Quantum Information Theory (Preskill Ch. 5)

Shannon's Noisy Channel Coding Theorem

Alice $\xrightarrow{\text{noisy channel}}$ Bob

$X = \{x, p(x)\}$ errors specified by $p(y|x)$ $Y = \{y, p(y)\}$

Alice & Bob need redundancy to communicate reliably over a noisy channel. **How much?**

Key Question: Can we always find a reliable code when the message length $n \to \infty$?

Binary alphabet: $\{0, 1\}$

$p(\text{no flip}) = 1 - p$

$p(\text{flip}) = p$

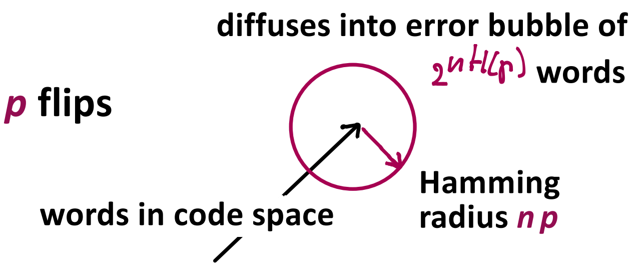
$\Rightarrow \begin{cases} p(0|0) = p(1|1) = 1 - p \\ p(0|1) = p(1|0) = p \end{cases}$

Basic Idea: Encode k bits of info in block of size n

We define the Code Rate $R = k/n$

Optimal Code: max # of bits must flip to interchange code words

n bits $\rightarrow$ np flips



Reliable Decoding: Error bubbles must not overlap

$\Rightarrow \underbrace{2^k \, 2^{nH(p)}}_{\text{\# of words required}} \leq 2^n \leftarrow$ total # of words in code space

Setting $k = nR$ and solving for the Code Rate we get

$$\boxed{R \leq 1 - H(p) \equiv C(p)} \leftarrow \text{Channel Capacity}$$

Quantum Information Theory (Preskill Ch. 5)

Quantum Information Theory:

Key Results from Classical Information Theory

* Message of letters drawn from ensemble $\{x, p(x)\}$

  Shannon Info $H(X) =$ # of incompressible bits per letter in limit $n \to \infty$

* Correlation between sent (X) and received (Y) messages

  Mutual Info
  $$I(X,Y) = H(X) - H(X|Y)$$
  $$= H(Y) - H(Y|X)$$

  # of bits of info about (X) learned from (Y)

Quantum Information Theory

➡ Need to generalize these concepts

Basic Scenario:

Alice sends letters drawn from the ensemble

$$\{\rho_x, p_x\} \Rightarrow \rho = \sum_x p_x \rho_x$$

Bob reads message by measuring the POVM

$$\{F_a\} \Rightarrow P(a) = \mathrm{Tr}(F_a \rho)$$

We define the von Neumann Entropy

$$\boxed{S(\rho) = -\mathrm{Tr}(\rho \log \rho)}$$

In the eigenbasis of ρ we have

$$\rho = \sum_\lambda \lambda |\lambda\rangle\langle\lambda| \Rightarrow$$
$$S(\rho) = -\sum_\lambda \lambda \log \lambda \, |\lambda\rangle\langle\lambda| = H(\Lambda)$$

Shannon Entropy of the ensemble $\Lambda = \{|\lambda\rangle, \lambda\}$

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Conclusion:

* If the alphabet consists of mutually orthogonal pure states then the quantum source reduces to a classical source
* In that case all signal states are perfectly distinguishable
* $S(\rho) = H(\Lambda)$

We can show the Von Neumann entropy quantifies

* The incompressible information content of a quantum source
* The quantum information content per quantum letter
* The classical information content per quantum letter (extractable by POVM)
* Entanglement of a bipartite pure state

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Mathematical properties of $S(\rho)$ (No proofs, see Preskill)

(1) **Purity** $\rho = |\psi\rangle\langle\psi| \rightarrow S(\rho) = 0$

(2) **Invariance** $S(U\rho U^{+}) = S(\rho)$

(3) **Maximum** ρ has d eigenvalues $\neq 0 \rightarrow S(\rho) \leq \log d$

(4) **Concavity** For $\lambda_1, \lambda_2, \dots \lambda_n \geq 0$, $\sum_i \lambda_i = 1 \rightarrow$

$$S(\lambda_1\rho_1 + \dots + \lambda_n\rho_n) \geq \lambda_1 S(\rho_1) + \dots \lambda_n S(\rho_n)$$

(vNE grows when ignorant of how the state was prepared)

(6) **Entropy of Measurement** (Q Meas adds randomness)

Measure $A = \sum_y a_y |a_y\rangle\langle a_y| \rightarrow$ outcomes $Y = \{a_y, p(a_y)\}$

