Quantum Tomography

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Tomography is a technique that reconstructs ensembles of information from an object being measured. Unlike its classical counterpart, quantum tomography is sensitive to state interactions which poses the risk of collapsing a qubit's wave function. That alone, significantly increases the complexity of a quantum analog to tomography techniques, that are ubiquitous in fields ranging from medicine to geophysics. This paper begins with a review of classical tomography in it's most recognized form. The radon transform and the central slice theorem are introduced as mathematical techniques that assist in understanding how one aspect of classical tomography: image reconstruction, is done. A transition is made to the quantum description where the approach to tomography is mainly focused on applications to quantum information science. The field of quantum tomography is examined through the review of: a tested experimental technique known as continuous measurements and a proposed theory based technique known as shadow tomography.

Classical Tomography

Tomography is typically associated with CT(computed tomography) scans in the medical field. The technique allows for the imaging of internal structures without interference from the underlying structures that would typically distort the reconstructed image. Tomography can be done using different sources ranging from RF to microwave radiation. The technique can be summarized by the following: take several measurements of an object and estimate the desired physical quantity through piecing together of the projected image. The radon transform is one mathematical technique for performing this task.

Radon Transform

The Radon transform can be performed in multiple dimensions. For simplicity, the 2-D version will be presented. Suppose there is an object that is represented by the function f(x,y). It's radon transform would then be denoted:

$$g(\rho,\phi) = \iint f(x,y)\delta(p-(xcos(\phi)+ysin(\phi))dxdy$$

where δ is the Dirac Delta Function and $g(\rho,\phi)$ is the projected data onto the ρ,ϕ basis. $g(\rho,\phi)$ is also known as a sinogram because a radon transform performed on an object that is off the origin, produces a sinusoidal pattern.

An example of the radon transform is shown in the next column.

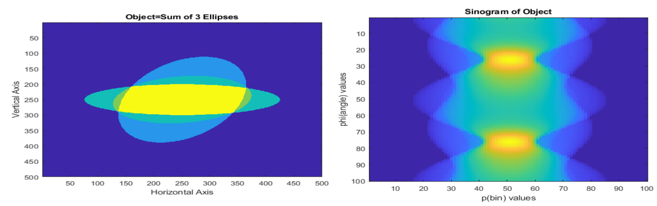

FIG. 1. The left image/object is a sum of three ellipses. The right image is the radon transform of the object. As noted, the radon transform of an off center object does in fact produce sinusoidal behavior.

Central Slice Theorem

The central slice theorem/projection slice theorem is a fundamental relationship in tomography that should at least be mentioned for a complete description of how image reconstruction works. The theorem states that the below two relationships are equivalent:

1. Taking the Radon transform of a two dimensional object and following that with the Fourier transform of the sinogram.
2. Taking a two dimensional Fourier transform and then taking the Radon transform.

The 2-D Fourier transform of the sinogram is:

$$G(\xi,\phi) = \int dpe^{-2\pi ip\xi} \int drf((r))\delta(p-\hat{n}\cdot r)$$

which shows that:

$$G(\xi,\phi) = F(\rho)|_{\rho=\xi\hat{n}}$$

Thus, the central slice theorem gives insight into the fact that by taking the 1-D Fourier transform of a sinogram at all angles, the 2-D Fourier Transform of the object will be complete, which means that the image can be recovered by taking a simple 2-D inverse Fourier transform.

Quantum Tomography(QT)

Due to the destructive nature of quantum measurements, any physical interaction with a qubit has the potential of altering the state in an undesirable way. This caveat of nature makes quantum tomographic methods very desirable in the field of quantum information science. Unlike its classical counterpart, quantum tomography deals with matrix representations of states, denoted ρ. ρ is a density matrix of DxD dimensional, mixed states. The goal of quantum tomographic methods is to find a way to give a classical approximation of the quantum state. Thus, it differs from the imaging methods in the sense that a quantum state is not being reproduced, but rather evaluated in such a way that the probability amplitude is maximized. The following sections will look at two proposed estimation schemes in quantum tomography through the lens of: an experimental set-up and a theoretically proposed methodology.

Experimental Quantum Tomography

A proposed method of ensuring state preservation is the use of multiple copies of an identical qubit. This however, drastically increases the size of the measurement system being observed. As the system size increases, the efficiency of the tomography method will decrease due to the cost of time being expended and the robustness of the states itself. One way to counteract this issue is the use of weak continuous measurements as proposed by Andrew Silberfard[1], for state estimation. The rest of this section will solely be focused on reviewing experimental results of this technique on pure states of cesium atoms. The below is an example of one such experimental set-up that utilizes continuous weak measurements.

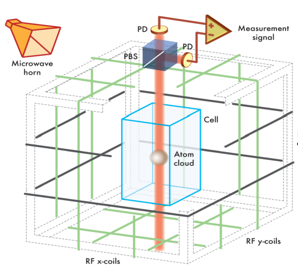

FIG. 2. Cesium atoms in plexiglass are being probed with radio frequency(RF) B-fields to measure the spin at the f=3 hyperfine state.[6]

Measurement Strategies

Innate to tomography is the potential for data corruption due to measurement errors. To minimize the presence of errors, the method of taking a measurement must be considered. For tomography with weak continuous measurements, positive operator valued measures(POVMs) are considered. As covered in a study by Sosa-Martinez et al.[Source], a comparison of eight POVMs ranging from SIC(symmetric informationally complete) to 4PB(4 polynomial bases). Utilizing maximum likelihood estimations as the data processing technique, it was identified that there are several trade-offs between accuracy and efficiency when it came to the various measurement methods within POVMs. Where one informationally complete class might perform better than the other in a specific area like a fully IC having more accurate results than the R1S-IC POVM, another IC-class might outperform the more accurate measure in terms of efficiency. The most important thing to note is that this study was performed on known, nearly pure states of cesium atoms which makes the likelihood of a high fidelity, much stronger. This shows that quantum tomography methods do face an in-built challenge of needing to account for trade-offs, especially since noise and error are ubiquitous in nature.

