Deflection of A Circular Scanning Mirror Under Angular Acceleration

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Abstract

During turn-around a scanning mirror may be subjected to severe angular acceleration which can cause surface deflection of sufficient magnitude to distort the image significantly. In this paper a solution is developed for a circular mirror for a particular method of application of the direction-reversing forces. Parametric curves are presented which allow computation of the deflection as a function of mirror size and point of load application. An example is computed in which the deflection is a significant fraction of the optical wave length.

1. Introduction

Many optical systems contain a scanning mirror which oscillates about an axis in the plane of the mirror, thereby causing the optical image to scan across a set of detectors located at the focal plane. When great precision and sensitivity are required, the optics become quite large and the mirror consequently becomes more flexible, especially when very light weight is required for airborne or spaceborne applications.

The effect of mirror bending is to distort the shape and increase the size of the optical blur function. The result is a decrease in precision in locating objects within the field of view. A lower limit to the blur circle (in radians) is the diffraction limit given by $2.44\ \lambda/D$, where λ is the wavelength and D is the aperture diameter. If distortion is to be negligible, the change in slope of the mirror surface must be small compared to this value. Depending on the deflection of the surface, the maximum slope will be 4 to 8 times w/D, where w is the maximum deflection and D is the diameter of the mirror (which is usually about the size of the aperture). Therefore, the requirement is $8\ w/D \ll 2.44\ \lambda/D$ or $w \ll 0.3\ \lambda$. A rule of thumb for designing optical systems is that the surface must be true to within 1/20 wavelength, or $w < \lambda/20$. For a system operating at $\lambda = 3\mu m$ the allowable deflection is $3 \times 10^{-6}/20 = 0.15\ \mu m$ or about 6μ inches. It will be seen that for large lightweight mirrors, this requirement may be a controlling element in the system design.

The situation treated here is that of steady angular acceleration about a diameter under the action of a moment applied by two concentrated forces of constant magnitude. This approximates the situation during the time the mirror is reversing direction at the end of a scan. The situation is pseudo-static; i.e., the system is assumed to be in steady state such that the applied forces are equilibrated by inertia forces due to the accelerating mass of the plate. Since angular velocities are low, the effects of centrifugal forces are neglected. Also neglected is the transient motion due to the impulsive application of the forces. Depending on the internal damping of the material, this effect can increase the instantaneous deflection by as much as a factor of 2.

The system is, of course, not operating during the time the mirror is changing direction. However, immediately upon release of the applied forces, the system must begin to operate. At that instant the deflections will be from one to two times those given by the pseudo-static analysis. The subsequent oscillation of the mirror is not treated here, but it can be expected to persist for several cycles at the natural frequency of the plate. For this reason the results obtained here must be viewed as upper limits to the deflection that can be expected during the usable portion of the scan.

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2. Problem Formulation

Consider a circular plate, as shown in Fig. 1, free to rotate about the x axis and acted upon by forces P, at positions defined by the polar coordinates b, γ, and causing an angular acceleration $\ddot{\psi}$ about the x axis. The force is related to the angular acceleration by

$$J\ddot{\psi} = 2Pb \sin \gamma \tag{1}$$

where J is the moment of inertia of the plate about the x axis. In terms of polar coordinates r, θ(θ measured positive counterclockwise from the x axis),*

$$J = \sigma h \int_0^{2\pi} \int_0^a (r \sin \theta)^2 \cdot r dr d\theta = \frac{\pi}{4}\ \sigma h a^4. \tag{2}$$

In a state of steady angular acceleration, each element of the plate is acted upon by an "inertia pressure" q equal to the negative of the linear acceleration of the element times σh, the mass of the plate per unit area.

$$q(r, \theta) = -\sigma h \ddot{\psi} r \sin \theta. \tag{3}$$

This is the ramp function sketched above. It is the combined effect of the forces P and the counteracting inertia pressure q which causes the mirror to bend.

The plate is assumed to be free at the boundary (r = a); i.e., the radial component of the bending moment and effective shear force must vanish at r = a.