S. E. of outcomes $H(Y) \geq S(\rho)$, w/ "=" for $[A, \rho] = 0$

(7) **Entropy of Preparation** (Mix N. O. states cannot recover full info)

Draw from $\{|\varphi_x\rangle, p_x\}$, $\rho = \sum_x p_x |\varphi_x\rangle\langle\varphi_x| \rightarrow H(X) \geq S(\rho)$

Quantum Information Theory (Preskill Ch. 5)

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Mathematical properties of $S(\rho)$ (No proofs, see Preskill)

(8) Subadditivity (info in whole ≤ sum of info in parts)

$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$ (classical $H(X,Y) \leq H(X) + H(Y)$ with "=" when X, Y uncorrelated)

(9) Triangle inequality (uncertainty about whole can be less than uncertainty about parts)

$S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|$ (classical $H(X,Y) \geq H(X), H(Y)$)

Quantum Information Theory (Preskill Ch. 5)

Quantum Data Compression

(Quantum analog of Shannons Noiseless Coding Theorem)

Starting Point: n – letter message drawn from $\{|\varphi_x\rangle, p_x\}$

need not be orthogonal

Each letter described by $\rho = \sum_x p_x |\varphi_x\rangle\langle\varphi_x|$

Message described by $\rho^n = \rho \otimes \rho \otimes \ldots \otimes \rho$

Basic Question: How redundant is this information?

– is there a "quantum code" which can compress to a smaller Hilbert space while retaining the fidelity of the encoded quantum information ?

Answer: Optimal Compression requires

$\log(\dim \mathcal{H}) = n S(\rho)$ qubits

(Schumacher's Theorem)

Corrollary: The von Neumann entropy is the # of qubits carried per letter in a message. We can always compress unless $\rho = \frac{1}{2}\mathbb{1}$

Example of how we might do this:

Alice sends a message using the alphabet $|\uparrow_z\rangle,\ p = 1/2$; $|\uparrow_x\rangle,\ p = 1/2$

$$\Rightarrow \quad \rho = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

(z or x basis)

Symmetry → eigenvectors are $\uparrow, \downarrow$ along $\vec{n} = \frac{1}{\sqrt{2}}(\vec{x} + \vec{z})$

eigenvectors
$$|0'\rangle = |\uparrow_{\vec{n}}\rangle = \begin{pmatrix} \cos \pi/8 \\ \sin \pi/8 \end{pmatrix}$$
$$|1'\rangle = |\downarrow_{\vec{n}}\rangle = \begin{pmatrix} \sin \pi/8 \\ -\cos \pi/8 \end{pmatrix}$$

eigenvalues
$$\lambda(0') = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2 \pi/8$$
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Overlap

$$\begin{cases} \langle 0'|\uparrow_z\rangle^2 = \langle 0'|\uparrow_x\rangle^2 = \cos^2 \pi/8 = 0.8535 \\ \langle 1'|\uparrow_z\rangle^2 = \langle 1'|\uparrow_x\rangle^2 = \sin^2 \pi/8 = 0.1465 \end{cases}$$

If Bob does not know what was sent, best guess is $|\psi\rangle = |0'\rangle$

$$\Rightarrow F = \langle\psi|\rho|\psi\rangle = \frac{1}{2}|\langle\uparrow_z|\psi\rangle|^2 + \frac{1}{2}|\langle\uparrow_x|\psi\rangle|^2 = 0.8535$$

Scenario:

Alice wants to send a 3-qubit message, but can transmit only 2 qubits. Could send Bob those qubits $(F = 1)$ and have Bob guess $|0'\rangle$ for the third

⇒ **Baseline Fidelity** (no tricks) $F = 0.8535$

How to improve:

Diagonalize ρ $\begin{cases} \text{Likely subspace } |0'\rangle \\ \text{Unlikely subspace } |1'\rangle \end{cases}$ for 1 bit

Let $|\varphi\rangle = \overbrace{|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle}^{\text{either } |\uparrow_z\rangle \text{ or } |\uparrow_x\rangle}$

Note: all possible $|\varphi\rangle$ have the same overlap with states of the type $|i\rangle|j\rangle|k\rangle$ where $i,j,k \in \{0', 1'\}$

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How to improve:

Diagonalize ρ: Likely subspace $|0'\rangle$, Unlikely subspace $|1'\rangle$ for 1 bit

Let $|\Psi\rangle = \underbrace{|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle}_{\text{either } |\uparrow_z\rangle \text{ or } |\uparrow_x\rangle}$

Note: all possible $|\Psi\rangle$ have the same overlap with states of the type $|i\rangle|j\rangle|k\rangle$ where $i,j,k \in \{0',1'\}$

