Solution Set 1

Problem I

Note: The notation used here is somewhat pedantic, but with a bit of care we can keep track. For clarity let's put hats on the operators and reserve the script $\mathcal{P}$'s for probabilities.

(a) The observables $\hat{X}(1)$ and $\hat{Y}(2)$ are projectors onto the states $|x_1\rangle$ and $|y_2\rangle$, respectively. We can guess and confirm by inspection ($\hat{X}(1)|0_1\rangle = |1_1\rangle\langle 1_1||0_1\rangle = 0|0_1\rangle$ etc.) that

Observables	Eigenvalues	Eigenvectors	Projectors
$\hat{X}(1)$	0	$\|0_1\rangle$	$\hat{P}_{x=0}(1) = \|0_1\rangle\langle 0_1\|$
	1	$\|1_1\rangle$	$\hat{P}_{x=1}(1) = \|1_1\rangle\langle 1_1\|$
$\hat{Y}(2)$	0	$\|0_2\rangle$	$\hat{P}_{y=0}(2) = \|0_2\rangle\langle 0_2\|$
	1	$\|1_2\rangle$	$\hat{P}_{y=1}(2) = \|1_2\rangle\langle 1_2\|$

(b) Product State $\quad |\Psi\rangle = (a_0|0_1\rangle + a_1|1_1\rangle)\,(b_0|0_2\rangle + b_1|1_2\rangle)$

Measure $\hat{X}(1)$

outcome 0 : $\quad \mathcal{P}(x{=}0) = \langle\Psi|\,\hat{P}_{x=1}(1)\otimes\hat{I}(2)|\Psi\rangle = |a_0|^2$

outcome 1 : $\quad \mathcal{P}(x{=}1) = \langle\Psi|\,\hat{P}_{x=1}(1)\otimes\hat{I}(2)|\Psi\rangle = |a_1|^2$

Measure $\hat{Y}(2)$

outcome 0 : $\quad \mathcal{P}(y{=}0) = \langle\Psi|\,\hat{I}(1)\otimes\hat{P}_{y=0}(2)|\Psi\rangle = |b_0|^2$

outcome 1 : $\quad \mathcal{P}(y{=}1) = \langle\Psi|\,\hat{I}(1)\otimes\hat{P}_{y=1}(2)|\Psi\rangle = |b_1|^2$

Entangled State $\quad |\chi\rangle = \alpha|0_1, 0_2\rangle + \beta|1_1, 1_2\rangle$

Measure $\hat{X}(1)$

outcome 0 : $\quad \mathcal{P}(x{=}0) = \langle\chi|\,\hat{P}_{x=0}(1)\otimes\hat{I}(2)|\chi\rangle = |\alpha|^2$

outcome 1 : $\quad \mathcal{P}(x{=}1) = 1 - \mathcal{P}(x{=}0) = |\beta|^2$

Measure $\hat{Y}(2)$

outcome 0 : $\quad \mathcal{P}(y{=}0) = \langle\chi|\,\hat{I}(1)\otimes\hat{P}_{y=0}(2)|\chi\rangle = |\alpha|^2,$

outcome 1 : $\quad \mathcal{P}(y{=}1) = 1 - \mathcal{P}(y{=}0) = |\beta|^2$

(c) As in part (a), we can guess and confirm the eigenvalues and eigenstates of $\hat{\mathrm{C}}$. E. g

Observable	Eigenvalues	Eigenvectors	Projectors
$\hat{\mathrm{C}}$	1	$\lvert 0_1 0_2\rangle$, $\lvert 1_1 1_2\rangle$	$\hat{\mathrm{P}}_{C=1}(1) = \lvert 0_1 0_2\rangle\langle 0_1 0_2\rvert + \lvert 1_1 1_2\rangle\langle 1_1 1_2\rvert$
	0	$\lvert 0_1 1_2\rangle$, $\lvert 1_1 0_2\rangle$	$\hat{\mathrm{P}}_{C=0}(1) = \lvert 0_1 1_2\rangle\langle 0_1 1_2\rvert + \lvert 1_1 0_2\rangle\langle 1_1 0_2\rvert$

Note that the outcomes indicate whether the qubit states are correlated ($C=1$) or anti-correlated ($C=0$).

(d) Case $\lvert\Psi\rangle$. We write the density operator $\hat{\rho}$ out explicitly:

$$\hat{\rho} = (a_0\lvert 0_1\rangle + a_1\lvert 1_1\rangle)\,(a_0^*\langle 0_1\rvert + a_1^*\langle 1_1\rvert)\,(b_0\lvert 0_2\rangle + b_1\lvert 1_2\rangle)\,(b_0^*\langle 0_2\rvert + b_1^*\langle 1_2\rvert)$$

