General Theory of Quantum Measurement (Preskill ch. 3)

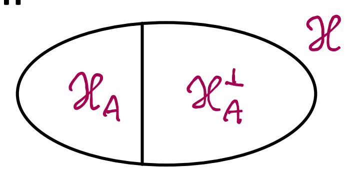
How to do it?

We can effectively do non-OM's in part of Hilbert space if we can add extra dimensions to $\mathcal{H}$:

$$\boxed{\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^{\perp} \quad \text{or} \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$$

Direct Sum Implementation

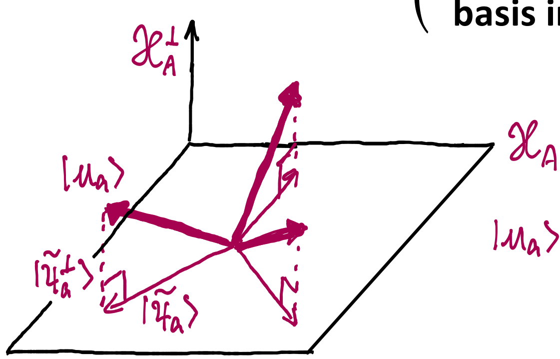
Let $\mathcal{H}_A \oplus \mathcal{H}_A^{\perp}$



Alice prepares states $\rho_A \in \mathcal{H}_A$

Bob (and/or Alice) makes OM $\{E_a\}$ in $\mathcal{H}$, $E_a = |u_a\rangle\langle u_a|$

Geometric visualization: $\left(\begin{array}{c}\text{like an over complete}\\ \text{basis in 2D subspace}\end{array}\right)$



$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^{\perp}\rangle$

We can now define effective measurement operators

$$F_a = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = Tr[E_a \rho_A] = Tr[F_a \rho_A]$$

Properties:

* Each F_a is Hermitian & non-negative $\Rightarrow P(m_a) \geq 0$
* Individual F_a are not projectors unless $\lambda_a = 1$
* $\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A \leftarrow$ identity on $\mathcal{H}_A$

POVM : **P**ositive **O**perator **V**alued **M**easure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a's are non-negative & $\sum_a F_a = \mathbb{1}$

General Theory of Quantum Measurement (Preskill ch. 3)

We can now define <u>effective</u> measurement operators

$$F_a = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow \mathcal{P}(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_a \rho_A]$$

<u>Properties</u>:

* Each F_a is Hermitian & non-negative $\Rightarrow \mathcal{P}(m_a) \geq 0$
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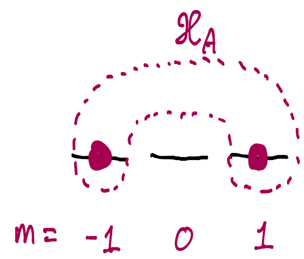
<u>POVM</u> : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a 's are non-negative & $\sum_a F_a = \mathbb{1}$

<u>Example</u> : POVM on Qubit encoded in Qutrit

$^{87}Rb(F=1)$ atomic HF state

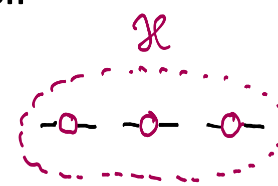


Unitary Transformation

$U: |\tilde{\psi}_1\rangle + |\tilde{\psi}_1^{\perp}\rangle \to |-1\rangle$

$U: |\tilde{\psi}_2\rangle + |\tilde{\psi}_2^{\perp}\rangle \to |0\rangle$

$U: |\tilde{\psi}_3\rangle + |\tilde{\psi}_3^{\perp}\rangle \to |1\rangle$



Alice prepares in $\mathcal{H}_A$

Bob measures $E_m = |m\rangle\langle m| \in \mathcal{H}$

Choose the map U

$\Rightarrow$ any 1 qubit, 3 outcome POVM we want

<u>Theorem</u>: Any POVM can be realized by adding to $\mathcal{H}_A$ an orthogonal complement $\mathcal{H}_A^{\perp}$