3. Solution

Timoshenko[1] presents a series solution for a circular plate, clamped at the edge, with a concentrated load a distance b from the center. To satisfy the free boundary conditions, it is necessary to take additional solutions of the homogeneous plate equations of the form $A_m\ r^m + B_m\ r^{m+2}$. The general solution also contains some terms which are singular at the origin (r = 0); however, in our case there is no force singularity applied there, hence these terms are not applicable.

Defining dimensionless variables $\rho = r/a$, $\beta = b/a$, and $\omega = 192\ Dw/\sigma h a^5 \ddot{\psi}$, the solution takes the form[2]

$$\omega = \rho^5 \sin \theta + 6 \sum_{m=1}^{\infty} [A_m \rho^m + B_m \rho^{m+2} - R_m] \cdot \frac{\sin m\gamma}{\sin \gamma} \sin m\theta \tag{4}$$

*σ is the mass density of the plate.

[1] Ref. 1, pp. 290, 291.

[2] In the remainder of the paper, D is the plate stiffness $Eh^3/12(1-\nu^2)$. h is the thickness of the plate, σ its density, E the modulus of elasticity and ν is Poisson's ratio.

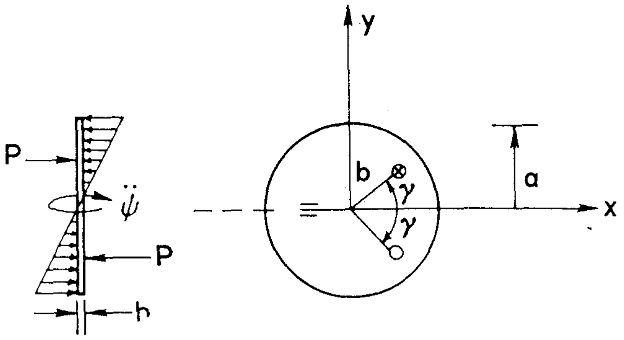

Fig. 1. Mirror geometry and nomenclature.

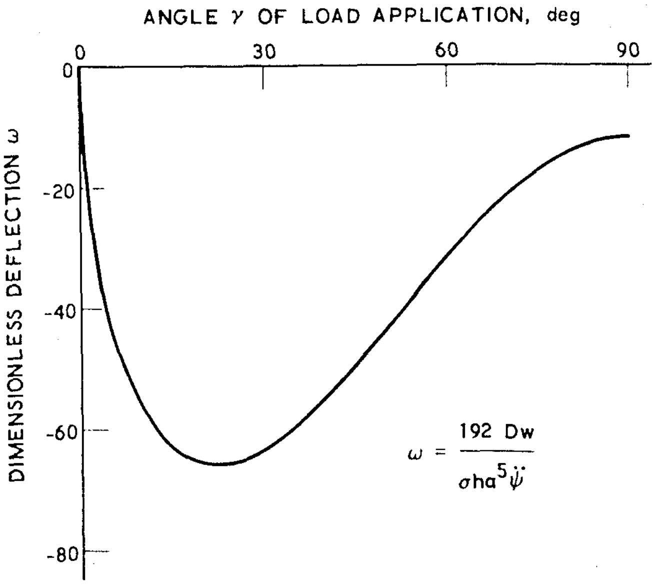

Fig. 2. Deflection under edge load vs point of load application.