Next we use

$$\begin{aligned}\hat{\rho}(1) &= \mathrm{Tr}_1[\hat{\rho}] = \textstyle\sum_{y=0,1}\langle y_2\rvert\hat{\rho}\lvert y_1\rangle \\ &= (a_0\lvert 0_1\rangle + a_1\lvert 1_1\rangle)\,(a_0^*\langle 0_1\rvert + a_1^*\langle 1_1\rvert)\textstyle\sum_{y=0,1}\underbrace{\langle y_2\rvert(b_0\lvert 0_2\rangle + b_1\lvert 1_2\rangle)\,(b_0^*\langle 0_2\rvert + b_1^*\langle 1_2\rvert)\lvert y_1\rangle}_{=\lvert b_0\rvert^2 + \lvert b_1\rvert^2 = 1} \\ &= (a_0\lvert 0_1\rangle + a_1\lvert 1_1\rangle)\,(a_0^*\langle 0_1\rvert + a_1^*\langle 1_1\rvert)\end{aligned}$$

Thus we get $\quad \hat{\rho} = \begin{bmatrix} \lvert a_0\rvert^2 & a_0 a_1^* \\ a_0^* a_1 & \lvert a_1\rvert^2 \end{bmatrix}$

Case $\lvert\chi\rangle$. Again, we write $\hat{\rho}$ out explicitly: $\quad \hat{\rho} = (\alpha\lvert 0_1 0_2\rangle + \beta\lvert 1_1 1_1\rangle)\,(\alpha^*\langle 0_1 0_2\rvert + \beta^*\langle 1_1 1_2\rvert)$

Thus, $\quad \hat{\rho}(1) = \sum_{y=0,1}\langle y_2\rvert\hat{\rho}\lvert y_2\rangle = \lvert\alpha\rvert^2\lvert 0_1 0_2\rangle\langle 0_1 0_2\rvert + \lvert\beta\rvert^2\lvert 1_1 1_1\rangle\langle 1_1 1_2\rvert = \begin{bmatrix} \lvert\alpha\rvert^2 & 0 \\ 0 & \lvert\beta\rvert^2 \end{bmatrix}$

Finally, we return to the probability of measurement outcomes $\mathcal{P}(x=0,1)$ and compute these using the reduced density operators.

Case $\lvert\Psi\rangle$

$$\mathcal{P}(x=0) = \mathrm{Tr}[\hat{\rho}(1)\,\hat{\mathrm{P}}_{x=0}(1)] = \mathrm{Tr}\left[\begin{pmatrix} \lvert a_0\rvert^2 & a_0 a_1^* \\ a_0^* a_1 & \lvert a_1\rvert^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \lvert a_0\rvert^2$$

$$\mathcal{P}(x=1) = \mathrm{Tr}[\hat{\rho}(1)\,\hat{\mathrm{P}}_{x=1}(1)] = \mathrm{Tr}\left[\begin{pmatrix} \lvert a_0\rvert^2 & a_0 a_1^* \\ a_0^* a_1 & \lvert a_1\rvert^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \lvert a_1\rvert^2$$

Case $\lvert\chi\rangle$ $\quad$ Same approach $\Rightarrow$ $\mathcal{P}(x=0) = \lvert\alpha\rvert^2$, $\mathcal{P}(x=1) = \lvert\beta\rvert^2$.

Problem II

(a) The preamble to part II(a) is mainly an excuse to introduce the basis states $|j,m_j\rangle$ for the two-spin system, where j is the quantum number for the total angular momentum, and m_j is quantum number for its projection onto the quantization axis. Note in particular the existence of the so-called singlet state, $|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle - |-\frac{1}{2},\frac{1}{2}\rangle\right)$, which has zero total angular momentum and zero projection onto the quantization axis. This particular entangled 2-spin state is invariant with respect to all spatial rotations and therefore essential for the Bell paradox.

The Problem that is actually asked is straightforward: We are given four states and their probability of occurrence in the ensemble. By definition,

$$\hat{\rho} = 0.4 \times |j=1, m_j=1\rangle\langle j=1, m_j=1| + 0.3 \times |j=1, m_j=0\rangle\langle j=1, m_j=0|$$
$$+ 0.2 \times |j=1, m_j=-1\rangle\langle j=1, m_j=1| + 0.1 \times |j=0, m_j=0\rangle\langle j=0, m_j=0|$$

in the $|j,m_j\rangle$ representation. This is clearly of the form $\hat{\rho} = \Sigma_k\, p_k\, |\psi_k\rangle\langle\psi_k|$, where more than one of the probabilities p_k are non-zero. That means we have a mixed state.

