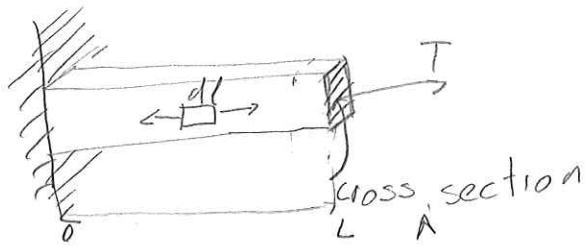
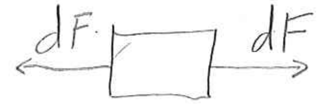
2) Force/deflection

$$\Delta z = \frac{F_z L_z}{EA}$$

Consider a beam in tension



Looking at differential element in the beam, the forces are equal but oppossite.



$\therefore$ stress is $\sigma = \frac{dF}{dA} \Rightarrow dF = \sigma dA$

$$F = \int_A \sigma dA = \sigma A \Rightarrow \sigma = \frac{F}{A} \quad \ldots(1)$$

$$\sigma = \varepsilon_0 E = \frac{\Delta L}{L} E \Rightarrow \Delta L = \int_0^L \frac{\sigma}{E} dl$$

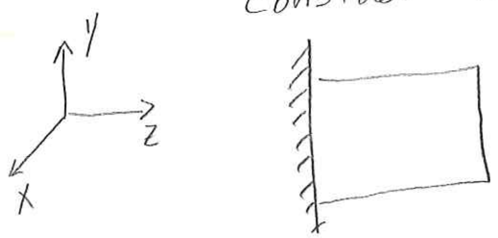
$$\Delta L = \frac{\sigma L}{E}$$

$$\Delta L = \frac{FL}{EA}$$

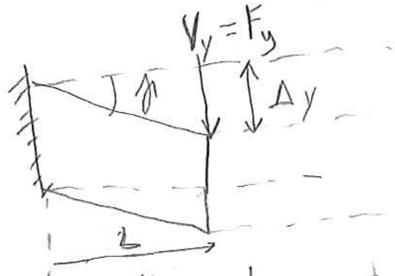
$$\boxed{\Delta L_z = \frac{FL}{EA}}$$

$$\Delta y_{shear} = \frac{F_y L_z}{GA}$$

consider a short beam



Consider a Force in y-direction



For a small element

$$\tau = \frac{dV}{dA} \Rightarrow V = \int_A \tau \, dA \Rightarrow \tau = \frac{V}{A}$$

$$\sin \gamma = \frac{\Delta y}{\Delta L} \xrightarrow{\text{small angle approx}} \gamma = \frac{\Delta y}{\Delta L} \Rightarrow \Delta y = \gamma \Delta L$$

$$\tau = G\gamma \Rightarrow \gamma = \frac{\tau}{G}$$

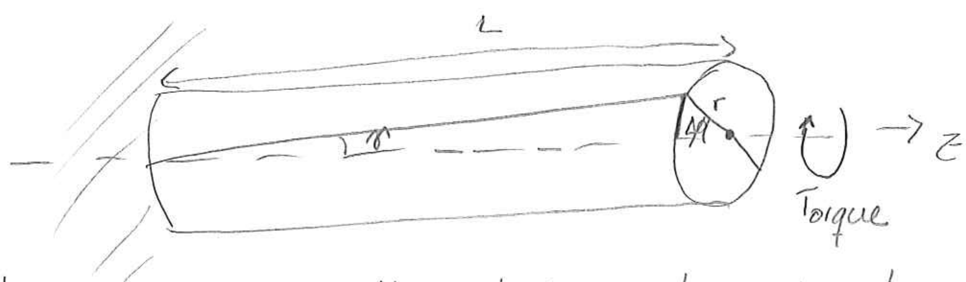
$$\Delta y = \int_0^L \frac{\tau}{G} \, dl = \frac{\tau L}{G} = \frac{VL}{GA}$$

$$\Delta y = \frac{LV}{AG}$$

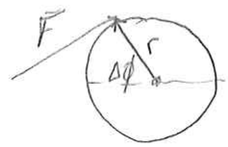
$$\Delta y = \frac{L_z F_y}{GA}$$

$$\Delta\phi_z = \frac{T_z L_z}{GJ}$$

Begin with a round shaft in Torsion



The torsion causes the shaft to twist an angle "γ" this also causes the end-cross sectional face to deflect



