}_2,\underline{p}_3,\underline{p}_1)$.

- Outgoing LOS reference frame $S(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ - (not shown) - The transformation of the camera set R through the optical system.

- Outer gimbal basis set $A(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ - Attached to the outer gimbal A and aligned with the set $(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ when angle ψ is zero. Vector $\underline{a}_2$ is parallel to the inner gimbal axis.

- Inner gimbal basis set $B(\underline{b}_1,\underline{b}_2,\underline{b}_3)$ - Fixed to the inner gimbal (and thus to mirror G) and aligned with $A(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ when θ is zero.

Vectors $\underline{m}_1$ and $\underline{m}_2$, the normals to mirrors M_1 and M_2 respectively, are defined as follows:

$$\underline{m}_1 = -k\underline{p}_2 + k\underline{p}_3 \qquad k = \left(\frac{1}{2}\right)^{\frac{1}{2}} \tag{25}$$

$$\underline{m}_2 = -k\underline{p}_1 - k\underline{p}_3 \tag{26}$$

The following equations, which can be assembled from observation of the system configuration, define basis transforms $[{}^R T^P]$ and $[{}^P T^N]$ respectively. Note that C_i and S_i stand for cos and sin of angle i respectively:

$$\begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} = \begin{bmatrix} (C_Y C_P) & (S_Y C_P) & -S_P \\ (C_Y S_P S_R - S_Y C_R) & (S_Y S_P S_R + C_Y C_R) & (C_P S_R) \\ (C_Y S_P C_R + S_Y S_R) & (S_Y S_P C_R - C_Y S_R) & (C_P C_R) \end{bmatrix} \begin{bmatrix} \underline{n}_1 \\ \underline{n}_2 \\ \underline{n}_3 \end{bmatrix} \tag{28}$$

The system component optical transformations $[O_D]$, $[O_{M1}]$, $[O_{M2}]$, and $[O_D]$ are defined in the P-basis in Eqs. 29 through 32. The derotation transformation $[O_D^P]$ is found by use of Eqs. 25 and A1 in Eq. A6. The fixed mirror transformations $[O_{M1}^P]$ and $[O_{M2}^P]$ are found by use of Eqs. 25 and 26 respectively and the plane-mirror transformation matrix of Eq. 5 of Reference 1. Development of the gimbaled mirror transformation $[O_G^P]$ also follows the methods of Reference 1 and is outlined in the example problem of Reference 2.

$$[O_D^P] = \begin{bmatrix} C_{2\rho} & 0 & -S_{2\rho} \\ 0 & 1 & 0 \\ -S_{2\rho} & 0 & -C_{2\rho} \end{bmatrix} \tag{29}$$

$$[O_{M1}^P] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{30}$$

$$[O_{M2}^P] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{31}$$

$$[O_G^P] = \begin{bmatrix} 1-2C_\theta^2 & -2S_\psi S_\theta C_\theta & 2C_\psi S_\theta C_\theta \\ -2S_\psi S_\theta C_\theta & 1-2S_\psi^2 S_\theta^2 & 2S_\psi C_\psi S_\theta^2 \\ 2C_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & 1-2C_\psi^2 S_\theta^2 \end{bmatrix} \tag{32}$$

8.1 Image rotation by use of the outgoing LOS reference frame

The derotation prism is set to its home position to calculate the unwanted system image rotation ν. With ρ (and thus λ) equal to zero, ν becomes equal to ϕ, and Eq. 9 can be used to find the unwanted system image rotation. Expanding Eq. 9 into a consistent basis yields:

$$\nu = \arctan\left(\frac{[[{}^P\mathbf{T}^N][\underline{g}^N]]^T[\underline{s}_3^P]}{[[{}^P\mathbf{T}^N][\underline{g}^N]]^T[\underline{s}_2^P]}\right) \tag{33}$$

With the derotation prism in its home position, the outgoing LOS vector set S is formed by passing the camera set R through the system transformation matrix as follows. Note that the matrix of column vectors $[\underline{r}_1, \underline{r}_2, \underline{r}_3]$ expressed in the P-basis is the transpose of the basis transformation of Eq. 25:

$$[\underline{s}_1\ \underline{s}_2\ \underline{s}_3] = [O_S^P][\underline{r}_1\ \underline{r}_2\ \underline{r}_3] \qquad [\mathbf{S}] = [O_S^P][\mathbf{R}]$$

$$[S] = [O_G^P][O_{M2}^P][O_{M1}^P][O_{DH}^P][{}^R\mathbf{T}^P]^T = \begin{bmatrix} -1+2C_\theta^2 & -2S_\psi S_\theta C_\theta & -2C_\psi S_\theta C_\theta \\ 2S_\psi S_\theta C_\theta & 1-2S_\psi^2 S_\theta^2 & -2S_\psi C_\psi S_\theta^2 \\ -2C_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & -1+2C_\psi^2 S_\theta^2 \end{bmatrix} \tag{34}$$