Problem III

(a) $\hat{\rho}$ is Hermitian and therefore has real-valued eigenvalues π_i and associated eigenvectors $|\chi_i\rangle$ that form an orthonormal basis in state space. That means we can write $\hat{\rho}$ on the form

$$\hat{\rho} = \sum_i \pi_i |\chi_i\rangle\langle\chi_i|$$

Also, $$\hat{\rho}^2 = \sum_{i,j} \pi_i |\chi_i\rangle\langle\chi_i|\pi_j|\chi_j\rangle\langle\chi_j| = \sum_i \pi_i^2 |\chi_i\rangle\langle\chi_i|$$

(b) **(b-1)** **(b-2)**

$$\hat{\rho} = \begin{bmatrix} \pi_1 & & & 0 \\ & \pi_2 & & \\ & & \ddots & \\ 0 & & & \pi_n \end{bmatrix}, \qquad \hat{\rho}^2 = \begin{bmatrix} \pi_1^2 & & & 0 \\ & \pi_2^2 & & \\ & & \ddots & \\ 0 & & & \pi_n^2 \end{bmatrix}$$

In the pure case $\hat{\rho} = |\psi\rangle\langle\psi|$, so $|\psi\rangle$ is an eigenvector of $\hat{\rho}$ with eigenvalue 1. Choose $|\chi_1\rangle = |\psi\rangle$. The remaining $|\chi_i\rangle \perp |\chi_1\rangle$ span the subspace associated with the $n-1$ fold degenerate eigenvalue 0. Then $\hat{\rho}$ has the form

(b-3)

$$\hat{\rho} = \begin{bmatrix} 1 & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0 \end{bmatrix}$$

We see immediately that a density matrix of the form (b-3) has $\mathrm{Tr}[\hat{\rho}] = \mathrm{Tr}[\hat{\rho}^2] = 1$.

However, a mixed density matrix of the form (b-1) has $\mathrm{Tr}[\hat{\rho}^2] = \sum_i \pi_i^2 \leq \left(\sum_i \pi_i\right)^2 = 1$

where we have used the Cauchi-Schwartz inequality and the "=" holds when one of the π_i is non-zero and the remaining $\pi_j = 0$.

(c) We have

$$\frac{d}{dt}\mathrm{Tr}[\hat{\rho}^2] = \mathrm{Tr}[\frac{d}{dt}\hat{\rho}^2] = \mathrm{Tr}[(\frac{d}{dt}\hat{\rho})\hat{\rho} + \hat{\rho}(\frac{d}{dt}\hat{\rho})] = \mathrm{Tr}[\frac{1}{i\hbar}[\hat{H},\hat{\rho}]\hat{\rho} + \frac{1}{i\hbar}\hat{\rho}[\hat{H},\hat{\rho}]]$$

$$= \mathrm{Tr}[\frac{1}{i\hbar}[\hat{H},\hat{\rho}]\hat{\rho} + \frac{1}{i\hbar}\hat{\rho}[\hat{H},\hat{\rho}]] = \frac{1}{i\hbar}\mathrm{Tr}[\hat{H}\hat{\rho}\hat{\rho} - \hat{\rho}\hat{\rho}\hat{H}] = \frac{1}{i\hbar}\mathrm{Tr}[\sum_{ijk} H_{ij}\rho_{jk}\rho_{ki} - \rho_{ij}\rho_{jk}H_{ki}]$$

Now, since the trace is basis independent, we can at all times calculate it in the basis where $\hat{\rho}$ is diagonal. In that case the only non-zero parts are those for which $i=j=k$, and

$$\frac{d}{dt}\mathrm{Tr}[\hat{\rho}^2] = \frac{1}{i\hbar}\mathrm{Tr}[\sum_i \hat{H}_{ii}\hat{\rho}_{ii}\hat{\rho}_{ii} - \hat{\rho}_{ii}\hat{\rho}_{ii}\hat{H}_{ii}] = 0$$

Thus $\mathrm{Tr}[\hat{\rho}^2]$ is constant in time, and a pure state cannot evolve into a mixed state and vice versa.

(d) Von-Neumann entropy $\quad S = -k_B \mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$

In basis $|\chi_i\rangle$ we have $\quad S = -k_B \mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -k_B \sum_i \pi_i \ln \pi_i$

Pure state:

In an n dimensional space, a pure state can be written as $\hat{\rho} = \lim_{\varepsilon\to 0} \begin{bmatrix} 1-(n-1)\varepsilon & & & 0 \\ & & \varepsilon & \\ & & & \ddots \\ 0 & & & \varepsilon \end{bmatrix}$

and thus $$-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -\lim_{\varepsilon\to 0}\left[(1-(n-1)\varepsilon)\ \ln(1-(n-1)\varepsilon) + \sum_{i=2}^{n}\varepsilon\ln\varepsilon\right] = 0$$

where in the last step we used $\lim_{\varepsilon\to 0}[\varepsilon \ln \varepsilon] = 0$. Thus pure states always have zero entropy.

Mixed state: We have

$$-\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -\lim_{\varepsilon\to 0}\left[\sum_{i=1}^{j}(\pi_i-\varepsilon)\ln(\pi_i-\varepsilon) + \sum_{j+1}^{n}\varepsilon\ln\varepsilon\right] = -\sum_{i=1}^{j}\pi_i \ln \pi_i \geq 0$$

It follows that for mixed states $S = -k_B \mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) > 0$

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Zhou	Yes
Condos	Yes
Tooley	Yes
Lopez	Yes
McCauley	Yes
O'Brien	Yes
Pacenti	Yes
Vats	No