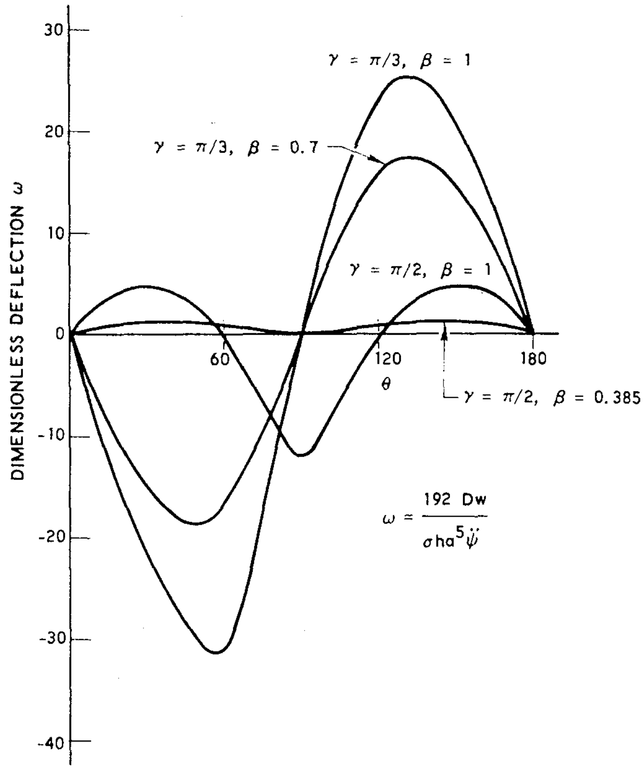

Fig. 3. Edge deflection vs θ.

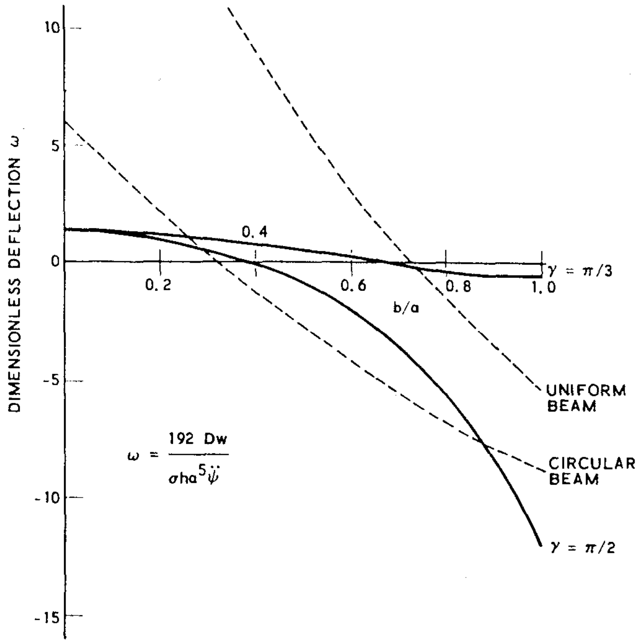

Fig. 4. Edge deflection at $\theta = \pi/2$ as function of load point.

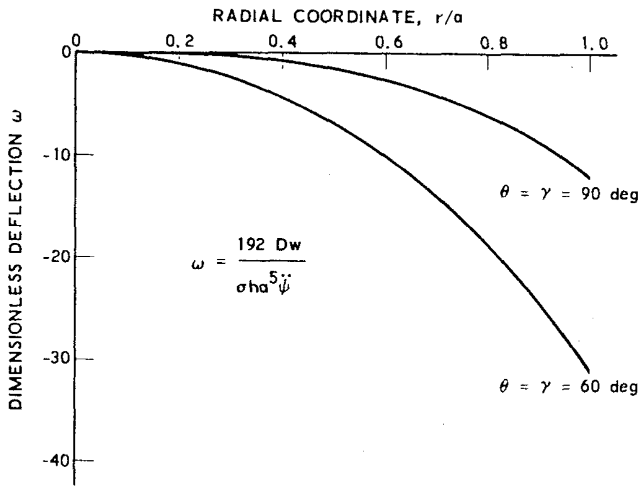

Fig. 5. Deflection along loaded radius for edge loading.

where the R_m terms correspond to the solution for the clamped plate subjected to opposing concentrated forces of magnitude given by Eq. (1) at $\rho = \beta$ and $\theta = \pm\gamma$; the first term is the particular solution for the inertia loading given by Eq. (3).[3] A_m and B_m are chosen to make the solution satisfy the boundary conditions and also the condition that the deflection and slope at the center are zero. These latter conditions are arbitrary, but are easily changed by a rigid body displacement (ω = const) or a rigid rotation ($\omega = A \sin\theta$ C $\cos\theta$) to correspond to the actual method of bearing support. The complete solution is given in Table 1. An APL computer program has been written to evaluate these expressions and the derivatives $\partial\omega/\partial\rho$, $\partial\omega/\rho\partial\theta$ as functions of β, γ, ρ, θ. Note that if values at the edge of the plate ($\rho = 1$) are desired, R_m and $\partial R_m/\partial\rho$ need not be computed since they are zero, corresponding to the clamped support condition.

