Best Input State of Estimating Phase Shift under Covariant Measurement

Qipeng Qian[1]

[1] *Program in Applied Mathematics, The University of Arizona, Tucson, Arizona 85721, USA*

I. INTRODUCTION

Estimating phase shifts is a fundamental task in quantum information processing because phase plays a critical role in the evolution and manipulation of quantum states. Precise phase estimation enables key applications such as quantum metrology, where it is used to achieve measurements with accuracy beyond classical limits, and quantum computing, where phase shifts are essential for gate operations and algorithms like Shor's and Grover's. In quantum communication, phase information encodes data in protocols like quantum key distribution (QKD). Furthermore, estimating phase shifts underpins foundational studies of quantum coherence and interference, making it indispensable for both theoretical and practical advancements in quantum technologies.

Over the past decade, extensive research has focused on probing the ultimate quantum limits for estimating the phase shift and identifying the states that reach these limits.

For the simplest single-mode input, [1] analyzed the optimal measurement and the optimal prob state under the Bayesian framework. [1] provides an analytical expression for the average cost of an estimation strategy with general density operator of pure states and arbitrary prior distribution of the phase parameter to be estimated. A specified cost function

$$C(\phi,\tilde{\phi}) = 4\sin^2\frac{\phi-\tilde{\phi}}{2} \tag{1}$$

and the corresponding optimal measurement are used to evaluate the performance of this strategy. However, the result of [1] involves calculating the singular value decomposition (SVD), which becomes particularly challenging for matrices with unknown variables, and even performing numerical analysis becomes extremely difficult as the matrix dimension increases.

For the two-mode case and without any prior information, [2] considered a situation where laser light, described by a coherent state $|\alpha\rangle$, and an arbitrary pure state $\sum_{n=0}^{k} c_n|n\rangle$ act as the initial 2-mode input state, which is then fed into the primary input port of a balanced beam splitter (50:50). The two optical paths after the beam splitter experience phase shifts and estimated under the criteria of the quantum Fisher information. [2] analytically demonstrated that, for a fixed mean photon number, squeezed states are optimal when analyzed through quadratures (which ends up in an uncertainty relation), without delving into the detailed form of the output state.

In this report, we adopt the setup of [2] as in FIG. 1, but consider the estimation within a Bayesian framework in which we are more interested. Using the same cost function Eq. (1) and considering only covariant measurement, we try to numerically find out the optimal input pure state in the second mode. This report is organized as follows: in Section II, we introduce the basic estimation strategy of Bayesian framework; in Section III, we define the covariant measurement and explain why we only consider it; in Section IV, we state the experimental results; in Section V we state our results and how we might improve it in the future.

II. ESTIMATION WITHIN BAYESIAN FRAMEWORK

The estimation strategy involves the output state $\hat{\rho}_\phi$, a POVM measurement $\Pi(\tilde{\phi})$, a cost function $C(\phi,\tilde{\phi})$, and a prior distribution $p(\phi)$, where ϕ is the phase to be estimated and $\tilde{\phi}$ is the estimator. The average cost of the estimation strategy has the form

$$\bar{C} = Tr\left[\int_0^{2\pi} d\tilde{\phi}\Pi(\tilde{\phi})\int_0^{2\pi} d\phi p(\phi)C(\phi,\tilde{\phi})\hat{\rho}_\phi\right]. \tag{2}$$

Here we consider $p(\phi)$ to be the uniform distribution on $[0,2\pi)$. For the output state, we write $\hat{\rho}_\phi = |\psi\rangle_\phi\langle\psi|_\phi$, where $|\psi\rangle_\phi$ takes the form

$$\begin{aligned}|\psi\rangle_{\phi_1,\phi_2} \;=\; & e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m\frac{\alpha^n}{\sqrt{n!}} \\ & \Big(\sum_{k=0}^{n}\sum_{l=0}^{m} f_{n,m}(k,l)e^{-i(k+l)\phi_1-i(m+n-k-l)\phi_2} \\ & |k+l\rangle_1\otimes|n+m-k-l\rangle_2\Big)\end{aligned} \tag{3}$$

