Derivation of line-of-sight stabilization equations for gimbaled-mirror optical systems

James C. DeBruin

Control Systems Technology Center
Texas Instruments, Inc
Dallas, Texas 75266

ABSTRACT

The gimbaled flat steering mirrors commonly used for pointing the outgoing line-of-sight of optical systems can also be driven to stabilize the line-of-sight, effectively isolating it from vehicle base motion. The stabilization equations provide the relative rates of the gimbal angles as functions of the angular velocity of the base. These equations are of use in feed-forward stabilization systems. Two algorithmic methods of deriving the stabilization equations are presented. These methods are distinguished from others by their use of a kinematic reference frame that is attached to the line-of-sight. The first method is completely general and can be applied to any system. The second is limited to systems of a specific configuration, but allows direct generation of uncoupled stabilization equations. Analysis of an aerial photography system is presented as an example.

1. INTRODUCTION

Gimbal-mounted mirrors are commonly used to point and stabilize the line-of-sight (LOS) of vehicle-mounted optical systems. A *stabilized* LOS maintains its commanded angular slew rates in inertial space, independent of any vehicle motion. If the commanded slew rates are zero, then the stabilized LOS maintains a constant pointing direction in inertial space. Gimbaled mirrors used for stabilization are themselves *not* stabilized, but must move relative to both the vehicle and inertial space. The *stabilization equations* relate the rates of the gimbal angles to the angular velocity of the vehicle.

Interest in the stabilization equations stems from the emergence of *feed-forward* stabilization systems. In such systems, the vehicle angular velocity is measured, usually with a triad of gyroscopes. The stabilization equations are solved to give the gimbal angle rates as a function of the measured angular velocity components. The calculated gimbal angle rates are then fed as commanded inputs to the gimbal servo loops.

Two methods of deriving the stabilization equations are presented in this paper. The first method is completely general, but often computationally complex. The second, while considerably simpler in its construction, is applicable only to a certain class of gimbal configurations. This second method also has the advantage of producing *uncoupled* stabilization equations. The *line-of-sight reference frame* is introduced as a kinematic construction used by both methods. An aerial photography system is analyzed using both methods as an example. A brief summary of other methods is also included.

0-8194-0671-6/92/$4.00

2. REVIEW OF ALTERNATE METHODS

Derivation of the LOS stabilization equation for a system using a single-axis gimbaled mirror is common in the literature.[1,2] The LOS is usually stabilized in such systems by coupling the mirror shaft to a parallel gyro-stabilized shaft via a two-to-one drive mechanism. Two axis stabilization can be achieved by mounting the entire mechanism on an orthogonal outer gimbal. This configuration differs from the general feed-forward system in that the gyroscope(s) follow the LOS and directly measure the disturbances normal to it. In a general feed-forward system, the gyroscopes are fixed in the vehicle, and thus the measured angular rates are independent of LOS pointing.

Rodden presents a method of finding the stabilization equations for a general feed-forward system that is based on differentiation of the vector equations of mirror reflection.[3] Though this method has not been *proven* to be equivalent to the methods presented herein, stabilization equations for several systems have been derived using both approaches, and equivalent results were obtained. The technique presented here has the advantage of using an analytical construction that has multiple applications in the analysis of plane-mirror optical systems. Comparisons of ease-of-use are left to the reader.

3. LINE-OF-SIGHT REFERENCE FRAMES

The combination of a line-of-sight vector and two mutually-normal image plane vectors collectively define a *line-of-sight reference frame.* For example, the line-of-sight vector $\underline{r}_1$ and the horizontal and vertical image vectors $\underline{r}_2$ and $\underline{r}_3$ of the camera *C* shown in Figure 1 together define the camera LOS reference frame *R*. Vector set $(\underline{r}_1,\underline{r}_2,\underline{r}_3)$ forms an orthonormal, right-handed triad.

The transformation of the incoming LOS vector $\underline{r}_1$ to the outgoing LOS $\underline{s}_1$ by the mirror *M* can be represented by the well-known equation:[4,5]

$$\underline{s}_1 = [\mathbf{M}]\ \underline{r}_1 \tag{1}$$

The matrix **M** is found from the mirror normal vector components as follows:

$$[\mathbf{M}] = \begin{bmatrix} 1-2n_1n_1 & -2n_1n_2 & -2n_1n_3 \\ -2n_2n_1 & 1-2n_2n_2 & -2n_2n_3 \\ -2n_3n_1 & -2n_3n_2 & 1-2n_3n_3 \end{bmatrix} \tag{2}$$

Eq.(1) can be generalized to include the image vectors as well:

$$\underline{s}_i = [\mathbf{M}]\ \underline{r}_i \qquad i=1,3 \tag{3}$$

As a result of the orthogonality of **M**, the *outgoing* LOS vector set $(\underline{s}_1,\underline{s}_2,\underline{s}_3)$ given by Eq.(3) will also be an orthonormal triad. Set *S*, however, will be left-handed, as shown in Figure 1. The change in handedness of the LOS reference frame is indicative of the *inversion/reversion* property of mirrors. The relative stability of *S* in inertial space determines the stability of the image on the camera focal plane.