For edge loading ($\beta = 1$), the solution at the edge ($\rho = 1$) becomes particularly simple.

[3]Ref. 1, pp. 285, 286.

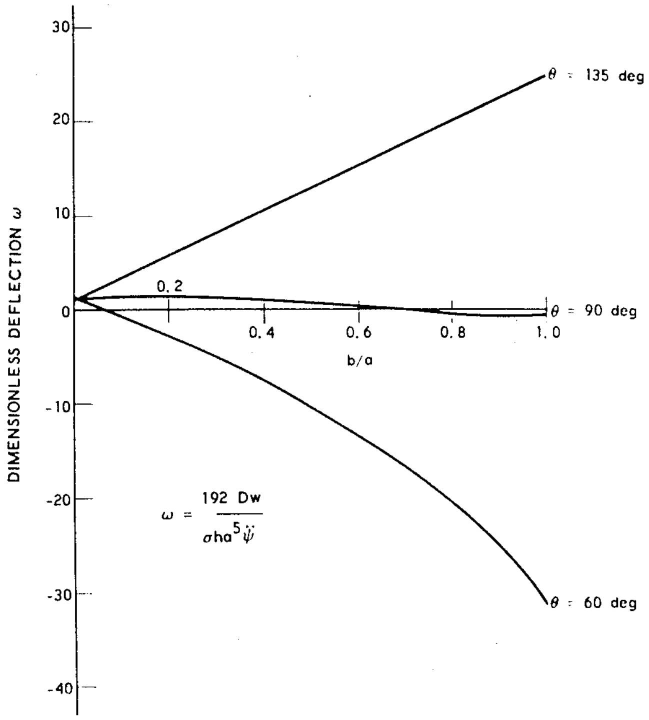

Fig. 6. Edge deflection vs b/a and θ for γ = 60 degrees.

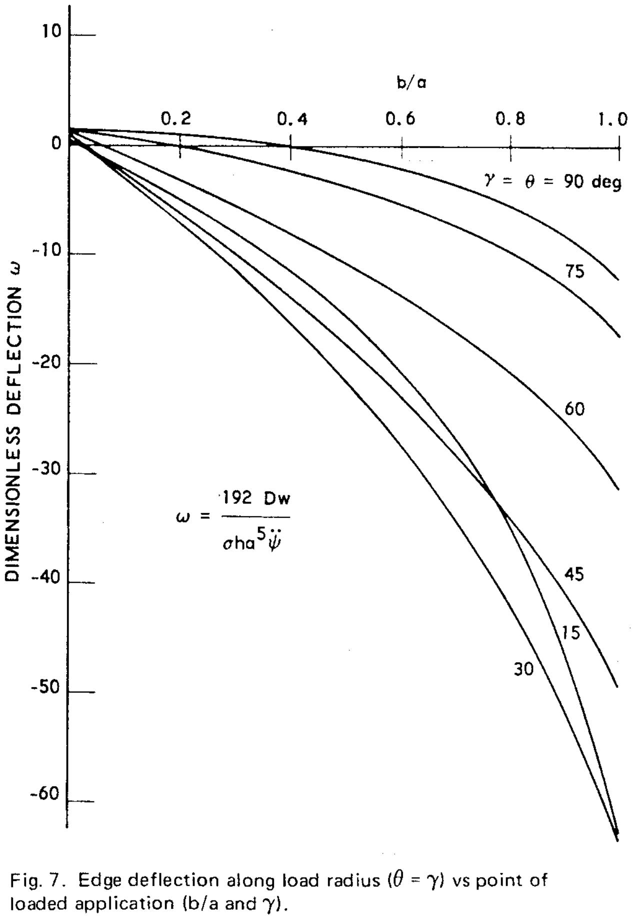

Fig. 7. Edge deflection along load radius ($\theta = \gamma$) vs point of loaded application (b/a and γ).