Eq. 33 can now be expanded by use of Eqs. 28 and 34 to give the unwanted system image rotation in terms of the system state variables (pitch, roll, ψ, θ):

$$\nu = \arctan\left(\frac{S_P(-2C_\psi S_\theta C_\theta) - C_P S_R(2S_\psi C_\psi S_\theta^2) + C_P C_R(-1+2C_\psi^2 S_\theta^2)}{-S_P(2S_\psi S_\theta C_\theta) + C_P S_R(-1+2S_\psi^2 S_\theta^2) + C_P C_R(-2S_\psi C_\psi S_\theta^2)}\right) \tag{35}$$

Note that, as expected, image rotation is independent of the aircraft yaw angle. Note also that when the system is in its home position, angle λ is equal to -90°. Eq. 13 then indicates that the derotation prism must be positioned to 45° to erect the image.

8.2 Image rotation by use of the multi-angle formulation

The example system matches the configuration shown in Figure 3. As such the formulation proceeds as outlined in Section 7:

1. The home-position optical transformations are found from Eqs. 29 and 32:

$$[O^P_{DH}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{36}$$

$$[O^P_{GH}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{37}$$

2. Set *A* is equivalent to set *R*, and sets *B* and *C* follow directly. Derotation prism image rotation angle λ is found by use of Eq. 17.

$$\begin{bmatrix} \underline{a}^P_1 & \underline{a}^P_2 & \underline{a}^P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{38}$$

$$\begin{bmatrix} \underline{b}^P_1 & \underline{b}^P_2 & \underline{b}^P_3 \end{bmatrix} = \begin{bmatrix} 0 & -S_{2\rho} & C_{2\rho} \\ 1 & 0 & 0 \\ 0 & -C_{2\rho} & -S_{2\rho} \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} \underline{c}^P_1 & \underline{c}^P_2 & \underline{c}^P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{40}$$

$$\lambda = \arctan\left(\frac{\underline{c}_2 \cdot \underline{b}_3}{\underline{c}_2 \cdot \underline{b}_2}\right) = \arctan\left(\frac{S_{2\rho}}{C_{2\rho}}\right) = 2\rho \tag{41}$$

3. The fixed-component optical transformation and LOS reference frame *D* are calculated by use of Eqs. 30, 31, and 40:

$$[\underline{d}^P_1\ \underline{d}^P_2\ \underline{d}^P_3] = [O^P_F][\underline{c}^P_1\ \underline{c}^P_2\ \underline{c}^P_3] = [O^P_{M2}][O^P_{M1}][\underline{c}^P_1\ \underline{c}^P_2\ \underline{c}^P_3] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{42}$$

4. Vector $\underline{u}_1$, the transformation of $\underline{d}_1$ through the gimbaled mirror, can be calculated by use of Eqs 32 and 42:

$$\underline{u}_1^P = [O_G^P]\underline{d}_1^P = \begin{bmatrix} 1-2C_\theta^2 & -2S_\psi S_\theta C_\theta & 2C_\psi S_\theta C_\theta \\ -2S_\psi S_\theta C_\theta & 1-2S_\psi^2 S_\theta^2 & 2S_\psi C_\psi S_\theta^2 \\ 2C_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & 1-2C_\psi^2 S_\theta^2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2C_\theta^2 \\ 2S_\psi S_\theta C_\theta \\ -2C_\psi S_\theta C_\theta \end{bmatrix} \tag{43}$$

Vectors $\underline{u}_2$ and $\underline{u}_3$ can be written in terms of the three components u_{1i} of $\underline{u}_1$ by using Eq. 18 (The values of u_{1i} are found in the last column of Eq. 43). There are an even number of reflections between the camera and the outgoing LOS, so set *U* is right-handed.

$$[\underline{u}_1^P \ \underline{u}_2^P \ \underline{u}_3^P] = \begin{bmatrix} u_{11} & -u_{11}u_{13} & u_{12} \\ u_{12} & -u_{12}u_{13} & -u_{11} \\ u_{13} & 1-u_{13}u_{13} & 0 \end{bmatrix} \tag{44}$$

5. Reference frame *V* is formed by setting ψ and θ equal to zero in Eq. 43 and substituting for u_{1i} in Eq. 44:

$$\left[\underline{v}_1^P \ \underline{v}_2^P \ \underline{v}_3^P \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{45}$$