4. ANGULAR VELOCITY OF AN LOS REFERENCE FRAME

The "unit-vector triad" definition of an LOS reference frame allows the *angular velocity* of an LOS frame to be defined as if it were a kinematic rigid body. For example, following the construction of Kane[6], ${}^{N}\underline{\omega}^{V}$, the angular velocity of an LOS frame *V* in some reference frame *N*, is given as follows:

$$ {}^{N}\underline{\omega}^{V} = \underline{v}_1\left(\frac{d}{dt}(\underline{v}_2)\cdot\underline{v}_3\right) + \underline{v}_2\left(\frac{d}{dt}(\underline{v}_3)\cdot\underline{v}_1\right) + \underline{v}_3\left(\frac{d}{dt}(\underline{v}_1)\cdot\underline{v}_2\right) \tag{4} $$

Equation (4) is valid as long as $\underline{v}_i$ are right-handed and fixed in *V* and the differentiation is with respect to the reference frame in which the angular velocity is defined (often referred to as the *base* reference frame, which for Eq.(4) is ***N***). Differentiation of the unit vectors $\underline{v}_i$ is straight forward if they are expressed by components in a vector basis that is fixed in the base reference frame. Eq.(4) can be used with a left-handed reference frame by simply negating any one of the unit vectors.

The construction of an expression for the angular velocity of a reference frame in a particular base frame is often aided by use of one or more intermediate reference frames and the *addition theorem*. For instance, if the angular velocity of LOS frame *V* in frame *P* is already known, then the angular velocity of *V* in *N* can be evaluated by finding the angular velocity of *P* in *N* and adding:

$$ {}^{N}\underline{\omega}^{V} = {}^{N}\underline{\omega}^{P} + {}^{P}\underline{\omega}^{V} \tag{5} $$

If a unit vector can be found that is fixed in two adjacent reference frames, then the frames are said two move with *simple* angular velocity. For instance, if a unit vector $\underline{k}$ is fixed in both reference frames *A* and *B*, and *B* is brought into alignment in *A* by a right-hand rotation through the angle θ, then the angular velocity of *B* in *A* is found as:

$$ {}^{A}\underline{\omega}^{B} = \dot{\theta}\underline{k} \tag{6} $$

5. LOS STABILIZATION

A line-of-sight is often defined as stabilized if the orthogonal components of the LOS angular velocity are zero.[7] This definition, however, does not cover the possibility of stabilization while tracking or slewing. Under these circumstances the *difference* between the commanded slew rates and the actual rates should be zero. Further, the angular velocity of a vector is not a generally defined kinematic quantity. The angular velocity of a LOS reference frame *is* defined though and can be used to state a clear mathematical definition of LOS stabilization:

> If the set $(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ forms the line-of-sight and image vectors of LOS reference frame *V*, and ${}^{N}\underline{\omega}^{V}$ is the angular velocity of *V* in reference frame *N*, and Ω_2 and Ω_3 represent the commanded LOS slew rates in *N* about the $\underline{v}_2$ and $\underline{v}_3$ vectors respectively, then the line-of-sight $\underline{v}_1$ is stabilized in *N* if the following two conditions are satisfied:

$$ {}^{N}\underline{\omega}^{V}\cdot\underline{v}_2 - \Omega_2 = 0 \qquad {}^{N}\underline{\omega}^{V}\cdot\underline{v}_3 - \Omega_3 = 0 \tag{7} $$

6. STABILIZATION EQUATIONS

The form of the stabilization equations follows directly from Equations (5) and (7). Let reference frame V be an outgoing LOS frame from an optical system attached to a vehicle P. Frame V is steered via a gimbaled mirror on P and thus has an angular velocity in P (${}^{P}\underline{\omega}^{V}$). The vehicle P moves in reference frame N (usually considered inertial) with angular velocity ${}^{N}\underline{\omega}^{P}$. Using Eq.(5), and setting the commanded slew rates Ω_i to zero (without loss of generality), Eqs.(7) become:

$$({}^{N}\underline{\omega}^{P} + {}^{P}\underline{\omega}^{V})\cdot\underline{v}_2 = 0 \qquad\qquad ({}^{N}\underline{\omega}^{P} + {}^{P}\underline{\omega}^{V})\cdot\underline{v}_3 = 0 \tag{8}$$

The angular velocity of P in N is a disturbance to the stabilization system and must be measured with the gyroscopes. The angular velocity of V in P is controllable and is a function of the gimbal angle rates $\dot{\theta}_i$. The stabilization equations are the control equations on $\dot{\theta}_i$ using vehicle angular velocity components ω_j as inputs such that Eqs.(8) are satisfied:

$$\dot{\theta}_i = f(\omega_j) \tag{9}$$

The formation of the stabilization equations proceeds as follows:

1) The angular velocity of the vehicle is assumed measured and known.
2) The angular velocity of the outgoing LOS frame in the vehicle reference frame is constructed.
3) The sum and inner products of Eq.(8) are computed.
4) The resulting equations are solved for $\dot{\theta}_i$.