If N F_a's are desired, where $N > \mathcal{D}im\, \mathcal{H}_A$

then we need $\mathcal{D}im\,(\mathcal{H}_A + \mathcal{H}_A^{\perp}) \geq N$

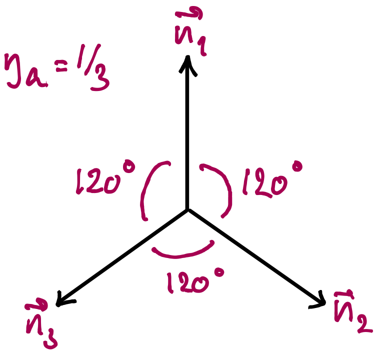
(Preskill 3.1.4)

General Theory of Quantum Measurement (Preskill ch. 3)

Toy Example: One Qubit POVM, illustrates different capabilities of OM & non-OM POVM's

Pick 3 unit vectors s. t. $\sum_a y_a \vec{n}_a = 0, \quad \sum_a y_a = 1$

$y_a = 1/3$



Measurement operators

$$F_a = 2 y_a |\uparrow_{\vec{n}_a}\rangle\langle\uparrow_{\vec{n}_a}| \quad \Rightarrow \quad \sum_a F_a = \mathbb{1}$$

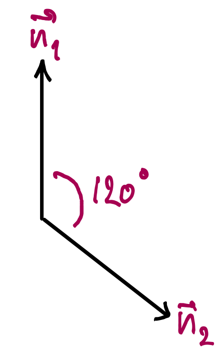
For the above & following, note that

$$|\uparrow_{\vec{n}_2}\rangle = \cos(60°)|\uparrow_{\vec{n}_1}\rangle + \sin(60°)|\downarrow_{\vec{n}_1}\rangle = \tfrac{1}{2}|\uparrow_{\vec{n}_1}\rangle + \tfrac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

$$|\uparrow_{\vec{n}_3}\rangle = \cos(60°)|\uparrow_{\vec{n}_1}\rangle + \sin(-60°)|\downarrow_{\vec{n}_1}\rangle = \tfrac{1}{2}|\uparrow_{\vec{n}_1}\rangle - \tfrac{\sqrt{3}}{2}|\downarrow_{\vec{n}_1}\rangle$$

Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability



How can Bob best tell the difference ?

OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Alice sends:
- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 3/4$

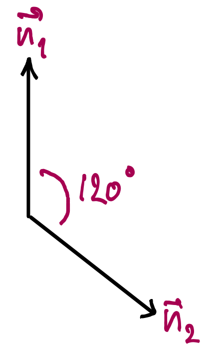
Bob's guess:
- $|\uparrow_{\vec{n}_1}\rangle$ (for both outcomes $|\uparrow_{\vec{n}_1}\rangle$)
- $|\uparrow_{\vec{n}_2}\rangle$ (for outcome $|\downarrow_{\vec{n}_1}\rangle$)

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$

General Theory of Quantum Measurement (Preskill ch. 3)

Application: Discriminating between non-orthogonal states

Alice prepares $|\uparrow_{\vec{n}_1}\rangle, |\uparrow_{\vec{n}_2}\rangle$ w/equal probability



How can Bob best tell the difference ?

OM in $\{|\uparrow_{\vec{n}_1}\rangle, |\downarrow_{\vec{n}_1}\rangle\}$ basis ?

Bob's guess

Alice sends:

- $|\uparrow_{\vec{n}_1}\rangle$ → Bob gets $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1$ — Bob's guess: $|\uparrow_{\vec{n}_1}\rangle$
- $|\uparrow_{\vec{n}_2}\rangle$ → Bob gets:
  - $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$ — Bob's guess: $|\uparrow_{\vec{n}_1}\rangle$
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 3/4$ — Bob's guess: $|\uparrow_{\vec{n}_2}\rangle$

Note: Bob can never know for sure he received $|\uparrow_{\vec{n}_1}\rangle$

Fidelity of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \underset{(a)}{\tfrac{1}{2} \times 1} + \underset{(b)}{\tfrac{1}{2}} \Big(\underset{(c)}{3/4 \times 1} + \underset{(d)}{1/4 \times 1/4} \Big) = \frac{29}{32} \simeq \underline{0.9063}$$

(Quite good)

(a) A sent $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/2$, B guesses $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1$ $(\mathcal{F} = 1)$

(b) A sent $|\uparrow_{\vec{n}_2}\rangle$ w/ $\mathcal{P} = 1/2$

(c) Given $|\uparrow_{\vec{n}_2}\rangle$ B gets $|\downarrow_{\vec{n}_1}\rangle$ & guesses $|\uparrow_{\vec{n}_2}\rangle$ w/ $\mathcal{P} = 3/4$ $(\mathcal{F} = 1)$