6. Reference frame *E* is calculated using Eqs. 37 and 45. The image rotation due to the fixed components is found by using Eqs. 42 and 45 in Eq. 19.

$$[\underline{e}_1^P \ \underline{e}_2^P \ \underline{e}_3^P] = [O_{GH}^P]^T [\underline{v}_1^P \ \underline{v}_2^P \ \underline{v}_3^P] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{46}$$

$$\gamma = \arctan\left(\frac{\underline{e}_2 \cdot \underline{d}_3}{\underline{e}_2 \cdot \underline{d}_2}\right) = \arctan\left(\frac{-1}{0}\right) = -90^\circ \tag{47}$$

7. Reference frame *F* is found by transforming *E* through the gimbaled mirror using Eqs. 32 and 46. The image rotation due to the gimbaled mirror is found as found using Eqs. 21 and 48. Note that $\underline{k}$ is aligned with $\underline{p}_3$.

$$[\underline{f}_1^P \ \underline{f}_2^P \ \underline{f}_3^P] = [O_G^P][\underline{e}_1^P \ \underline{e}_2^P \ \underline{e}_3^P] = \begin{bmatrix} -1+2C_\theta^2 & 2C_\psi S_\theta C_\theta & 2S_\psi S_\theta C_\theta \\ 2S_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & -1+2S_\psi^2 S_\theta^2 \\ -2C_\psi S_\theta C_\theta & 1-2C_\psi^2 S_\theta^2 & -2S_\psi C_\psi S_\theta^2 \end{bmatrix} \tag{48}$$

$$\sigma = \arctan\left(\frac{\underline{k}^P \cdot \underline{f}_3^P}{\underline{k}^P \cdot \underline{f}_2^P}\right) = \arctan\left(\frac{-2S_\psi C_\psi S_\theta^2}{1-2C_\psi^2 S_\theta^2}\right) \tag{49}$$

8. Eq. 24 will be used to calculate angle τ. As such, the components of set *W* do not have to be explicitly calculated.

9. Expanding Eq. 24 into a consistent reference frame yields:

$$\tau = \arctan\left(\frac{[[{}^{P}T^{N}][g^{N}]]^{T}[\underline{u}_3^{P}]}{[[{}^{P}T^{N}][g^{N}]]^{T}[\underline{u}_2^{P}]}\right) \tag{50}$$

Using subscripted component notation, Eq. 49 expands to:

$$\tau = \arctan\left(\frac{T_{13}u_{12} - T_{23}u_{11}}{-T_{13}u_{11}u_{13} - T_{23}u_{12}u_{13} + T_{33}(1-u_{13}u_{13})}\right) \tag{51}$$

$$\tau = \arctan\left(\frac{-S_P(2S_\psi S_\theta C_\theta) - C_P S_R(-1+2C_\theta^2)}{C_P S_R + 2C_\psi S_\theta C_\theta\left[-S_P(-1+2C_\theta^2)+C_P S_R(2S_\psi S_\theta C_\theta)+C_P C_R(2C_\psi S_\theta C_\theta)\right]}\right) \tag{52}$$

10. The derotation command signal can be generated using Eqs 47, 49, and 52 in Eq. 15. Total system image rotation can be calculated by substituting Eqs. 41, 47, 49, and 52 into Eq. 14.

9. APPENDIX: DEROTATION PRISM OPTICAL TRANSFORMATION

A generic derotation prism *D* is shown in Figure A1, along with incoming and outgoing LOS reference frames $R(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ and $S(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ respectively. The derotation prism is mounted with its rotation axis aligned with the incoming LOS $\underline{r}_1$. Further, the derotation prism is fixed within its rotary mount such that the outgoing LOS $\underline{s}_1$ is parallel to $\underline{r}_1$ for all rotation angles ρ. There are two orientations, 180° apart, in which the prism will invert an image but not rotate it. Either of these can serve as the home position without loss of generality. The angle ρ is then defined by use of vector set *R* and the right-hand rule as shown.