Note: for any $|\Psi\rangle$ drawn from Alice and Bobs alphabet, we have

Likely subspace Λ

$$|\langle 0'0'0'|\Psi\rangle|^2 = \cos^6\pi/8 = \underline{0.6219}$$

$$|\langle 0'0'1'|\Psi\rangle|^2 = |\langle 0'1'0'|\Psi\rangle|^2 = |\langle 1'0'0'|\Psi\rangle|^2 = \cos^4\tfrac{\pi}{8}\sin^2\tfrac{\pi}{8} = \underline{0.1067}$$

Unlikely subspace $\Lambda^\perp$

$$|\langle 0'1'1'|\Psi\rangle|^2 = |\langle 1'0'1'|\Psi\rangle|^2 = |\langle 1'1'0'|\Psi\rangle|^2 = \cos^2\tfrac{\pi}{8}\sin^4\tfrac{\pi}{8} = \underline{0.0183}$$

$$|\langle 1'1'1'|\Psi\rangle|^2 = \sin^6\tfrac{\pi}{8} = \underline{0.0031}$$

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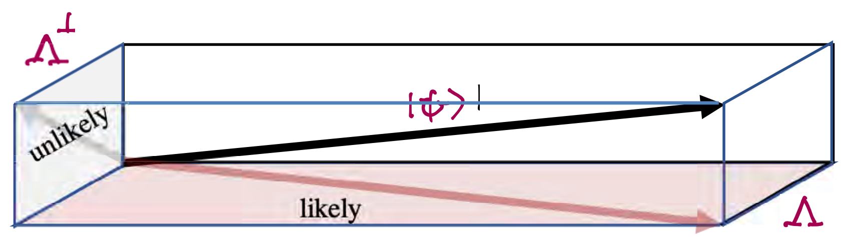
This structure of message space suggests Alice should send only the part $\in \Lambda$, which fits in 2 qubits ($\mathrm{Dim}\,\Lambda = 4$). She can do this by performing a measurement that projects her 3–qubit state onto either Λ or $\Lambda^\perp$

$$\Rightarrow \begin{cases} P_{\text{Likely}} = 0.6219 + 3\times 0.1067 = 0.9419 \\ P_{\text{Unlikely}} = 3\times 0.0183 + 0.0031 = 0.0581 \end{cases}$$

Geometric illustration of a space with likely and unlikely subspaces.



The state vector (black) has a large projection on the "likely subspace" (pink), and a much smaller projection on the "unlikely subspace" (grey). Accordingly, when we project onto the likely subspace and ignore the unlikely subspace, we don't loose much information.

Quantum Information Theory (Preskill Ch. 5)

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To do this Alice can apply the Unitary Transformation U that maps

$U: |\psi_{Likely}\rangle \rightarrow |\cdot\rangle|\cdot\rangle|0'\rangle$

$U: |\psi_{Unlikely}\rangle \rightarrow |\cdot\rangle|\cdot\rangle|1'\rangle$

She measures the 3rd qubit

$$\Rightarrow \begin{cases} |0\rangle \rightarrow \text{projects onto } \Lambda \\ |1\rangle \rightarrow \text{projects onto } \Lambda^\perp \end{cases}$$

If Alice's outcome is $|0\rangle$ she sends $|\psi_{comp}\rangle = |\cdot\rangle|\cdot\rangle$

Bob decompresses by appending $|0'\rangle$ and undoing U

$\Rightarrow |\psi_{Bob}\rangle = U^{-1}(|\psi_{comp}\rangle|0'\rangle)$

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$$\Rightarrow |\Phi_{Bob}\rangle = U^{-1}(|\Psi_{comp}\rangle|0'\rangle)$$

If Alice's outcome is $|1'\rangle$ she sends $U|0'0'\rangle$

$$\Rightarrow |\tilde{\Phi}_{Bob}\rangle = U^{-1}(|0'0'\rangle|0'\rangle) = |0'0'0'\rangle$$

This leaves Bob with the state

$$\rho_{Bob} = E|\Phi\rangle\langle\Phi|E + \langle\Phi|1-E|\Phi\rangle|0'0'0'\rangle\langle 0'0'0'|$$

($E =$ projection on Λ)

which has Fidelity

$$\mathcal{F} = \langle\Phi|\rho_{Bob}|\Phi\rangle$$

$$= \langle\Phi|E|\Phi\rangle + \langle\Phi|1-E|\Phi\rangle|\langle\Phi|0'0'0'\rangle|^2$$

$$= 0.9419^2 + 0.0581 \times 0.6219 = \underline{0.9234} > \underline{0.8535}$$

As with classical data compression, longer messages allow for more compression or compression without loss.

In Quantum Communication one has the option of choosing an alphabet where the individual letters are mixed states. This makes it much more challenging to find bounds for compressibility and code rates. See Preskill Chapter 5 for more information.