The two methods presented here of finding the stabilization equations differ only in the manner in which ${}^{P}\underline{\omega}^{V}$ is calculated (step 2 above).

6.1. Method One: direct differentiation

The first method of calculating the angular velocity of the outgoing LOS frame in the vehicle proceeds directly:

1) Designate a camera LOS frame as described. Note that the choice of orthogonal image vectors is arbitrary.
2) Calculate the mirror transform(s) as required. The transform for the gimbaled mirror(s) will be a function of the gimbal angles.
3) Transform the camera LOS set to the outgoing vector set ($\underline{v}_1, \underline{v}_2, \underline{v}_3$) using Eq.(3) for each mirror.
4) Calculate ${}^{P}\underline{\omega}^{V}$ directly using Eq.(4). Note that differentiation of the vectors is easier if they are expressed in a vector basis that is fixed in the vehicle.

6.2. Method Two: intermediate reference frames

The second method of finding ${}^{P}\underline{\omega}^{V}$ involves finding a series of adjacent reference frames between the vehicle and the outgoing LOS frame, each of which moves with simple angular velocity in the next frame. The addition theorem is then used to find ${}^{P}\underline{\omega}^{V}$. This method is known to work if, for each gimbaled mirror and for all mirror orientations, a vector ($\underline{u}$) exists that is both fixed in the plane of the mirror and orthogonal to the incoming line-of-sight.

For systems that meet the above criteria, the method can be summarized as follows:

1) Define a reference frame (U) in which the incoming LOS and $\underline{u}$ are both fixed. Calculate the mirror transform in the U basis.
2) Express the incoming LOS in this basis and calculate the outgoing LOS $\underline{v}$.
3) Define a LOS reference frame V using $\underline{v}$ and $\underline{u}$.
4) Calculate the angular velocity of V in U using Eq.(4). Frame V will move with simple angular velocity in U.
5) Calculate the angular velocity of U in the vehicle by working outward through the gimbals.

Note: In general, the outgoing LOS reference frames defined by step 3) of both methods are not the same. The common designator V is used to indicate that they both satisfy the definition of an outgoing LOS reference frame. As it happens, the two frames will generally have a simple angular velocity relative to each other about the line-of-sight. This implies that the angular velocity component of an LOS frame about the line-of-sight is inconsequential in the calculation of the stabilization equations. This can be verified by noting that stabilization condition of Eq.(7) can be alternately written:

$$ {}^{N}\underline{\omega}^{V} - ({}^{N}\underline{\omega}^{V} \cdot \underline{v}_1)\underline{v}_1 = 0 \tag{10} $$

Note that the term being subtracted is the line-of-sight component.

7. EXAMPLE: AN AERIAL PHOTOGRAPHY SYSTEM

Figure 2 shows a gimbaled-mirror aerial photography system. The camera C is mounted internally to the floor of aircraft P. The camera LOS $\underline{c}_1$ is reflected off of mirror M and pointed at the target as outgoing LOS $\underline{v}_1$. Mirror M is brought into alignment in the aircraft by rotations ψ of the outer gimbal A and θ of the inner gimbal, to which M is rigidly attached. The components of the aircraft angular velocity in the inertial reference frame N are measured as $(\omega_1,\omega_2,\omega_3)$ along the forward, right, and down directions of the aircraft respectively. The following unit vector bases, as shown in Figure 3, are introduced to define the geometry and aid in the analysis of the system. Each forms a right-handed orthonormal triad unless otherwise indicated:

- Airframe set $(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ - Fixed in the aircraft and aligned in the forward, right, and down directions respectively.

- Camera LOS set $(\underline{c}_1,\underline{c}_2,\underline{c}_3)$ - Aligned with set $(-\underline{p}_1,-\underline{p}_2,\underline{p}_3)$.

- Outer gimbal set $(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ - Attached to the outer gimbal A and aligned with the set $(\underline{p}_1,\underline{p}_2,\underline{p}_3)$ when angle ψ is zero. Vector $\underline{a}_2$ is parallel to the inner gimbal axis.

- Mirror set $(\underline{m}_1,\underline{m}_2,\underline{m}_3)$ - Fixed to the steering mirror M (and thus to the inner gimbal) and aligned with $(\underline{a}_1,\underline{a}_2,\underline{a}_3)$ when θ is zero. Vector $\underline{m}_1$ is coincident with the vector normal to the mirror surface.