Cases of Incomplete Data

Bearing in mind that noise infiltration or cases of incomplete data are possible, it is crucial that the data reconstruction algorithms are able to be robust in cases where information needs to be estimated. Two such algorithms being considered are: the least squares method, and compressed sensing.

The least squares approach will give a state estimation of:

$$\hat{\rho} = argmin_p \sum_i [M_i - KTr(\rho O_i)]^2$$

subject to: $Tr(\rho) = 1, \rho^\dagger = \rho, \rho \geq 0$.

The compressed sensing algorithm gives a state estimation of:

$$\hat{\rho} = argmin_p Tr(\rho)$$

subject to: $\sum_i [M_i - KTr(\rho O_i)]^2 \leq \epsilon, \rho^\dagger = \rho, \rho \geq 0, Tr(\hat{\rho}) = 1$.

The two algorithms showed similar performances in state fidelity with compressed sensing being more robust to added error. More work is being done in this area as algorithm development and experimental set-up refinement are still continuously being improved.

Theoretical Quantum Tomography

Inasmuch as the preceding subsection is rooted in tried and tested/testable methods, theoretical quantum tomography deals with mathematical and algorithmic schemes that are still being explored and attempted to be implemented. This subsection deals not only with the reconstruction of a quantum state, but more importantly, the estimation of a state's behavior given certain expectation values. To preface the scheme of shadow tomography and its natural outcome of classical shadows, the concept of a gentle measurement must first be explored.

Gentle Measurement

In papers by Winter and Wilde[9,11], a proposition of the following scheme is proposed: if a measurement of a density matrix of mixed states(ρ) has a probability of $P \geq 1-\epsilon$, then another measured state ρ' will be within a trace distance of $\sqrt{\epsilon}$. To simplify, if a mixed state is measured to be in a certain eigenstate with a relatively large probability, then a measurement of the same state following the first, will yield something essentially the same if the second measured state is within a certain distance of that basis vector. The concept is even extended to the bound that taking M measurements will still allow for a probability of falling within the first eigenstate of $P \geq 1-2M\sqrt{\epsilon}$. This bound is often coupled with the idea of amplification where measuring x copies of the original state ρ will only result in an error that falls off exponentially(e^{-x}) since the probability bound gives a high likelihood of obtaining the eigenstate measurement. As optimistic as the gentle measurement scheme seems, the caveat is that the algorithm is a very specific version of shadow tomography where the expectation value of the density state must fall outside of a value $a \geq c$ and $a \leq c-\epsilon$. Thus, some work must be done to meet the requirements for a proper shadow tomography scheme.

Shadow Tomography

Shadow tomography is a scheme slightly different from what was presented in the experimental subsection. Where tomographic measurements are usually thought to have the goal of reconstructing a specific piece of information, shadow tomography instead focuses on the expectation value of a state. To be precise, given an unknown D dimensional state(ρ), if at least two dynamic measurement outcomes are provided, then shadow tomography will attempt to estimate the overall behavior of the mixed state within a certain error(ϵ) that has a $P \geq \frac{2}{3}$. Not every state of the density matrix needs to be known, the scheme is more focused on the overall behavior of the state like whether a certain qubit will pass through a successive number of gate operations. It was originally proposed by S. Aaronson[10] that the number of copies of ρ would need to scale as $O(\frac{log(M)^4 log(D)}{\epsilon^2})$ in order to achieve this goal. It must be stated that at this time, the shadow tomography method has not incorporated a gentle measurement advantage yet. But, the scaling would later be improved on to approximately O(log(M)). The computational complexity(k) of shadow tomography after incorporating the quantum analogue of a differential privacy technique known as private multiplicative weight, led k to scale as $O(\frac{log(M)^2 log(D)^2}{\epsilon^8})$. The addition of QPWM(Quantum Private Multiplicative Weights) also proved to be synonymous with an incorporation of gentle measurements. It should be noted that though the scheme looks hopeful in reaching the end goal of shadow tomography, there are existing concerns for how this can be implemented experimentally and what the exact computational cost of performing such a technique is.

Summary

Tomography is technique in which information is reconstructed given certain measurements on an object of interest. Classical tomography schemes like CT scans in the medical field deal with image reconstruction. These types of tomography techniques take for granted the fact that the objects they are measuring do not suffer from the same level of sensitivity that a quantum state does. Due to the destructive effect of a measurement, the quantum analogue of tomography as focused on in this paper ends up being a tool tailored for quantum information science. The techniques of quantum tomography are still being researched, but two ideas are looked at here: an experimental method of using weak continuous measurements and a theoretical scheme of shadow tomography. The experimental tests of the weak continuous measurement scheme show that quantum tomography has its challenges and trade-offs. The way in which a measurement is taken and the types of algorithms implemented to reconstruct a quantum state must be carefully considered in order to maximize results. More work with this tomographic method is being expanded on. In shadow tomography, a promising outlook for the number of measurements needed to implement the algorithm is shown, but the computational complexity has not yet been fully accounted for. Shadow tomography looks to be hopeful method of reconstructing quantum states, but more research needs to be done into making the scheme practical. A better sample complexity, efficient computational time, and convenient measurement scheme at an experimental level must be worked towards in order to utilize such a proposition. In all, quantum tomography like other quantum technology fields is still a work in progress. There are many wins along the way and equally many challenges yet to be overcome.

Citations

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