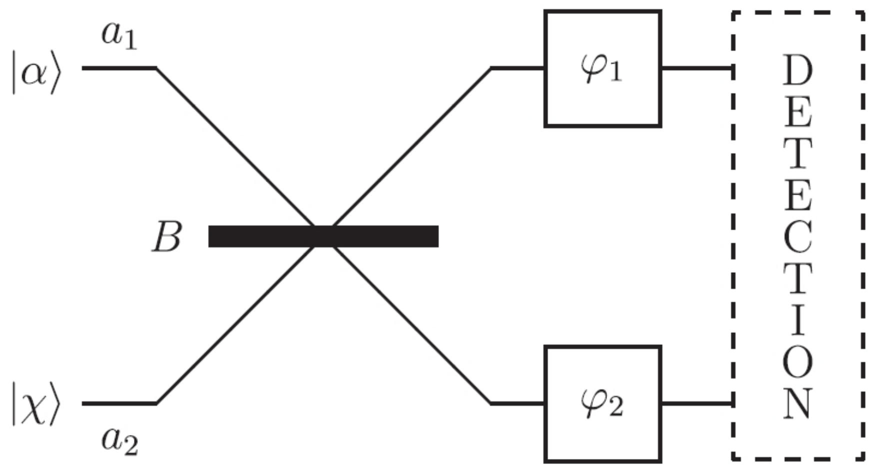

FIG. 1. The set up for measurement of phase shift.

where

$$f_{n,m}(k,l) = \frac{(-1)^{m-l}\sqrt{(k+l)!(m+n-k-l)!}\binom{n}{k}\binom{m}{l}}{2^{\frac{n+m}{2}}\sqrt{(n!)(m!)}}. \quad (4)$$

The detailed derivation is shown in Appendix A.

We can observe that $|\psi\rangle_\phi$ is infinite-dimensional and so is $\hat{\rho}_\phi$, which is hard to use in order to calculate $\bar{C}$. To overcome this case, we truncate our input states in both modes, i.e. truncate the coherent state in the first mode up to fock state $|n_{up}\rangle$ and the pure state in second mode to fock state $|m_{up}\rangle$. The final dimension of $\hat{\rho}_\phi$ is $(n_{up}+m_{up}+1)^2 \times (n_{up}+m_{up}+1)^2$.

III. COVARIANT MEASUREMENT

As previously mentioned, determining the optimal measurement using the method proposed by [1] requires calculating the SVD of a matrix constructed from the density matrix and the prior distribution. This approach is challenging in our case because the density matrix contains unknown parameters, and its high dimensionality makes numerical computation impractical in general.

To simplify the calculation, we decide to consider optimal measurement within the range of covariant measurement.

Definition 1. *Let G be a parametric group of transformations of a set Θ and $g \to V_g$ be a (continuous) projective unitary representation of G in a Hilbert space. Let $M(d\theta)$ be a measurement with values in Θ. The measurement $M(d\theta)$ is covariant with respect to representation $g \to V_g$ if*

$$V_g^\dagger M(B) V_g = M(B_{g^{-1}}), \quad (5)$$

for any B belongs to the σ-field of Borel subsets of Θ, where

$$B_g = \{\theta : \theta = g\theta', \theta' \in B\}. \quad (6)$$

The following theorem proved by [3] shows that how covariant measurement can simplify calculation in our case.

Theorem. *The covariant measurement $M(\phi)$*

$$\langle n|M(\phi)|m\rangle = \frac{1}{2\pi} e^{i(n-m)\phi} \quad (7)$$

is the optimal measurement of angle of rotation ϕ for states that can be written as

$$e^{-i\phi J}|\psi\rangle\langle\psi|e^{i\phi J}, \quad (8)$$

where J is the operator of spin angular momentum, and for any even 2π-periodic cost function.

It's clear that our cost function is an even 2π-periodic cost function, which induces the optimal measurement is the $M(\phi)$ in the above theorem. Thus, we can plug in $M(\phi)$ into the $\Pi(\phi)$ in Eq. (2) to have an analytic form of $\bar{C}$.

Finally, the goal is to find out the optimal coefficients of the truncated input pure state, i.e. $c_i = |c_i|e^{i\theta_i}$.

IV. EXPERIMENTS

In this section we describe our experiment set-up and explain the results.

First, in this report we only estimate ϕ_1 show in Fig. 1, so we set $\phi_2 = -\phi_1$. Second, for the truncation mentioned in the previous section, we set $n_{up} = 5$ and $m_{up} = 2$. We also assume that for the coherent input state, $\alpha = 0.1$, for which our truncation length n_{up} is suitable. What left is all the coefficients. Using normalization condition, we have one constraint on modules of the coefficients, i.e. $|c_i|$'s. Additionally, we introduce the mean photon number N of the input pure state to have the second constraint. So, we end up with $2(m_{up}+1)-2=4$ parameters to be optimized.