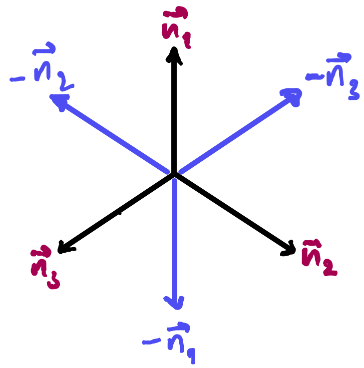
(d) Given $|\uparrow_{\vec{n}_2}\rangle$ B gets $|\uparrow_{\vec{n}_1}\rangle$ & guesses $|\uparrow_{\vec{n}_1}\rangle$ w/ $\mathcal{P} = 1/4$ $(\mathcal{F} = 1/4)$

General Theory of Quantum Measurement (Preskill ch. 3)

Instead

Bob does the POVM

$$\boxed{F_a = \frac{2}{3}|\downarrow_{\vec{n}_a}\rangle\langle\downarrow_{\vec{n}_a}|}$$



Alice sends
- $|\uparrow_{\vec{n}_1}\rangle \rightarrow$ Bob gets
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $P = 0$
  - $|\downarrow_{\vec{n}_2}\rangle$ w/ $P = 1/2$
  - $|\downarrow_{\vec{n}_3}\rangle$ w/ $P = 1/2$
- $|\uparrow_{\vec{n}_2}\rangle \rightarrow$ Bob gets
  - $|\downarrow_{\vec{n}_1}\rangle$ w/ $P = 1/2$
  - $|\downarrow_{\vec{n}_2}\rangle$ w/ $P = 0$
  - $|\downarrow_{\vec{n}_3}\rangle$ w/ $P = 1/2$

Bob gets
- $|\downarrow_{\vec{n}_1}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec{n}_2}\rangle$
- $|\downarrow_{\vec{n}_2}\rangle \rightarrow$ Bob knows Alice sent $|\uparrow_{\vec{n}_1}\rangle$
- $|\downarrow_{\vec{n}_3}\rangle \rightarrow$ Bob doesn't know (DK)

Fidelity of Bob's guess (Prob. his guess if correct)

$$\mathcal{F}_{POVM} = \underbrace{\frac{1}{2}}_{P(\text{know})}\times 1 + \underbrace{\frac{1}{2}}_{P(\text{don't know})}\left(\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{4}\right) = \frac{13}{16} = \underline{0.8125}$$

(a) (b) (c)

(a) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B knows which one w/ $P = 1/2$ $(\mathcal{F} = 1)$

(b) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B DK, correct guess w/ $P = 1/2$ $(\mathcal{F} = 1)$

(c) A sent $|\uparrow_{\vec{n}_1}\rangle$ or $|\uparrow_{\vec{n}_2}\rangle$, B DK, wrong guess w/ $P = 1/2$ $(\mathcal{F} = 1/4)$

Note: If in (c) Bob guesses $|\downarrow_{\vec{n}_3}\rangle$ w/ $\mathcal{F} = 3/4$ he gets a slightly better fidelity of

$$\mathcal{F}_{POVM} = \frac{14}{16} = \underline{0.8750}$$

However: if Bob sticks with **Heralded Success** he will have a subensemble w/ $\mathcal{F}_{POVM} = 1$!

Check out R. M. B Clarke et al., "Experimental Realization of Optimal Detection Strategies for Overcomplete States". You can find it on the OPTI 646 website under the "Reading" Tab.

General Theory of Quantum Measurement (Preskill ch. 3)

Rewind: how to implement a non-OM ?

* The Postulates of QM tells us we can do OM's in a given Hilbert space

* We can effectively do non-OM's in part of $\mathcal{H}$ if

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^\perp \quad \text{or} \quad \underbrace{\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B}$$

Look at this option !

