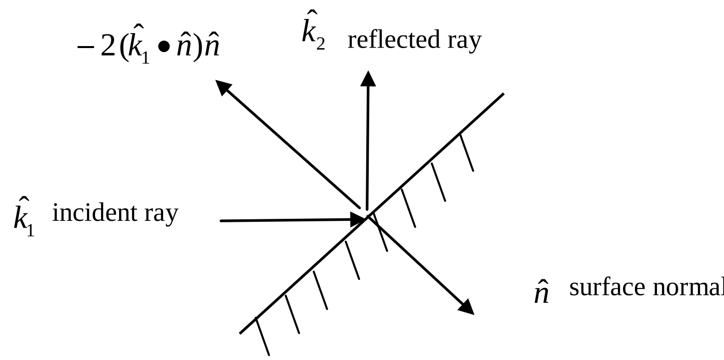
OPTI 421/521 – Introductory Optomechanical Engineering

6. Mirror matrices

Matrix formalism is used to model reflection from plane mirrors.

Start with the vector law of reflection:

$$\hat{k}_2 = \hat{k}_1 - 2(\hat{k}_1 \bullet \hat{n})\hat{n}$$



The hats indicate unit vectors
k_1 = incident ray
k_2 = reflected ray
n = surface normal

For a plane mirror with its normal vector **n** with (x,y,z) components (n_x,n_y,n_z)

Using the standard vector representation with unit vectors,

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$$

The matrix representation of this vector is

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

The vector law of reflection can be written in matrix form as

$$k_2 = M\,k_1$$

Where the mirror matrix M is calculated to be

$$M = I - 2n \cdot n^T$$

M can be expanded as

$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{pmatrix} n_x & n_y & n_z \end{pmatrix}$$

or

$$M := \begin{bmatrix} 1 - 2 \cdot n_x^2 & -2 \cdot n_x \cdot n_y & -2 \cdot n_x \cdot n_z \\ -2 \cdot n_x \cdot n_y & 1 - 2 \cdot n_y^2 & -2 \cdot n_y \cdot n_z \\ -2 \cdot n_x \cdot n_z & -2 \cdot n_y \cdot n_z & 1 - 2 \cdot n_z^2 \end{bmatrix}$$

After calculating this mirror matrix, any vector k_1 gets changed by reflection from the mirror to a new vector k_2, calculated by simple matrix multiplication

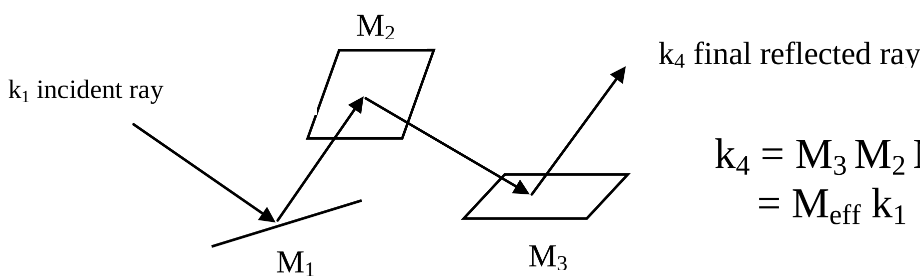
$$k_2 := M \cdot k_1$$

If the initial vector k_1 is the direction the ray incident on the mirror, then k_2 is the direction of the reflected ray.

A series of reflections is modeled by successive mirror matrix multiplications. If light bounces off mirror 1, then 2 then 3, the net effect of these three reflections is

$$k_4 := M_3 \cdot M_2 \cdot M_1 \cdot k_1$$

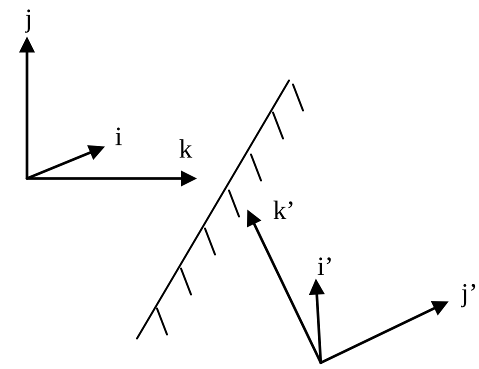
which reduces to a single effective mirror matrix $M_{eff} := M_3 \cdot M_2 \cdot M_1$



$$k_4 = M_3 M_2 M_1 k_1$$
$$= M_{eff} k_1$$

So the effect of any set of mirrors can be reduced to a single 3x3 matrix.