moment $= r \times F$

The force is a "shear force" $F = V$

$\tau = \gamma(r) G \qquad V = \tau \, dA$

$\gamma = \frac{\tau}{G}$

moment $= r \times V = r \times \tau \, dA$

For volume element with length L

$L\gamma \approx r\Delta\phi \Rightarrow \frac{L\tau}{G} = r\Delta\phi$

$$\tau = \frac{G r \Delta\phi}{L}$$

$$\text{Torque} = \int dM = \int r \times \tau(r)\, dA = \int r \left(\frac{r\Delta\phi}{L_z}\right) G \, dA = \frac{\Delta\phi G}{L_z}\int r^2 dA$$

Definition
Polar moment of inertia $J \equiv \int r^2 dA$

$$\Rightarrow T = \frac{\Delta\phi}{L_z} GJ \Rightarrow \Delta\phi = \frac{T_z L_z}{GJ}$$

$$\Delta\theta_x = \frac{M_x L_z}{E I_{xx}}$$

$$\varepsilon \cdot dL = x \cdot d\phi$$

$$\varepsilon = x \frac{d\phi}{dL}$$

$$\sigma = E\varepsilon = E x \frac{d\phi}{dL}$$

Look at a moment acting on a beam of length L

Each element contributes

$$dM = dF \cdot x = \sigma dA \cdot x$$

Total moment

$$M_{Total} = \int dM = \int \sigma dA x = \int E \frac{d\phi}{dL} x^2 dA$$

$$M_{Total} = \frac{E d\phi}{dL} \int x^2 dA$$

but

$$I_{xx} = \int x^2 dA$$

$$M_{Total} = E \frac{d\phi}{dL} I_{xx}$$

$M \rightarrow$ constant

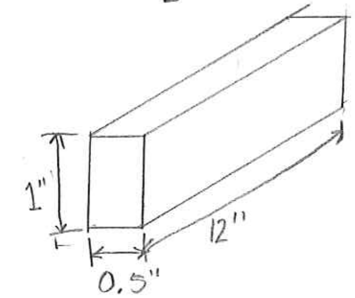
$$d\phi = \frac{dL \, I_{xx}}{E}$$

$$\Delta\phi = \frac{M}{E I_{xy}} \int dL$$

$$\Delta\phi = \frac{ML}{E I_{xy}}$$

3. Beam deflection – rectangular cross section

Beam with dimensions 0.5" wide x 1" high x 12" long, made from aluminum ($E = 10$ Msi)



a) i) cross sectional area A

$$A = (1")(0.5") = 0.5 \text{ inch}^2$$

ii) Second moments in both directions I_x and I_y

$$I = \frac{bh^3}{12}$$

$$I_y = \frac{(0.5)(1)^3}{12} = 0.042 \text{ in}^4$$

$$I_x = \frac{(1)(0.5)^3}{12} = 0.0104 \text{ in}^4$$

iii) Equivalent polar moment of inertia K

$$K = \frac{1}{3} h b^3 \left(1 - 0.58 \frac{b}{h}\right)$$

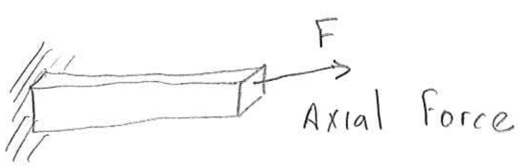
$b < h$ $\qquad b = 0.5"$, $h = 1"$

$$K = \frac{1}{3}(1)(0.5)^3 \left(1 - 0.58\left(\frac{0.5}{1}\right)\right) = 0.0296 \text{ in}^4$$

b) Angular deflection, maximum linear deflection, and the maximum material stress.