The following optical transformation can be derived from the geometry of Figure A1 and the internal prism configuration. The result is the same for all on-axis derotators:

$$[O_D^R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -C_{2\rho} & S_{2\rho} \\ 0 & S_{2\rho} & C_{2\rho} \end{bmatrix} \tag{A1}$$

Note that $[O_D]$ is superscripted "R" to indicate that the transformation is expressed in the vector basis *R*. The outgoing LOS reference frame *S* is found by transforming *R* through the derotation prism:

$$[\underline{s}_1^R\ \underline{s}_2^R\ \underline{s}_3^R] = [O_D^R][\underline{r}_1^R\ \underline{r}_2^R\ \underline{r}_3^R]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -C_{2\rho} & S_{2\rho} \\ 0 & S_{2\rho} & C_{2\rho} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -C_{2\rho} & S_{2\rho} \\ 0 & S_{2\rho} & C_{2\rho} \end{bmatrix} \tag{A2}$$

The columns of the last matrix of Eq. A2 represent the vector components of the outgoing LOS reference frame S as expressed in the R basis:

$$\begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \underline{s}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -C_{2\rho} & S_{2\rho} \\ 0 & S_{2\rho} & C_{2\rho} \end{bmatrix} \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix} \quad . \tag{A3}$$

Let home-position reference frame $T(\underline{t}_1, \underline{t}_2, \underline{t}_3)$ be aligned with S when the derotation prism is in its home position (ρ=0):

$$\begin{bmatrix} \underline{t}_1 \\ \underline{t}_2 \\ \underline{t}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix} \tag{A4}$$

Eq. 12 is derived by calculating the rotation λ of image vector $\underline{s}_2$ from its home position for a given prism angle ρ:

$$\lambda = \arctan\left(\frac{\underline{s}_2 \cdot \underline{t}_3}{\underline{s}_2 \cdot \underline{t}_2}\right) = \arctan\left(\frac{S_{2\rho}}{C_{2\rho}}\right) = 2\rho \tag{A5}$$

Note that if a vector basis other than R is chosen to express the system and component optical transformations, then the matrix of Eq. A1 must be transformed. As expected, the equation for angle λ is independent of the vector basis chosen in the derivation. If needed, however, transformation $[O_D]$ can be expressed in any other basis P by using Eq. 27 of Reference 1:

$$[O_D^P] = [{}^P T^R][O_D^R]^T[{}^R T^P] \tag{A6}$$

10. REFERENCE LIST

1. DeBruin, J.C. and Johnson, D.B., "Line-of-Sight Reference Frames: A Unified Approach to Plane-Mirror Optical Kinematics," to be published in *Acquisition, Pointing, Tracking VI*, Masten, M., Ed., Proc. SPIE, Vol. 1697, 1992.

2. DeBruin, J.C., "Derivation of Line-of-Sight Stabilization Equations for Gimbaled-Mirror Optical Systems," *Active and Adaptive Optical Components*, Ealey, M., Ed., Proc. SPIE, Vol. 1543, 1992, pp. 236-247.

3. Hopkins, R.E., in *Optical Design*, MIL-HDBK-141, U.S. Government Printing Office, Washington, 1962, Sec. 13.

4. Hopkins, R.E., in *Applied Optics and Optical Engineering*, Vol. 3, Kingslake, R., Ed., Academic Press, New York, 1965.

5. Swift, D.W., "Image Rotation Devices -- A Comparative Survey," *Optics and Laser Technology*, Vol. 4, No. 4, 1972, pp. 175-188.

6. Durie, D.S., "A Compact Derotater Design," *Optical Engineering*, Vol. 13, No. 1, Jan./Feb. 1974, pp. 19-22.

7. Walker, B.H., "Pechan Derotation Prism -- Application & Alignment Notes," Optical Engineering, Vol. 13, No. 6, 1974, pp. G233-G234.

8. Boot, A., "Evolution of Map Display Optical Systems," *Max Born Centenary Conf.*, Proc. SPIE, Vol. 369, 1983, pp 512-520.

9. Stetson, K.A, and Elkins, J.N., "Optical System for Dynamic Analysis of Rotating Structures," United Technologies Research Center, East Hartford, CT, National Technical Information Center Report No. AFAPL-TR-77-51, 1977.

10. Fagan, W.F. and Waddell, P., "Industrial Application of Image Derotation," *Industrial Applications of Laser Technology*, Proc. SPIE, Vol. 398, 1983, pp. 193-198.

11. Freitag, M. and MacLeod, S., "Effect of Scene Rotation on Target Acquisition," *Proc. 18th Annual Meeting*, Human Factors Society, 1974, pp. 146-152.

12. Dolan, A., "Traz-an Interactive Ray-Tracing Computer Program Integrated with a Solid-Modeling CAD System," *International Lens Design Conference (1985)*, Proc. SPIE, Vol. 554, 1986, pp. 31-36.

13. Walles, S. and Hopkins, R.E, "The Orientation of the Image Formed by a Series of Plane Mirrors," *Applied Optics*, Vol. 3, No. 12, Dec. 1964, pp. 1447-1452.