- Outgoing LOS set $(\underline{v}_1,\underline{v}_2,\underline{v}_3)$ - The transformation of the camera set C through the mirror M. Set V is left-handed.

The angular velocity of the aircraft in *N* can be expressed in vector form using the three measured components:

$$^{N}\underline{\omega}^{P} = \omega_1 \underline{p}_1 + \omega_2 \underline{p}_2 + \omega_3 \underline{p}_3 \tag{11}$$

The stabilization equations for this system will be generated by summing Eq.(11) with the angular velocity of the respective outgoing LOS reference frames as found by the two methods of this paper.

7.1. Method One: direct differentiation

Following the steps outlined in Section 6.1:

1) The Camera frame *C* has been designated.
2) The following two basis transforms are required (Note that C_i and S_i stand for cos(*i*) and sin(*i*) respectively):

$$\begin{bmatrix} \underline{m}_1 \\ \underline{m}_2 \\ \underline{m}_3 \end{bmatrix} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 1 & C_\theta \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\psi & S_\psi \\ 0 & -S_\psi & C_\psi \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \tag{13}$$

The mirror normal $\underline{m}_1$ can be expressed in the *P* basis using Eqs.(12) and (13):

$$\underline{m}_1 = C_\theta \underline{p}_1 + S_\psi S_\theta \underline{p}_2 - C_\psi S_\theta \underline{p}_3 \tag{14}$$

The mirror transformation matrix **M** can be found using Eq.(14) in Eq.(2):

$$[\mathbf{M}] = \begin{bmatrix} 1-2C_\theta^2 & -2S_\psi S_\theta C_\theta & 2C_\psi S_\theta C_\theta \\ -2S_\psi S_\theta C_\theta & 1-2S_\psi^2 S_\theta^2 & 2S_\psi C_\psi S_\theta^2 \\ 2C_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & 1-2C_\psi^2 S_\theta^2 \end{bmatrix} \tag{15}$$

3) Since matrix **M** above is formulated in the *P* basis, the vectors of the camera LOS set must be expressed as column vectors in this basis:

$$[\ \underline{c}_1\ \underline{c}_2\ \underline{c}_3\ ] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

The outgoing LOS set *V* is formed, also in column form, by using Eqs.(15) and (16) in Eq.(3):

$$[\ \underline{v}_1\ \underline{v}_2\ \underline{v}_3\ ] = [\mathbf{M}][\ \underline{c}_1\ \underline{c}_2\ \underline{c}_3\ ] \tag{17}$$

Completing the calculations and transposing produces the more familiar row format:

$$\begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \end{bmatrix} = \begin{bmatrix} -1+2C_\theta^2 & 2S_\psi S_\theta C_\theta & -2C_\psi S_\theta C_\theta \\ 2S_\psi S_\theta C_\theta & -1+2S_\psi^2 S_\theta^2 & -2S_\psi C_\psi S_\theta^2 \\ 2C_\psi S_\theta C_\theta & 2S_\psi C_\psi S_\theta^2 & 1-2C_\psi^2 S_\theta^2 \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \tag{18}$$

4) The outgoing LOS vectors of Eq.(18) are in convenient form for use in Eq.(4), since the time derivatives of $\underline{p}_i$ in the P reference frame are equal to zero. Only the $\underline{v}_2$ and $\underline{v}_3$ terms need be calculated. Since V is a left-handed set, vector $\underline{v}_3$ is negated.

$${}^{P}\underline{\omega}^{V}\cdot\underline{v}_2 = \left(\frac{d}{dt}(-\underline{v}_3)\cdot\underline{v}_1\right) = 2S_\psi S_\theta C_\theta\dot{\psi} - 2C_\psi\dot{\theta} \tag{19}$$

$${}^{P}\underline{\omega}^{V}\cdot\underline{v}_3 = -\left(\frac{d}{dt}(\underline{v}_1)\cdot\underline{v}_2\right) = 2C_\psi S_\theta C_\theta\dot{\psi} + 2S_\psi\dot{\theta} \tag{20}$$

Equations (19) and (20) end the method-specific steps in calculating the angular velocity of the LOS frame in the vehicle. To find the complete stabilization equations, the angular velocity of the aircraft as given by Eq.(11) must first be transformed to the V basis. The matrix of Eq.(18) represents the vector basis transformation from the P to the V basis and can be used for this transformation::

$${}^{N}\underline{\omega}^{P}\cdot\underline{v}_2 = 2S_\psi S_\theta C_\theta\omega_1 + (-1+2S_\psi^2 S_\theta^2)\omega_2 - 2S_\psi C_\psi S_\theta^2\omega_3 \tag{21}$$

$${}^{N}\underline{\omega}^{P}\cdot\underline{v}_3 = 2C_\psi S_\theta C_\theta\omega_1 + 2S_\psi C_\psi S_\theta^2\omega_2 + (1-2C_\psi^2 S_\theta^2)\omega_3 \tag{22}$$