Applied load: $F = 10$ lbs or $M = 10$ in-lb

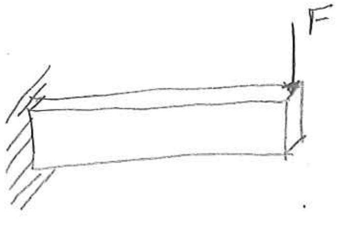
A)



$$\Delta L = \frac{FL}{EA}$$

- Angular deflection: $\delta = 0$
- Maximum linear deflection: $\Delta L = \frac{FL}{EA} = \frac{(10)(12)}{(10 \times 10^6)(0.5)} = 24 \mu in$
- Stress: $\sigma = \frac{\Delta L}{L} E = \frac{\left(\frac{F\cancel{L}}{\cancel{E}A}\right)\cancel{E}}{\cancel{L}} = \frac{F}{A} = \frac{10}{0.5} = 20 psi$

B)



In-plane

- Angular deflection

From tables
$C_1 = 3$ $\quad C_2 = 2$

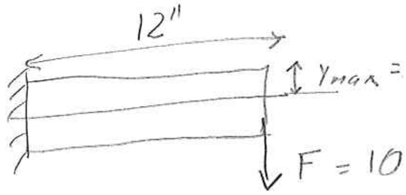
$$\delta = \frac{F l^3}{C_1 E I_y} = \frac{(10)(12)^3}{3(10 \times 10^6)(0.042)} = 0.0137 \text{ in}$$

$$\theta = \frac{F l^2}{2 E I_y} = \frac{(10)(12)^2}{2(10 \times 10^6)(0.042)} = 0.0017 \text{ rad}$$

- Maximum linear deflection

$\Delta L = 0$

- Stress:



$$\sigma = \frac{M y_{max}}{I_y}$$

$$\sigma = \frac{(12)(10)(0.5")}{0.042} = 1428.57 \text{ psi}$$

c)

Out of the plane

- Angular deflection

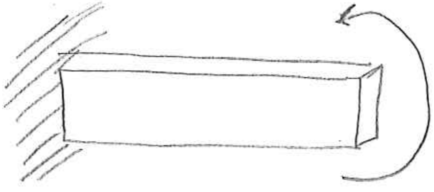
$$\delta = \frac{F l^3}{C_1 E I_x} = \frac{(10)(12)^3}{3(10 \times 10^6)(0.0104)} = 0.0553 \text{ in}$$

$$\theta = \frac{F l^2}{2 E I_x} = \frac{(10)(12)^2}{2(10 \times 10^6)(0.0104)} = 0.0069 \text{ rad}$$

- Maximum linea deflection: $\Delta L = 0$

- Stress $\sigma = \dfrac{M y_{max}}{I_x} = \dfrac{(10)(12)(0.25)}{0.0104} = 2884.6$ psi

D)



From tables $c_1 = 2$ $c_2 = 1$

- Angular deflection $M = FL \Rightarrow F = \frac{M}{L}$

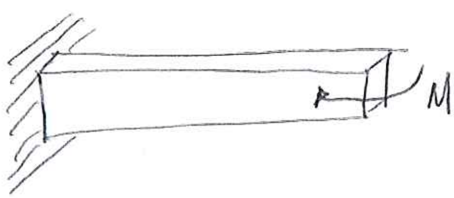
$$\delta = \frac{Ml^2}{2EI_y} = \frac{(10)(12)^2}{2(10\times10^6)(0.042)} = 0.0017 \text{ in}$$

$$\theta = \frac{Ml}{1EI_y} = \frac{(10)(12)}{(10\times10^6)(0.042)} = 0.000286 \text{ rad}$$

- Maximum Linear deflection $\Delta L = 0$

- Stress $\sigma = \frac{M\,y_{max}}{I_y} = \frac{(10)(0.5)}{(0.042)} = 119$ psi

E)



- Angular deflection

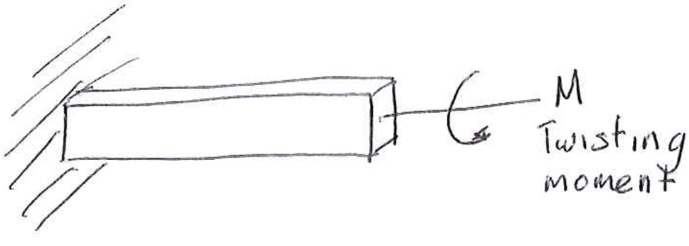
$$\delta = \frac{Ml^2}{2EI_x} = \frac{(10)(12)^2}{2(10\times10^6)(0.0104)} = 0.0069 \text{ in}$$

$$\theta = \frac{Ml}{EI_x} = \frac{(10)(12)}{(10\times10^6)(0.0104)} = 0.00115 \text{ rad}$$