We numerically find out the minimum of $\bar{C}$ in the feasible set using grid search.

V. CONCLUSIONS

In this report, we try to find the optimal probe states for quantum-enhanced interferometry using a laser power source under the Bayesian framework. We consider only covariant measurement to make numerical experiment doable.

Appendix A: Analytical form of the output state to be estimated

In this Appendix, derived the analytical form of the output state in our set up shown in Fig. 1.

For a balanced beam splitter followed by 2 phase shift operator on each mode, the transformation of the annihilation operators for two input modes $\hat{a}_1$ and $\hat{a}_2$ is given by:

$$\hat{b}_1 = \frac{e^{i\phi_1}}{\sqrt{2}}\left(\hat{a}_1 + \hat{a}_2\right), \quad \hat{b}_2 = \frac{e^{i\phi_2}}{\sqrt{2}}\left(\hat{a}_1 - \hat{a}_2\right),$$

where $\hat{a}_1$, $\hat{a}_2$ and ϕ_1, ϕ_2 are the annihilation operators and phase shift for modes $1, 2$ respectively, and $\hat{b}_1$ and $\hat{b}_2$ are the output modes.

Also, the beam splitter applies the transformation to the creation operators:

$$\hat{b}_1^\dagger = \frac{e^{i\phi_1}}{\sqrt{2}}\left(\hat{a}_1^\dagger + \hat{a}_2^\dagger\right), \quad \hat{b}_2^\dagger = \frac{e^{i\phi_2}}{\sqrt{2}}\left(\hat{a}_1^\dagger - \hat{a}_2^\dagger\right).$$

Then, we need to understand how the beam splitter acts on the Fock state $|n\rangle_A \otimes |m\rangle_B$, which can be written as

$$|n\rangle_A \otimes |m\rangle_B = \frac{(\hat{a}_1^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_2^\dagger)^m}{\sqrt{m!}}|0\rangle_A \otimes |0\rangle_B.$$

Using the representation of $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ with $\hat{b}_1^\dagger, \hat{b}_2^\dagger$, we get the state after the beam splitter and phase shift is

$$\hat{U}_{BS}|n\rangle_A \otimes |m\rangle_B = \frac{\left(\frac{e^{-i\phi_1}\hat{b}_1^\dagger + e^{-i\phi_2}\hat{b}_2^\dagger}{\sqrt{2}}\right)^n \left(\frac{e^{-i\phi_1}\hat{b}_1^\dagger - e^{-i\phi_2}\hat{b}_2^\dagger}{\sqrt{2}}\right)^m}{\sqrt{n!m!}}|0\rangle_A \otimes |0\rangle_B.$$

Expanding the exponentials, we obtain the final output state as

$$\hat{U}_{BSPS}|n\rangle_1 \otimes |m\rangle_2 = \sum_{k=0}^{n}\sum_{l=0}^{m} f_{n,m}(k,l)e^{-i(k+l)\phi_1 - i(m+n-k-l)\phi_2}|k+l\rangle_1 \otimes |n+m-k-l\rangle_2,$$

where

$$f_{n,m}(k,l) = \frac{(-1)^{m-l}\sqrt{(k+l)!(m+n-k-l)!}\binom{n}{k}\binom{m}{l}}{2^{\frac{n+m}{2}}\sqrt{(n!)(m!)}}$$

are coefficients that depend on the binomial expansions.

For our input states, we have

$$|\psi_{\text{out}}\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sum_{m=0}^{\infty} c_m \hat{U}_{BSPS} |n\rangle_1 \otimes |m\rangle_2.$$

Thus, the output state becomes

$$|\psi_{\text{out}}\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m \frac{\alpha^n}{\sqrt{n!}} \left(\sum_{k=0}^{n} \sum_{l=0}^{m} f_{n,m}(k,l) e^{-i(k+l)\phi_1 - i(m+n-k-l)\phi_2} |k+l\rangle_1 \otimes |n+m-k-l\rangle_2 \right).$$

[1] Rafał Demkowicz-Dobrzański, "Optimal phase estimation with arbitrary a priori knowledge," Physical Review A—Atomic, Molecular, and Optical Physics **83**, 061802 (2011).

[2] Matthias D Lang and Carlton M Caves, "Optimal quantum-enhanced interferometry using a laser power source," Physical review letters **111**, 173601 (2013).

[3] Alexander S Holevo, *Probabilistic and statistical aspects of quantum theory*, Vol. 1 (Springer Science & Business Media, 2011).