$$\omega = -\frac{7+\nu}{3+\nu}\sin\theta - \frac{24}{1-\nu}\sum_{m=2}^{\infty}\frac{m+2}{m^2(m-1)}\cdot\left[1-\frac{m-1}{m+1}\cdot\frac{2}{3+\nu}\right]\frac{\sin m\gamma}{\sin\gamma}\sin m\theta. \qquad (5)$$

4. Results

Equations (4) and (5) have been evaluated for various loading situations for $\nu = 0.3$, and the results are shown in Figs. 2 through 7. The maximum deflection usually occurs at the edge for θ slightly less than γ. Somewhat pathological exceptions to this rule of thumb occur for $\gamma = \pi/2$ and β near 0.4 and for γ near zero. In the first case, the end deflection is near zero and the maximum occurs elsewhere; however, the deflection is small everywhere compared to the case of edge loading. If this situation were practical to implement, it would result in the optimum situation from an optical point of view. Of course, the stress near the point of load application would be significantly larger than in the case of edge loading. In the second case, the solution approaches that of a pure moment applied at the x axis, and the maximum ($\omega = 13.1$ for $\nu = 0.3$) occurs at $\rho = 1$, $\theta = \pi/2$.

For comparison, results are shown in Fig. 4 based on two beam models. The results are consistent using the various models and serve to confirm that no gross error has occurred in the computations.

5. Application to a Particular Mirror Design

The results in the figures are in terms of the dimensionless deflection factor ω which must be multiplied by $w_p = \sigma h a^5 \ddot{\psi}/192D$ to give the deflection in inches. In this section, this factor will be evaluated for one proposed mirror design. The cross section is shown in Fig. 8. The same cross section is seen in two orthogonal directions, i.e., the ribs form a waffle-iron pattern. The plate is actually orthotropic, but the effect has not been considered in this analysis.

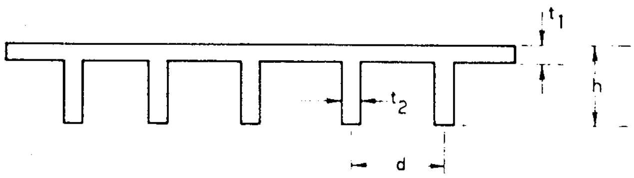

Fig. 8. Mirror cross-section.

The plate stiffness $D = Eh^3/12(1-\nu^2)$ is actually $I_d/(1-\nu^2)$, the moment of inertia per unit length divided by $(1-\nu^2)$. By computing the moment of inertia of one of the T sections and dividing by d,

$$I_d = \frac{1}{3}\left[(t_1-c)^3 + c^3\right] + \frac{1}{3}\frac{t_2}{d}\left[(h-c)^3 - (t_1-c)^3\right] \qquad (6)$$

where c is the distance of the centroid of the section from the face of the mirror, given by

$$c = \frac{1}{2}\left[dt_1^2 + t_2(h^2 - t_1^2)\right]. \qquad (7)$$

For a particular mirror design, a = 16.5 in., h = 3.5 in., t_1 = 0.25 in., t_2 = 0.125 in., and d = 2.1 in., which gives c = 0.827 in. and I_d = 0.507 cu in. For a total weight W of 36 lb and $\nu = 0.3$, the deflection factor w_p becomes

$$w_p = \frac{\sigma h a^5 \psi}{192D} = \frac{Wa^3 \psi(1-\nu^2)}{192\pi g l_d}$$

$$= 0.167 \times 10^{-6} \text{ in.} \approx \frac{1}{6}\ \mu \text{ in.} \qquad (8)$$

So the deflection for this example in μ inches is obtained approximately by dividing ω in the figures by 6.

6. Conclusions

Figure 7 shows that it is generally preferable to apply the forces as far from the edge as is practically possible and as near as possible to $\gamma = \pi/2$. However, as the point of load application moves nearer to the center of the plate, the forces required to generate a given angular acceleration increase, causing high stresses near the center of the plate and requiring more complex methods of load transfer from the supporting structure. For these reasons, values of b/a much less than 0.7 have not been proposed for implementation.