- Maximum Linear deflection $\Delta L = 0$

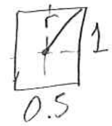
- Stress $\sigma = \frac{(10)(0.25)}{(0.0104)} = 240.38$ psi

F)



Angular deflection about its axis

$$\tau = \gamma G = \Delta\phi G = \frac{Tr}{K}$$



$r = \sqrt{0.5^2 + 0.25^2}$

$r = 0.559$

$$\Delta\phi = \frac{TL}{GJ}$$

$J = K = 0.02958 \text{ in}^4$ From (iii)

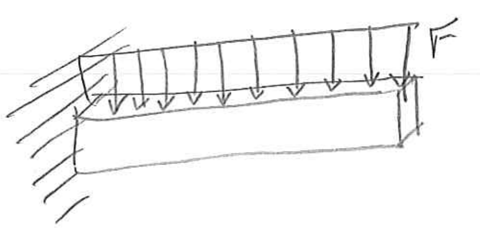
$$G = \frac{E}{2(1+\nu)} = \frac{10 \times 10^6}{2(1+0.33)} = 3.76 \text{ MSi}$$

$$\Delta\phi = \frac{(10)(12)}{(3.76 \times 10^6)(0.02958)} = 0.00107 \text{ rad}$$

$\nu_{al} = 0.33$

$$T = \frac{(10)(0.56)}{0.02958} = 189.3$$

G)



From tables

$C_1 = 8$, $C_2 = 6$

Angular deflection

$$\delta = \frac{F l^3}{C_1 E I_y} = \frac{(10)(12)^3}{8(10 \times 10^6)(0.042)} = 0.0051 \text{ in}$$

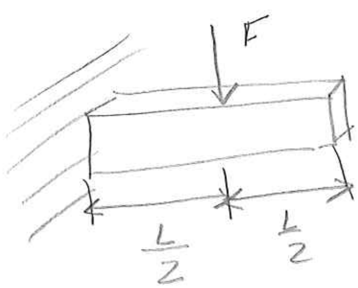
$$\theta = \frac{F l^2}{6 E I_y} = \frac{(10)(12)^2}{6(10 \times 10^6)(0.042)} = 0.00057 \text{ rad}$$

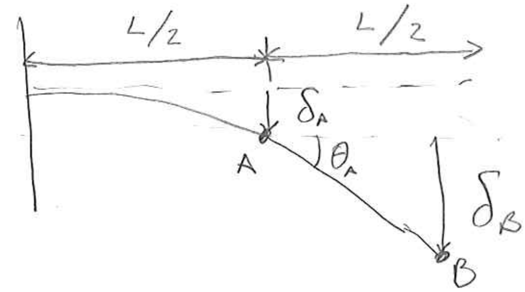
Maximum linear deflection: $\Delta L = 0$

Stress $$\sigma = \frac{F y_{max} L/2}{I_y} = \frac{(10)(0.5)(6)}{0.042} = 714.28 \text{ psi}$$

$L = 12$ $\quad F = 10\ lb$

H)





From tables $\quad C_1 = 3, \quad C_2 = 2$

Point A

$$\delta_A = -\frac{F\left(\frac{L}{2}\right)^3}{3EI} = -\frac{FL^3}{24EI} \quad , \quad \theta_A = \frac{F\left(\frac{L}{2}\right)^2}{2EI} = \frac{FL^2}{8EI}$$

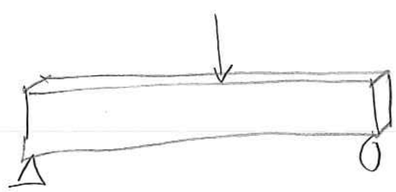
Point B

$$\delta_B = -\theta_A \cdot \left(\frac{L}{2}\right) = -\frac{FL^2}{8EI}\left(\frac{L}{2}\right) = -\frac{FL^3}{16EI}$$

Total displacement

$$\delta = \delta_A + \delta_B = -\frac{FL^3}{EI}\left(\frac{1}{24} + \frac{1}{16}\right) = -\frac{FL^3}{EI}\left(\frac{2+3}{48}\right) = -\frac{5FL^3}{48EI_y} = -0.0043\ \text{inch}$$