The mirror matrix shows the reflected coordinates, not just the incident ray. Initial coordinates (i,j,k) get reflected to a new set (i',j',k')



For example, a mirror with its normal in the z direction would be described by M_z

$$M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A set of coordinates would be reflected so that

$$x' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad z' = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

An incident ray traveling in the +z direction will be reflected to travel in the –z direction. Images of the x and y axes do not change.

Image orientation can be computed by transforming the "up" and "right" axes in object space using the mirror matrix M to find the orientation and parity in image space. Each direction of the coordinate system is transformed by the mirror matrix

Parity

The parity of this one mirror is of course odd (–1). The image of a right handed coordinate system will appear to be left handed in the reflection. This means that clockwise rotation about any basis vector will appear counter-clockwise in the image.

In general, the determinant of the mirror matrix gives the parity of the system.

- An even number of reflections will cause the image to be right-handed, or to have parity = det(M) = 1.
- A system with an odd number of reflections will cause the image to be left-handed, or to have parity = det(M) = -1.

Mirrors with any orientation can be defined using rotations. The matrix method uses well defined coordinate transformations which use simple matrix multiplications. The effect of rotating a mirror M, or system of mirrors that has equivalent matrix M is

$$M_r = R \cdot M \cdot R^T$$

where M_r is the new matrix and R is the rotation matrix given below

x rotation $$R_x := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

y rotation $$R_y := \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

z rotation $$R_z := \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(The rotation matrices have the special property that $R^T = R^{-1}$. Transpose operation, swap rows with columns: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

In many cases you can write down the mirror matrix by inspection. You can trace the x, y, and z unit vectors through the prism by reflecting the vectors one at a time using the bouncing pencil paradigm. In fact, you only need to trace two axes through and use the parity to get the third.

Use these coordinates to evaluate how object motion relates to image motion, both for translation and rotation. Remember to reverse the direction of rotation if the system has –1 parity.

To find the effect of small rotations of any prism, apply the rotation transformations to the prism matrix $\mathbf{M_p}$

$$M_r = R_x(\alpha)\, M_p\, R_x(\alpha)^T$$

This new matrix defines the new line of sight as well as any image rotation

For small angles (jitter), you can use the small angle approximation and apply a perturbation:

$$\sin\alpha \approx \alpha$$

$$\cos\alpha \approx 1$$

If you take the incident light direction as z, the last row gives z', the new direction after the perturbation. For image rotation, compare the perturbed directions of x' and y' with the unperturbed values. For example, a 45° mirror, giving 90° reflection.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

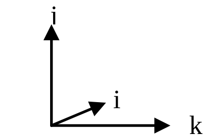
Rotate about z axis, (the incident line of sight)

$$M'(\gamma) \approx \begin{bmatrix} 1 & -\gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & \gamma & \gamma \\ \gamma & 0 & -1 \\ \gamma & -1 & 0 \end{bmatrix}$$

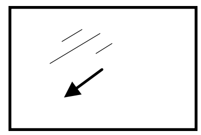
Upon rotation, the new propagation direction z' is deviated by an amount γ into the x direction. The change in the x-y plane defines image rotation. These coordinates are reflected to the x-(-z) plane. The mirror rotation rotates this coordinate frame by an amount γ about the z' (or –y) axis.

Some common types of mirrors:

Free space $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

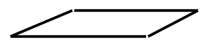


x mirror $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



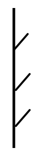
insensitive to x rotation
2θ for y and z rotations

y mirror $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



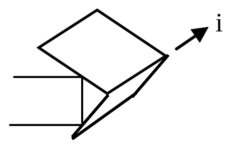
insensitive to y rotation
2θ for x and z rotations

z mirror $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



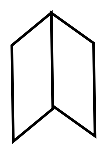
insensitive to z rotation
2θ for x and y rotations

90° x roof $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



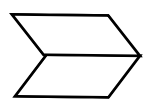
insensitive to x rotation
2θ for y and z rotations

90° y roof $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



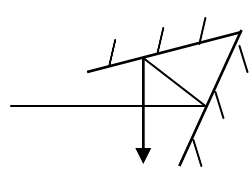
insensitive to y rotation
2θ for x and z rotations

90° z roof $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



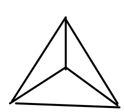
insensitive to z rotation
2θ for x and y rotations

45° x roof $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$



90° deviation
insensitive to x rotation
θ for y and z rotations

cube corner $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$



retro-reflects
insensitive to all rotations