The stabilization equations are formed by substituting Eqs.(19) through (22) into Eqs.(8):

$$2S_\psi S_\theta C_\theta\dot{\psi} - 2C_\psi\dot{\theta} + 2S_\psi S_\theta C_\theta\omega_1 + (-1+2S_\psi^2 S_\theta^2)\omega_2 - 2S_\psi C_\psi S_\theta^2\omega_3 = 0 \tag{23}$$

$$2C_\psi S_\theta C_\theta\dot{\psi} + 2S_\psi\dot{\theta} + 2C_\psi S_\theta C_\theta\omega_1 + 2S_\psi C_\psi S_\theta^2\omega_2 + (1-2C_\psi^2 S_\theta^2)\omega_3 = 0 \tag{24}$$

Note that Eqs.(23) and (24) form a system of coupled linear equations in $\dot{\psi}$ and $\dot{\theta}$. The equations must be uncoupled for use as feed-forward commands to the gimbal servo rate loops. This can be accomplished by Gauss elimination, Cramer's rule, etc:

$$\dot{\theta} = -\frac{1}{2}(C_\psi\omega_2 + S_\psi\omega_3) \tag{25}$$

$$\dot{\psi} = -\omega_1 + \cot(2\theta)(S_\psi\omega_2 - C_\psi\omega_3) \tag{26}$$

As mentioned previously, the method just described is algebraically complex, even for the simple example presented. The use of a symbolic algebra program such as MACSYMA or REDUCE can significantly decrease the time spent calculating and checking the results. As an example, the REDUCE runstream written to calculate Eqs.(19) and (20) from Eqs.(4) and (18) is presented in the Appendix along with the output. One need only to attempt this calculation by hand to appreciate the power of symbolic algebra in this application.

7.2. Method Two: intermediate reference frames

Following the steps outlined in Section 6.2:

1) A vector $\underline{u}$ is sought that is fixed in the mirror and perpendicular to the incoming LOS for all gimbal angles. Equations (12) and (13) can be used to show that vector $\underline{m}_2$ satisfies this requirement. Further, both the incoming LOS vector $\underline{c}_1$ (= $-\underline{p}_1$ = $-\underline{a}_1$) and $\underline{m}_2$ (= $\underline{a}_2$) are fixed in the outer gimbal reference frame *A*, and thus *A* will serve as the desired reference frame *U*. The mirror normal in the *A* basis is found from Eq.(12):

$$\underline{m}_1 = C_\theta \underline{a}_1 - S_\theta \underline{a}_3 \tag{27}$$

The mirror transformation in the *A* basis follows directly:

$$[\mathbf{M}] = \begin{bmatrix} 1-2C_\theta^2 & 0 & 2S_\theta C_\theta \\ 0 & 1 & 0 \\ 2S_\theta C_\theta & 0 & 1-2S_\theta^2 \end{bmatrix} = \begin{bmatrix} -C_{2\theta} & 0 & S_{2\theta} \\ 0 & 1 & 0 \\ S_{2\theta} & 0 & C_{2\theta} \end{bmatrix} \tag{28}$$

2) The expression of the incoming LOS in the *A* basis and the calculation of the outgoing LOS proceeds directly:

$$\underline{v}_1 = [\mathbf{M}]\ \underline{c}_1 = [\mathbf{M}]\ (-\underline{a}_1) = \begin{bmatrix} -C_{2\theta} & 0 & S_{2\theta} \\ 0 & 1 & 0 \\ S_{2\theta} & 0 & C_{2\theta} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

$$\underline{v}_1 = C_{2\theta} \underline{a}_1 - S_{2\theta} \underline{a}_3 \tag{30}$$

3) The LOS reference frame *V* is defined using $\underline{v}_1$, $\underline{u}$ as $\underline{v}_2$, and their vector product as $\underline{v}_3$:

$$\underline{v}_3 = \underline{v}_1 \times \underline{v}_2 = (C_{2\theta} \underline{a}_1 - S_{2\theta} \underline{a}_3) \times (\underline{a}_2) \tag{31}$$

$$\underline{v}_3 = S_{2\theta} \underline{a}_1 + C_{2\theta} \underline{a}_3 \tag{32}$$

Reference frame *V* can be expressed in matrix format as:

$$\begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \underline{v}_3 \end{bmatrix} = \begin{bmatrix} C_{2\theta} & 0 & -S_{2\theta} \\ 0 & 1 & 0 \\ S_{2\theta} & 0 & C_{2\theta} \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \tag{33}$$