14. Sirtsov, G.P., "On the Transformation of Vectors by an Optical System of Three Plane Mirrors," *Soviet J. Optical Technology*, Vol. 44, No. 1, 1977, pp. 22-24.

15. Sirtsov, G.P., "On the Transformation of Vectors by an Optical System of Four Plane Mirrors," *Soviet J. Optical Technology*, Vol. 46, No. 2, 1979, pp. 78-81.

16. Sirtsov, G.P., "Synthesis of Plane-Mirror Systems," *Soviet J. Optical Technology*, Vol. 47, No. 7, 1980, pp. 392-394.

17. Sirtsov, G.P., Lobasov, M.A., and Moskvicheva, L.M., "Using a Computer to Calculate the Vector Triad in an Optical System Containing Mirrors and Reflective Prisms," *Soviet J. Optical Technology*, Vol. 48, No. 11, 1981, pp. 656-657.

18. Sirtsov, G.P., "Use of a Matrix Method to Transform the Vector Triad in an Optical System Containing Plane Mirrors," *Soviet J. Optical Technology*, Vol. 50, No. 1, 1983, pp. 19-22.

19. Royalty, J., "Development of Kinematics for Gimballed Mirror Systems," *Acquisition, Tracking, and Pointing IV*, Gowrinathan, S., Ed., Proc. SPIE, Vol. 1304, 1990, pp. 262-274.