I)



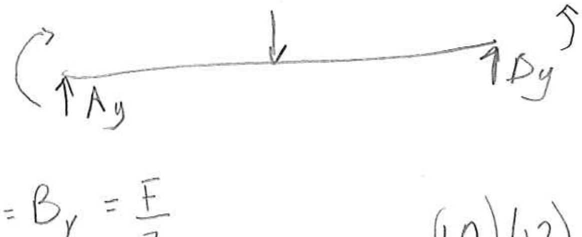
$C_1 = 48 \quad C_2 = 16$

$$\delta = \frac{Fl^3}{C_1 EI} = \frac{(10)(12)^3}{48(10\times10^6)(0.042)} = 0.00086\ \text{in}$$

$$\theta = \frac{Fl^2}{C_2 EI} = \frac{(10)(12)^2}{16(10\times10^6)(0.042)} = 0.00021\ \text{rad}$$

$$\sigma_{max} = \frac{M_{max}\, y_{max}}{I_y}$$

Free body Diagram



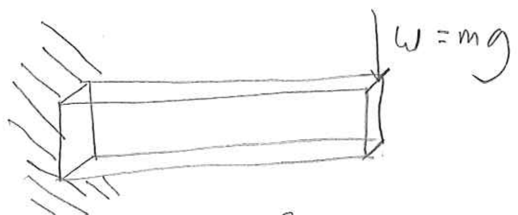
$$\Sigma F = 0 = -F + A_y + B_y \implies A_y = B_y = \frac{F}{2}$$

$$M_{max} = \frac{L}{2}\left(\frac{F}{2}\right) = \frac{FL}{4}$$

$$\implies \sigma_{max} = \frac{(10)(12)(0.5)}{4(0.042)}$$

$$\boxed{\sigma_{max} = 357.14\ \text{psi}}$$

c) 10 lb is placed at the end of the beam



$g = 386 \text{ inch}/\text{sec}^2$

$m = \frac{F}{g} = \frac{10}{386} = 0.0258 \frac{lb \cdot s^2}{in}$

Resonant frequency for 3 modes

$W = \sqrt{\frac{k}{m}}$ $\quad k = \frac{F}{\delta}$

From tables $\delta = \frac{F l^3}{3EI}$

$W_n = \sqrt{\frac{g}{\delta x}}$

$k = \frac{F}{\frac{F l^3}{3EI}} = \frac{3EI}{l^3} \Rightarrow W_n = \sqrt{\frac{3EI}{l^3 M}}$

$f_n = \frac{1}{2\pi} W$

- x-bending $\quad W_n = \sqrt{\frac{3EI_x}{l^3 M}}$

$W_n = \sqrt{\frac{3(10\times10^6)(0.0104)}{(12)^3(0.0258)}} = 83.65 \text{ rad}/\text{sec}$

$f_n = \frac{1}{2\pi} W_n = \underline{13.31 \text{ Hz}}$

- y-bending $\quad W_n = \sqrt{\frac{3EI_y}{l^3 m}}$

$W_n = \sqrt{\frac{3(10\times10^6)(0.042)}{(12)^3(0.0258)}} = 168.11 \text{ rad}/\text{sec}$

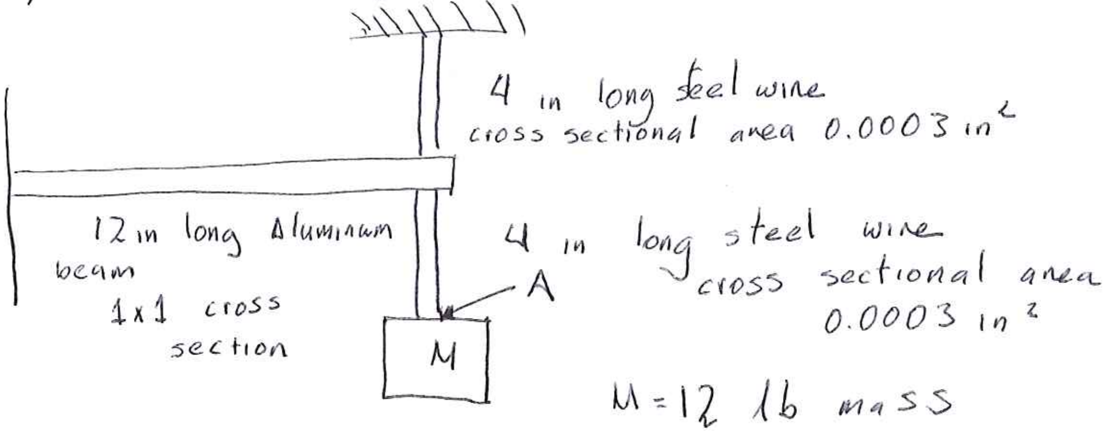
$f_n = \frac{1}{2\pi} W_n = \underline{26.76 \text{ Hz}}$