4) Equation (33) can be used in Eq.(4) to show that the LOS frame *V* moves with simple angular velocity in *A*:

$$^{A}\underline{\omega}^{V} = 2\dot{\theta}\underline{v}_2 \tag{34}$$

5) The angular velocity of the outer gimbal reference frame in the vehicle can be found using Eqs.(4) and (13).

$$^{P}\underline{\omega}^{A} = \dot{\psi}\underline{a}_1 \tag{35}$$

Equations (34) and (35) are summed using the addition theorem to give the angular velocity of the LOS frame in the vehicle:

$$^{P}\underline{\omega}^{V} = {^{P}\underline{\omega}^{A}} + {^{A}\underline{\omega}^{V}} = \dot{\psi}\underline{a}_1 + 2\dot{\theta}\underline{v}_2 \tag{36}$$

Equation (36) can be expressed in a consistent basis by solving Eq.(33) for $\underline{a}_1$ and substituting:

$$^{P}\underline{\omega}^{V} = \dot{\psi}(C_{2\theta}\underline{v}_1 + S_{2\theta}\underline{v}_3) + 2\dot{\theta}\underline{v}_2 \tag{37}$$

Equation (37) ends the method-specific steps of calculating the angular velocity of *V* in *P*. Though the method has been presented here rather formally, many of the intermediate results, such as Eqs.(34) and (35), can usually be reached by observation of the system configuration.

Forming the stabilization equations from here proceeds as before. The angular velocity of the vehicle in the inertial reference frame can be expressed in the *V* basis by combination of Eqs.(11), (13), and (33):

$$^{N}\underline{\omega}^{P} = (\ldots)\underline{v}_1 + (C_{\psi}\omega_2 + S_{\psi}\omega_3)\underline{v}_2 + (S_{2\theta}\omega_1 - S_{\psi}C_{2\theta}\omega_2 + C_{\psi}C_{2\theta}\omega_3)\underline{v}_3 \tag{38}$$

Equations (37) and (38) are substituted into Eqs.(8) to yield:

$$2\dot{\theta} + C_{\psi}\omega_2 + S_{\psi}\omega_3 = 0 \tag{39}$$

$$-S_{2\theta}\dot{\psi} + S_{2\theta}\omega_1 - S_{\psi}C_{2\theta}\omega_2 + C_{\psi}C_{2\theta}\omega_3 = 0 \tag{40}$$

Equations (25) and (26) follow directly from Eqs.(39) and (40).

Note that the second method led directly to the uncoupled stabilization equations (39) and (40). Further, all the algebra involved was handled easily by hand. As a final note on applicability, seven of the nine classes of gimbaled mirrors (Z1, Z2, and H1 through H5) described by Casey and Phinney in their overview paper[8] meet the configuration requirement of the method.

8. CONCLUSIONS

Two methods of generating the line-of-sight stabilization equations of gimbaled mirror systems were presented in this paper. Both make use of line-of-sight reference frames, which allow a mathematical definition of line-of-sight stabilization. The first method, while completely general, is algebraically complex. The use of computer symbolic algebra greatly facilitates this method. The second method, while not completely general, is applicable to many common systems, is algebraically simple by comparison, and directly produces uncoupled stabilization equations. The line-of-sight reference frame methodology introduced herein is also useful in image rotation, boresight coefficient, and mechanical tolerance analyses. Accompanying work on these topics is in progress by the author.

9. APPENDIX: SYMBOLIC ALGEBRA RUNSTREAM AND RESULTS

9.1. REDUCE runstream

```
% RUNSTREAM FOR LOS STABILIZATION
ON GCD;
% MULTI-USE OPERATORS
% DEFINE A DEXTRAL UNIT VECTOR SET
OPERATOR UVS$
FOR ALL A1,A2,A3  LET UVS(A1,A2,A3)=
<<LET A1*A1=1$LET A2*A2=1$LET A3*A3=1$
  LET A1*A2=0$LET A2*A3=0$LET A3*A1=0$
  FACTOR A1,A2,A3$>>$
% ANGLE DEFINITION
OPERATOR ANGLE$
FOR ALL THETA,SIN,COS LET ANGLE(THETA,SIN,COS)=
<<LET DF(SIN,T)=COS*DF(THETA,T)$
  LET DF(COS,T)=-SIN*DF(THETA,T)$>>$
% DOT OPERATOR
OPERATOR DOT,D$
FOR ALL U1 LET DOT(U1)= <<LET DF(U1,T)=D(U1)$>>$
% AERIAL PHOTOGRAPHY SYSTEM
UVS(P1,P2,P3);
ANGLE(THE,ST,CT);
ANGLE(PSI,SS,CS);
DOT(THE);
DOT(PSI);
A:=2*SS*ST*CT;
B:=2*CS*ST*CT;
C:=2*SS*CS*ST**2;
V1:=(2*CT**2-1)*P1+A*P2-B*P3;
V2:=A*P1+(2*SS**2*ST**2-1)*P2-C*P3;
V3:=B*P1+C*P2+(1-2*CS**2*ST**2)*P3;
LET SS**2+CS**2=1;
LET ST**2+CT**2=1;
W2:=DF(-V3,T)*V1;
W3:=-DF(V1,T)*V2;
END;
```

9.2. REDUCE output

```
W2  =   2*(D(PSI)*SS*CT*ST - D(THE)*CS)

W3  =   2*(D(PSI)*CS*CT*ST + D(THE)*SS)
```

10. REFERENCES

1. B. Ellison and J. Richi, "Inertial stabilization of periscopic sights band driven three axis gimbal", *Proceedings of the SPIE*, Vol. 389, pp 107-120, SPIE, Bellingham, 1983.