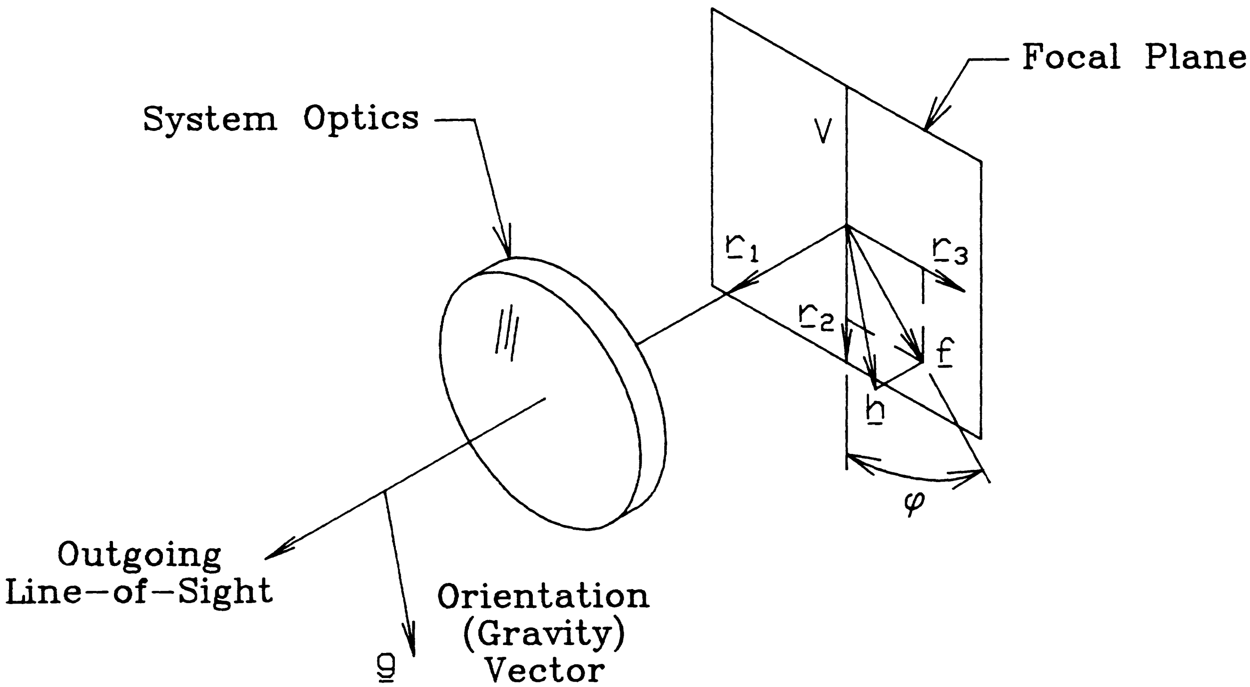

Fig. 1. Back Projection of Orientation Vector

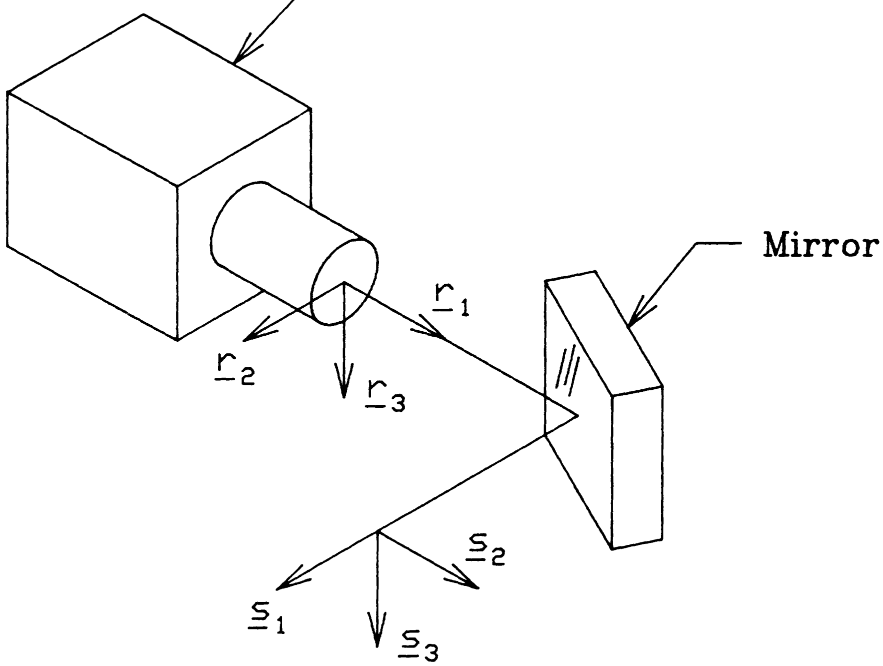

Fig. 2. Line-of-Sight Reference Frames