The maximum deflections for the example given range from 2 to 10 μ inches, which is of the order of $\lambda/20$ (= 6 μ inches for $\lambda = 3\ \mu$m). For the geometry suggested for that design (γ = 60 deg, β = 1), the deflection is 5 μ inches.

Note that the complete deflection surface of the mirror is available from the solution in Table 1. This information is necessary to evaluate the optical effects of the deformation. A computer program is available for computing these results as required. The program also computes the local changes in slope necessary for optimal ray tracing. Furthermore, with some additional work, the program can be extended to compute the mirror stresses as given in the appendix.

Table 1. Solution

$$\omega = \rho^5 \sin\theta + 6 \sum_{m=1}^{\infty} [A_m \rho^m + B_m \rho^{m+2} - R_m] \frac{\sin m\gamma}{\sin\gamma} \sin m\theta$$

$$B_1 = \frac{2}{3+\nu}\left(\frac{1-\nu}{6} + \beta^2\right),\ A_1 = 0$$

$$B_m = \frac{2}{3+\nu}\beta^{m-1}\left(\frac{2}{m} - \frac{\beta^2}{m+1}\right) \qquad (m > 1)$$

$$A_m = B_m\left(\frac{4/m}{1-\nu} - 1\right) - \frac{4(m+2)}{(1-\nu)\,m^2\,(m-1)}\beta^{m-1} \qquad (m > 1)$$

$$R_1 = -\frac{\beta^2}{2\rho} - (1-\beta^2)\rho + \left(1 - \frac{\beta^2}{2}\right)\rho^3 - 2\rho\ell n\rho \qquad (\rho > \beta)$$

$$= -(1-\beta^2)\rho - \frac{1}{2}(1-\beta^2)^2\rho^3/\beta^2 - 2\rho\ell n\beta \qquad (\rho < \beta)$$

$$R_m = \beta^{m-1}\rho^m\left[\frac{\beta^2}{m} - \frac{1}{m-1} + \frac{\rho^2}{m} - \frac{\beta^2\rho^2}{m+1}\right] + \frac{\beta^{m-1}}{\rho^m}\left[\frac{\rho^2}{m(m-1)} - \frac{\beta^2}{m(m+1)}\right] \qquad (\rho > \beta)\ (m > 1)$$

$$= \beta^{m-1}\rho^m\left[\frac{\beta^2}{m} - \frac{1}{m-1} + \frac{\rho^2}{m} - \frac{\beta^2\rho^2}{m+1}\right] + \frac{\rho^m}{\beta^{m+1}}\left[\frac{\beta^2}{m(m-1)} - \frac{\rho^2}{m(m+1)}\right] \qquad (\rho < \beta)\ (m > 1)$$

Appendix I. Differential Equations and Boundary Conditions

For small deflections, the equation describing the deflection w of the middle surface of a plate is

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = q(x,y,t)$$

where D is the plate stiffness $[Eh^3/12(1-\nu^2)]$ for a uniform plate (for a honeycomb or a ribbed structure, an equivalent thickness must be used), q is the load per unit area, and ∇^2 is the Laplacian operator:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

In polar coordinates, the moments per unit length are given by

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right]$$

$$M_{r\theta} = (1-\nu)D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right)$$

and the transverse shear stress resultants are

$$Q_r = -D\frac{\partial}{\partial r}(\nabla^2 w),$$

$$Q_\theta = -\frac{D}{r}\frac{\partial}{\partial\theta}(\nabla^2 w).$$

At the boundary, the moment M_r and the effective shear must vanish.

$$M_r(r=a) = 0,$$

$$V_r(r=a) = Q_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} = 0. \qquad (9)$$

The way in which the twisting moment combines with the transverse shear is described by Timoshenko.[3]

[3] Ref. 1, pp. 83, 84

Reference

1. Timoshenko and Woinkowsky-Krieger, *Theory of Plates and Shells,* 2nd Edition, McGraw-Hill Book Co., Inc. (1959).