2. M.K. Masten and J.M. Hilkert, "Electromechanical system configurations for pointing, tracking, and stabilization applications", *Proceedings of the SPIE*, Vol. 779, pp 75-87, SPIE, Bellingham, 1987.

3. J.J. Rodden, "Mirror line of sight on a moving base", AAS 89-030.

4. J.C. Polasek, "Matrix analysis of gimbaled mirror and prism systems", *Journal of the Optical Society of America*, Vol. 57-10, pp 1193-1201, 1967.

5. L. Levi, *Applied Optics*, Vol. I, p. 347, Wiley, New York, 1968.

6. T.R. Kane and D.A. Levinson, *Dynamics: Theory and Applications*, pp 1-25, McGraw-Hill, New York, 1985.

7. W.L. Wolfe and G.J. Zissis, *The Infrared Handbook*, p 22-18, The Office of Naval Research, Washington, 1985.

8. W.L. Casey and D.D. Phinney, "Representative pointed optics and associated gimbal characteristics", *Proceedings of the SPIE*, Vol. 887, pp 116-123, SPIE, Bellingham, 1988.

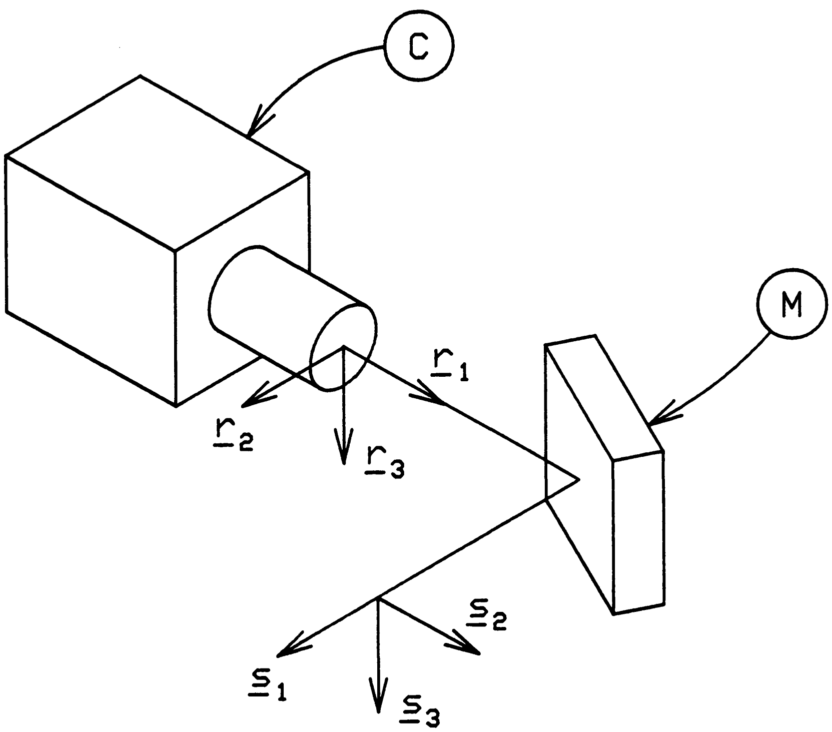

Fig.1. Line–of–Sight Reference Frames

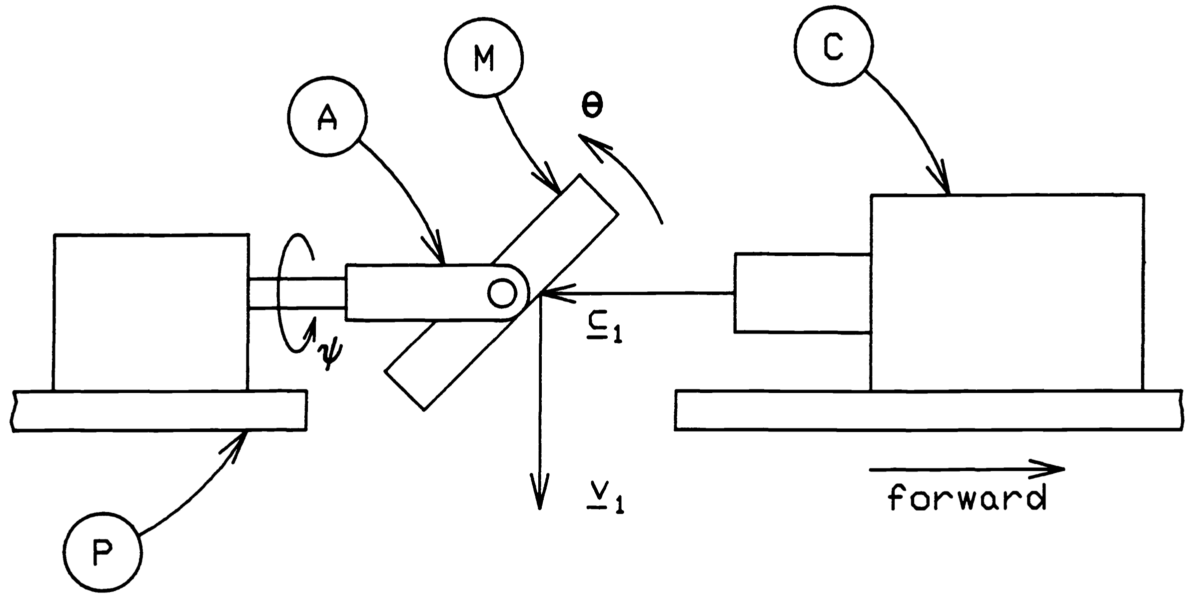

Fig.2. Aerial Photography System

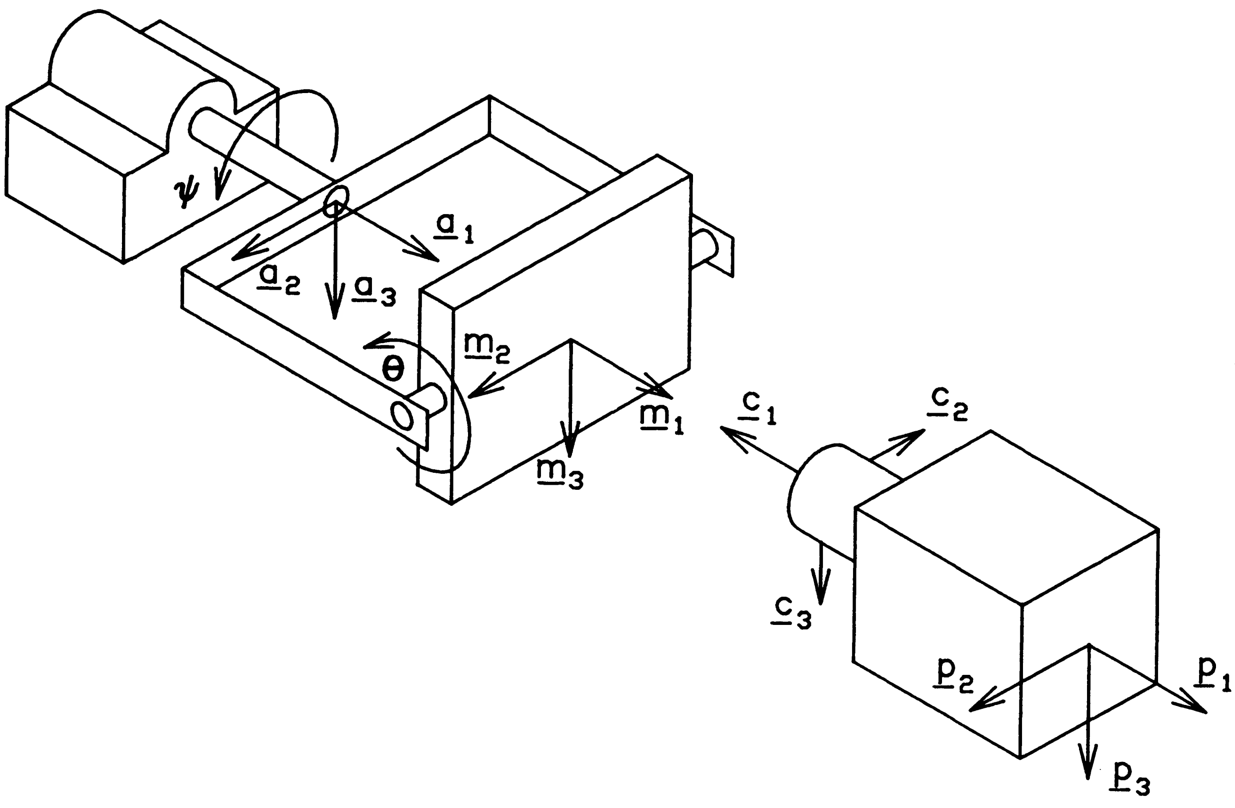

Fig.3. Analytical Framework