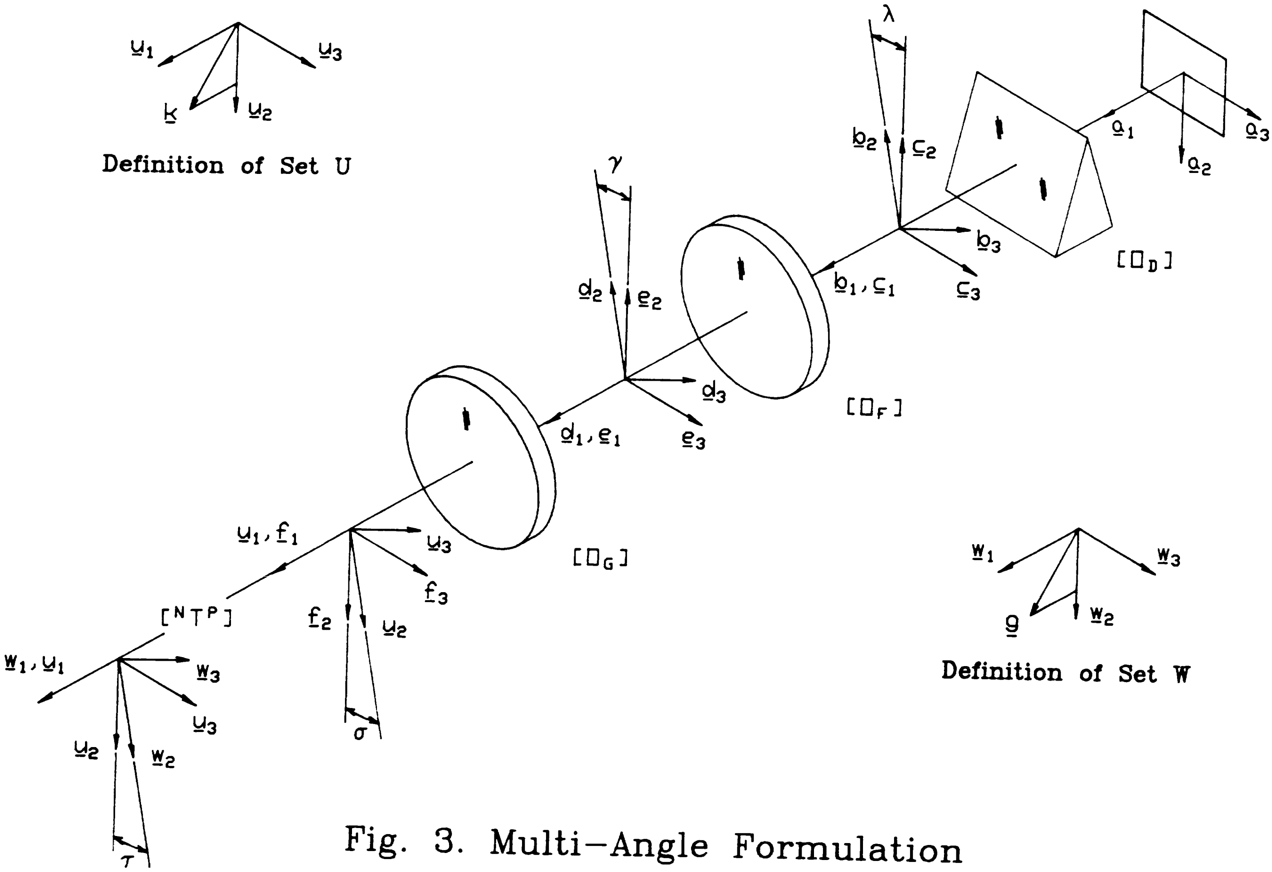

Fig. 3. Multi–Angle Formulation

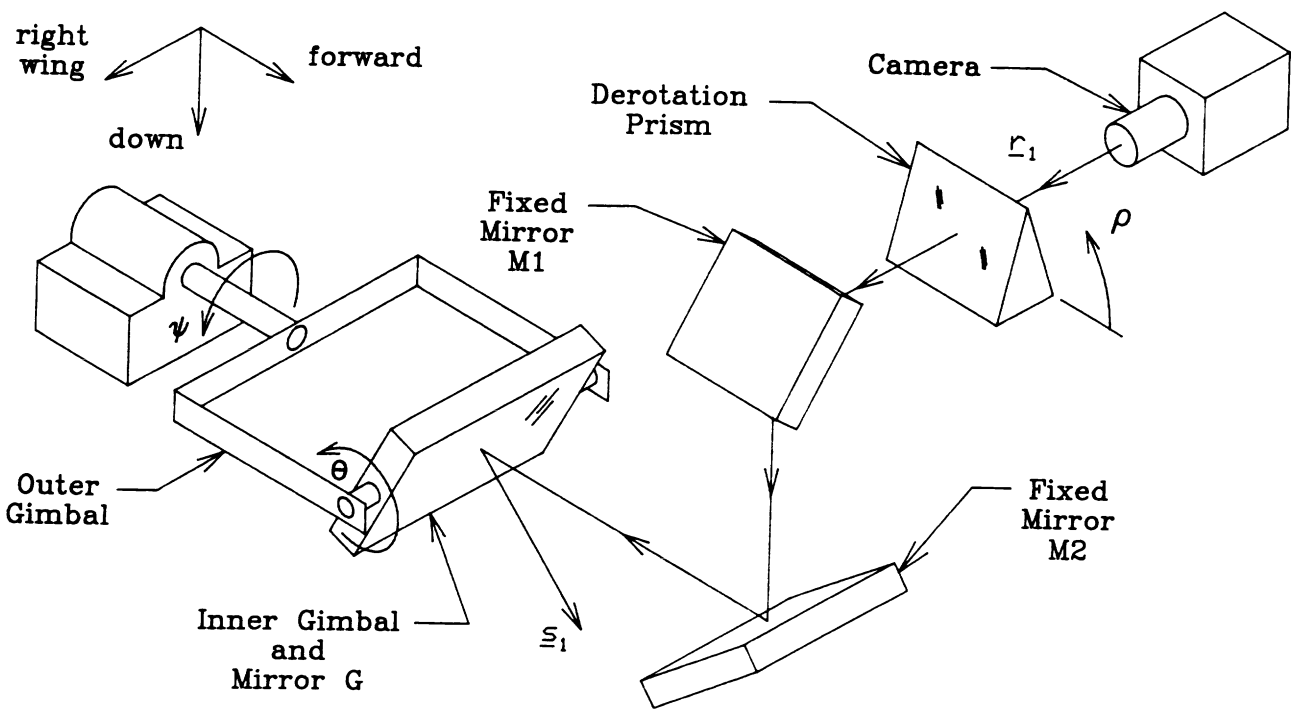

Fig. 4. Aerial Photography System

