# Quantum Entanglement

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Entanglement is a powerful feature of quantum mechanics. Since entanglement is a relatively unexplored concept and it is a resource, having a framework to study its non-local properties is essential to develop a deep understanding of them and techniques to use them. We present mathematical ideas used to classify entangled states using subspaces of Hilbert spaces and simple measures used to quantify entanglement. Our literature review indicates that entanglement by itself has become a subfield of quantum information theory. New developments allow us to glimpse the potential of entanglement as a driver of quantum technologies.

## I. Introduction

Quantum Mechanics is known for its counterintuitive and unconventional set of ideas that can be used to describe microscopic systems. Entanglement is one of the quantum phenomena that challenges our physical intuition and stimulates our curiosity, and it has become one of the main ingredients of quantum technologies. It is crucial to the development of robust, reliable, and powerful quantum communication systems and quantum computers. Entanglement is regarded as a quantum resource that does not have a classical counterpart. Therefore, to boost our ability to develop quantum technologies, we want to be able to generate and quantify entanglement. Most quantum states are entangled or have some entanglement. That fact is evident from a mathematical point of view. The state of a qubit can be visualized using the Bloch sphere, which is a two dimensional surface in a three dimensional space. Two Bloch spheres can be used to model separable states of two qubits. In such a case, the two Bloch spheres can be seen as 2D subspaces embedded in a 4D Hilbert space where the corresponding possible entangled states of those two qubits can be found. Relative to the full dimensionality of the 4D space, the measure of the 2D spaces ought to be zero–the subspaces can be seen as sets of points. Thus, the number of separable states is relatively small. Moreover, since the dimension of the Hilbert space associated with a collection of $n$ qubits is $2^n$, the number of entangled states increases exponentially.

In this paper, entanglement is discussed and some of the metrics used to quantify it are presented. The quantification of entanglement of a two qubit state is not regarded as a challenge, but the problem is non-trivial when a multipartite system is considered. Entanglement has multiple aspects and properties, and some of them are not well understood.

## II. More on Entanglement

Entanglement can be defined as non-local correlations of subsystems of a quantum system. Thus, it can be said entanglement is a macroscopic property of a collection of subsystems. A quantum state is said to be entangled if it cannot be expressed as a product of irreducible vectors associated with single qubits. More generally, an entangled state is a non-convex element of a Hilbert space $H$.

Mathematically speaking, entanglement is a feature of the structure of the tensor product of Hilbert spaces. Consider

$$H = H_1 \otimes \ldots \otimes H_n$$

with $n$ being the number of subsystems. An entangled state in $H$ cannot be found in $H_i$ for any i, and by definition, it cannot be decompose into vectors in a collection of $H_i$'s. Thus, entanglement is a collective property of multiple subspaces. For any subset $M$ of $I = \{1, \ldots, n\}$ with more than one element we can consider the set of all convex combinations in the corresponding subspace of $H$. The complement of such a set is the set of entangled states. Moreover, the set of convex combinations is a subspace of $H' = H_k \otimes \ldots \otimes H_{k+l}$ with $k, \ldots, k+l \in M$ and $|M| = l$. Since the number of subsets of $I$ increases as $n$ increases, the number of possible subcollections of $H_i's$ increases. The problem has been solved for $l < 4$, [9], but classifying the possible separable subspaces of $H$ for greater values of $l$ is not an easy task.

Perhaps a simpler approach is one that involves invariants and symmetries of spaces or vectors [3][4]. Vectors can be grouped using groups, which are algebraic objects used to study symmetries. Let $G_i$ be a subgroup of the group of linear transformations of $H_i$, $GL(H_i)$. Any $g \in G_i$ must preserve the local structure of $H_i$. $G = \Pi_i G_i$ can partition $H$ because equivalence classes can be defined using elements of $G$. Two vectors in $H$ are said to be equivalent if $g \in G$ can transform one into the other ($h = gh'\, h, h' \in H$). This approach reduces the number of states that must be studied because representatives of classes form a smaller vector space $\widetilde{H}$. An additional layer of abstraction is used to simplify the problem further. $\widetilde{H}$ can be partitioned using invariants to define another set of equivalence classes. $\tilde{h}, \tilde{h}' \in \widetilde{H}$ are equivalent if they have the same set of invariants. $\widetilde{H'}$ is the set of class representatives, and its cardinality is equal to the number of invariants of interest. The purpose of this classification is to study entanglement properties of groups of entangled states. Local unitary transformations do not disentangle or entangle states. Thus, $G$ groups states without changing its entanglement properties. It follows that $\widetilde{H}$ is the set of states that share those properties. $\widetilde{H'}$ is the set of groups of states in $\widetilde{H}$ that have the same invariants. This algebraic approach also requires the use of maps defined on subspaces whose tensor product is equal to $H$. For example, one of those maps is defined on $H_1 \subset H$ such that $H = H_1 \otimes H_2$ with $H_2 \subset H$. The number of possible pairs of subspaces adds complexity to this approach. Thus, a complete classification of entangled states in an arbitrary finite Hilbert space is still challenging.