- z-bending $\quad \delta z = \frac{FL}{EA}$ $\quad k_z = \frac{F}{\delta z} = \frac{EA}{L}$

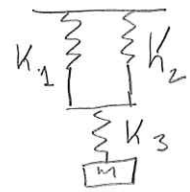
$W_n = \sqrt{\frac{EA}{Lm}} = \sqrt{\frac{(10\times10^6)(0.5)}{(12)(0.0258)}} = 4018.7 \text{ rad}/\text{sec}$

$f_n = \frac{1}{2\pi} W_n = \underline{639.59 \text{ Hz}}$

4) Stiffness, resonand Frequency



a) Stiffness



Aluminum beam

$E_{AL} = 10 \times 10^6 \ lb/in^2$

$l = 12 \ in$

$A = 1 \ in^2$

$I_y = \frac{bh^3}{12} = \frac{1}{12} \ in^4$

$\delta = \frac{Fl^3}{3EI_y}$

$$k = \frac{F}{\delta} = \frac{\cancel{F}}{\frac{\cancel{F}l^3}{3EI_y}} = \frac{3EI_y}{l^3}$$

$$k = \frac{3(10 \times 10^6)(0.0833)}{(12)^3}$$

$$\underline{k_1 = 1446.18 \ lb/in}$$

Steel wires

$E_{st} = 30 \times 10^6 \ lb/in^2$

$L = 4 \ in$

$A = 0.0003 \ in^2$

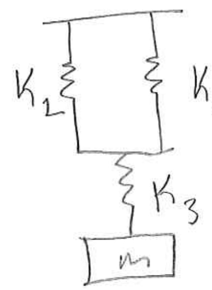
$\delta L = \frac{FL}{EA}$

$$k = \frac{F}{\delta L} = \frac{\cancel{F}}{\frac{\cancel{F}L}{EA}} = \frac{EA}{L}$$

$$k_2 = k_3 = \frac{(30 \times 10^6)(0.0003)}{4}$$

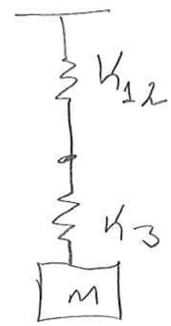
$$\underline{k_1 = k_2 = 2250 \ lb/in}$$

b)



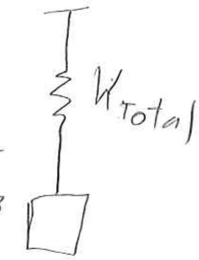
1 and 2 are in parallel $\Longrightarrow$

$K_{12} = K_1 + K_2$



K_{12} and K_3 are in series

$$\frac{1}{K_{Total}} = \frac{1}{K_{12}} + \frac{1}{K_3}$$



$$K_{12} = 1446.18 + 2250 = 3696.18 \text{ lb/in}$$

$$\frac{1}{K_{Total}} = \frac{1}{K_{12}} + \frac{1}{K_3} = \frac{1}{3696.18} + \frac{1}{2250} = 0.000715$$

$$\underline{K_{TOTAL} = 1398.6 \text{ lb/in}}$$

Total deflection

$$K = \frac{F}{\delta} \Rightarrow \delta = \frac{F}{K} = \frac{mg}{K}$$

$F = 12$ lbs

$$\delta = \frac{12}{1398.6} = 0.0086 \text{ in}$$

c) Resonant frequency

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{K}{\frac{\delta K}{g}}} = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{386}{0.0086}}$$

$\omega_n = 212.10$ rad/sec

$$\underline{F_n = \frac{1}{2\pi}\omega_n = 33.76 \text{ Hz}}$$