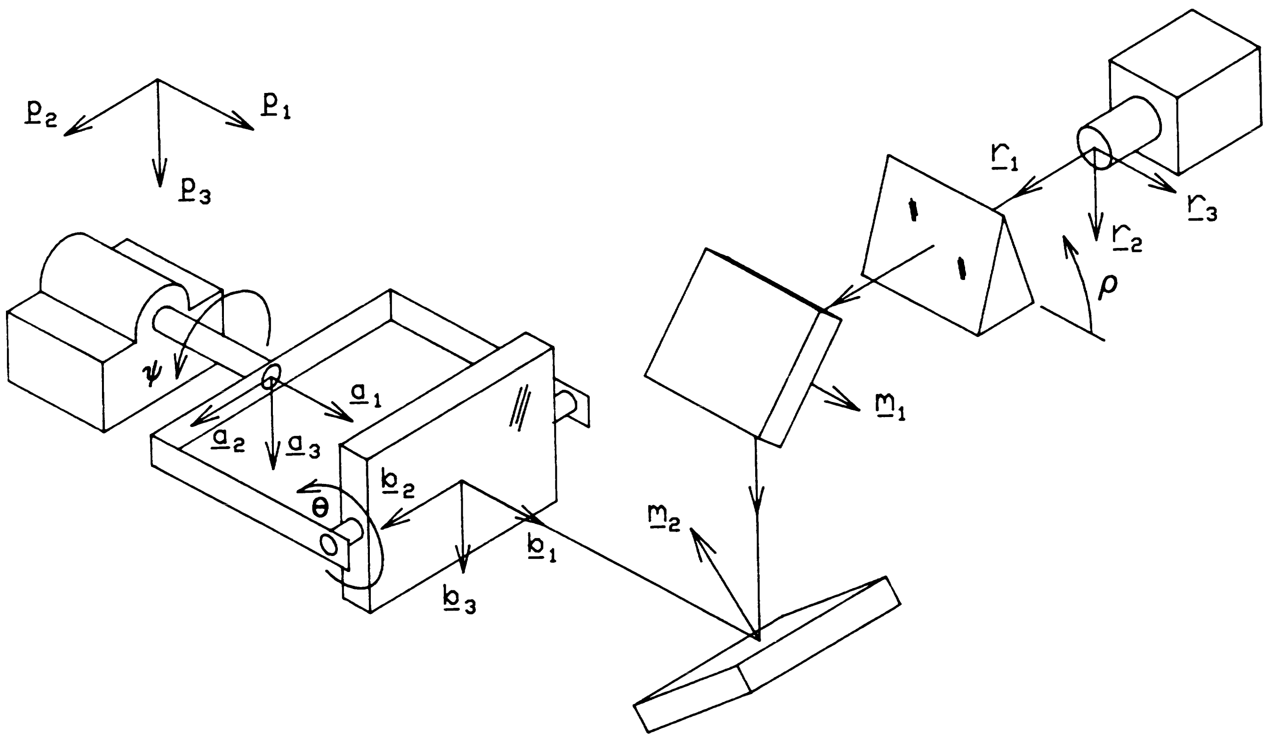

Fig. 5. Analytical Framework

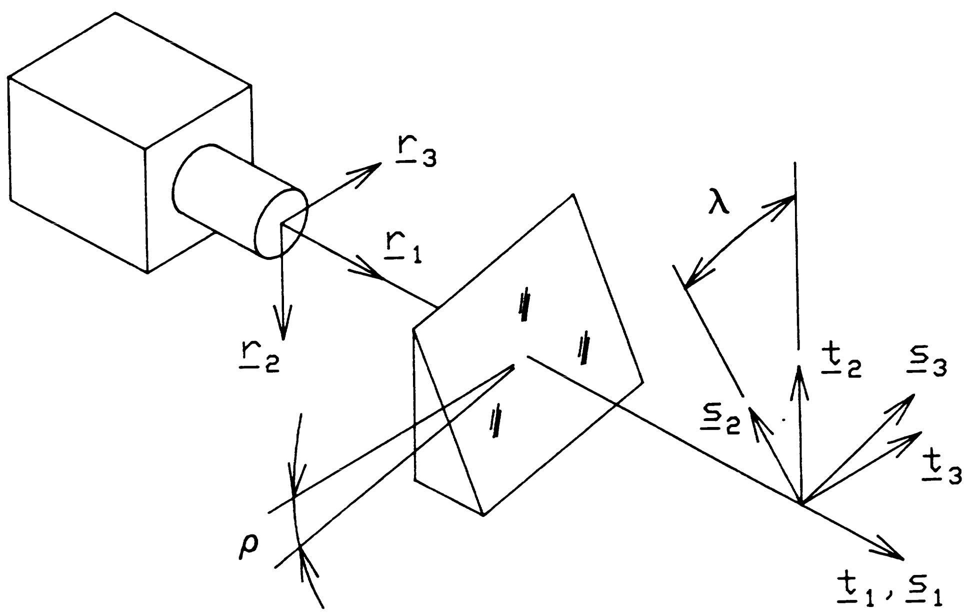

Fig. A1. Derotation Prism Geometry