The density matrix $\rho$ of a quantum state is a Hermitian operator, and it maps quantum states to other quantum states. Therefore, it is used to study entangled states [1]. Consider the vector space $V$ of Hermitian operators, which can be spanned by products of the Pauli matrices and the identity operator that act on the subsystems of the system of interest. It is possible to use invariants, defined using subspaces of $V$, to capture the strength of non-local correlations. The main idea is to decompose $\rho$ into projections on subspaces of $V$, which are spanned by subsets of the basis $B$ of $V$. For example, $V_S$ is the span of elements in $S \subset B$, which act non-trivially on a subset of parities. It can be shown that the trace of the squared projection, $\xi^2$, of $\rho$ onto $V_S$ scaled by the dimension, $d$, of $V$ does not change if local unitary transformations act on $\rho$. In other words, $L(\rho) = d\, Tr(\xi^2(\rho))$ is invariant under local unitary operators. Recall that such operators do not change the entanglement of a system. $L(\rho)$ is called the strength length of the projection of $\rho$, and in some cases can be used to identify entangled states.

Other approaches that involve symmetries and invariants have been developed. Permutations of components of a quantum state that leave the state unchanged can be used to study entangled states [7]. A different approach is to use equivalent local unitary operations on different components of a state to define symmetries of it and identify entangled states [6].

## III. Entanglement Measures

In the previous section, tools for the identification and classification of entangled states are presented. We would also like to be able to quantify the amount of entanglement in a quantum state. A system is either entangled or separable. For high dimensional quantum states, there is not a universal separability condition. Also, entanglement is a continuous quantity, systems can have different degrees of entanglement, and there are states that are maximally entangled, [11], which means their correlations are as strong as they can be. There are many entanglement measure, but none of them is general enough to measure non-local correlations in all systems. Let $\rho$ be a density matrix of a quantum state. An entanglement measure $E$ should have the following properties [8].

- $E(\rho) = 0$ iff $\rho$ describes a separable state
- $E(\rho)$ must be invariant under local unitary transformations
- $E(\rho)$ must not increase under local operations and classical communications

Some entanglement measures are limited to simple systems. For example, the Schmidt rank can be used to identify entangled states that are pure bipartite systems. It is defined by the Schmidt decomposition of a state

$$|\psi>_{AB} = \sum_i \lambda_i |i>_A\, |j>_B$$

with $|i>_A, |j>_B$ being basis elements of the subsystems and $\lambda_i$ a function of the eigenvalues of both components [5]. If the sum has more than one term, then $|\psi>_{AB}$ is entangled. Concurrence is the name of a measure that is more general than the Schmidt rank as the bipartite system of interest does not have to be pure [2]. The following is its definition.

$$C(\rho) = max\{0, 2\max_i \lambda_i - \sum_i \lambda_i\}$$

with $\lambda_i$'s being the square roots of the eigenvalues of $\rho(\sigma_{1y} \otimes \sigma_{2y})\rho * (\sigma_{1y} \otimes \sigma_{2y})$. The range of $C(\rho)$ is $[0, 1]$. The closer its value is to 1 the more entangled the corresponding state is.

The amount of entanglement that can be distilled into pure entanglement using local operations and classical communication can be measured. It is called distillable entanglement, [5], and its definition is given by

$$E_D(\rho) = sup\{r: \lim_{n\to\infty}(inf || \Lambda(\rho^{\otimes n}) - \phi^{+}{}_{2^{rn}} ||_1) = 0\}$$

with $r$ being the rate of distillation, $n$ the number of copies of the state associated with $\rho$, $\phi^{+}{}_{2^{rn}}$ the desired state, and $\Lambda$ the symbolic representation of local operations and classical communication. The objective is to obtain an approximation of the desired state using $n$ copies of a less entangled state. Therefore, $E_D$ captures how much entanglement is stored in the final state. The upper bound of $E_D$ is called logarithmic negativity, and it is a function of another entanglement measure. Such a measure is called negativity, [2][5], and it captures how entangled one of the subsystems and the rest of the system are. Thus, it can also localize entanglement. Its definition is given by

$$N = \sum_{\lambda<0} \lambda$$

with $\lambda$ being the eigenvalue of the partial transpose of one of the subsystems.

There exist families of measures. For example, the Renyi entropies form a family of entanglement measures [15]. The Von Neumann entropy is a well known member of the Renyi family [8][15]. It captures the amount of entanglement between components of a bipartite system. Its definition is given by

$$S(\rho) = Tr(\rho \, log_2 \, \rho)$$

Significantly more entanglement measures exist, so the list of functions given above can be easily extended. Entanglement of formation provides the amount of entanglement needed to construct a state [2]. Entanglement witnesses which are operators can be used to identify and classify entangled states without full descriptions of those states [14]. Another more modern technique involves using machine learning [10]. More specifically algorithms such as the maximum likelihood and deep learning have been used to quantify entanglement without full descriptions of states. Results indicate that machine learning techniques are more effective when knowledge about the measurement projectors is used to train neural networks. The number of tools used to study entanglement reflect its arcane and complex nature.

## IV. Conclusion

Although different types of entanglement are not discussed in this paper, it can be said that there is more than one type [12]. Some states have non-local correlations that are more stable than non-local correlations of other systems. There are entangled states that have long range distributions of their correlations. Therefore, it can be said that the process of studying entangled states has three components.

- **Identification:** It has to be determined whether or not a quantum system is entangled or if some of its components are.
- **Characterization:** Properties of non-local correlations such as their distributions must be described.
- **Quantification:** Effective ways to measure entanglement are needed to quantify it and use it as a resource effectively.

Quantum technologies have great potential that will be exploited as researchers and engineers explore the richness of quantum mechanics. Though it is impossible to predict how the quantum revolution will progress, we can assume that it will cause a wave of innovation and discoveries that at the moment seem to be exotic or impossible. Entanglement is already being used to build a quantum engine [13]. Thus, it might be fair to say that entanglement is going to be the superpower of quantum technologies.

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