Motivation:

* We cannot count on our system to be embedded in a larger Hilbert space
* A more realistic implementation is to juxtapose system A with a second system B and doing OM's in the resulting tensor product space

Systems A & B, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_{AB} = \rho_A \otimes \rho_B$

Set of orthogonal E_a acting in $\mathcal{H}$, $\sum_a E_a = \mathbb{1}$

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

$\mathcal{H}_B$, ρ_B & an OM $\{E_a\}$ in $\mathcal{H}_a = \mathcal{H}_A \otimes \mathcal{H}_B$ s. t.

$$\mathcal{P}(m_a) = \mathrm{Tr}_{AB}[E_a(\rho_A \otimes \rho_B)] = \mathrm{Tr}_A[F_a \rho_A]$$

where $F_a = \mathrm{Tr}_B[E_a \rho_B]$

and $\rho_{AB} \to \rho'_{AB}(m_a) = \dfrac{E_a(\rho_A \otimes \rho_B)E_a}{\mathcal{P}(m_a)}$

Math details

$$\mathrm{Tr}_{AB}[E_a(\rho_A \otimes \rho_B)] = \mathrm{Tr}_A\big[\mathrm{Tr}_B[(\rho_A \otimes \rho_B)E_a]\big]$$

$$= \mathrm{Tr}_A\big[\rho_A \mathrm{Tr}_B[\rho_B E_a]\big] = \mathrm{Tr}_A[F_a \rho_A]$$

where $F_a = \mathrm{Tr}_B[E_a \rho_B]$

Hint: Make frequent use of the Trace invariance under cyclic permulation.

Also: for any operators X_A, Y_B, we have

$$\mathrm{Tr}_B(X_A \otimes Y_B) = X'_A$$

Tracing over System B leaves a new operator that acts only on System A, and vice versa!

General Theory of Quantum Measurement (Preskill ch. 3)

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

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The F_a have the properties of POVM elements:

* **Hermiticity**:

$$F_a^+ = Tr_B[E_a \rho_B]^+ = Tr_B[\rho_B^+ E_a^+] = Tr_B[E_a \rho_B] = F_a$$

* **Positivity**: E_a, ρ_B positive (eigenvalues ≥ 0) (i)

  ⇨ $E_a \rho_B$ positive, marginal $Tr_B[E_a \rho_B] = F_a$

* **Completeness**: $\sum_a F_a = \mathbb{1}$ (ii)

* **Non-orthogonality**: $\# F_a\text{'s} \begin{cases} > \mathcal{H}_A \\ \leq Dim(\mathcal{H}_A \otimes \mathcal{H}_B) \end{cases}$

Math Details

Eigenbasis of ρ_B. There are *d* of these if $\mathcal{H}_B$ is *d*-dimensional

(i) Let $\rho_B = \sum_\mu p_\mu |\mu\rangle_{BB}\langle\mu|$ ⇨ $F_a = \sum_\mu p_\mu \, {}_B\langle\mu|E_a|\mu\rangle_B$

⇨ ${}_A\langle\psi|F_a|\psi\rangle_A = \sum_\mu p_\mu \left({}_A\langle\psi| \otimes {}_B\langle\mu|\right) E_a \left(|\mu\rangle_B \otimes |\psi\rangle_A\right) \geq 0$

(ii) $\sum_a F_a = \sum_\mu p_\mu \, {}_B\langle\mu| \sum_a E_a |\mu\rangle_B = \mathbb{1}_A$

General Theory of Quantum Measurement (Preskill ch. 3)

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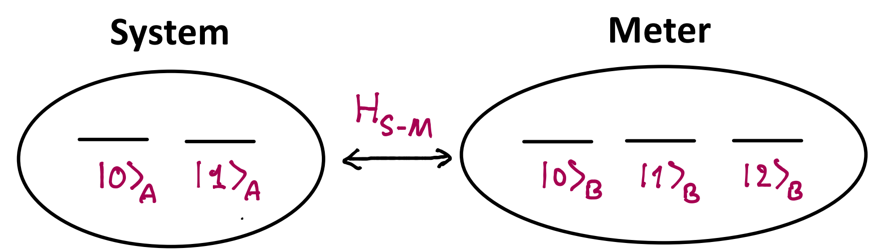

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ has TP basis

$|00\rangle_{AB}$ $|01\rangle_{AB}$ $|02\rangle_{AB}$

$|10\rangle_{AB}$ $|11\rangle_{AB}$ $|12\rangle_{AB}$

start meter in $|0\rangle_B$

The interaction drives a unitary map, for example

$$|00\rangle_{AB} \to \sum_{j=0}^{2} a_j |0\rangle_A |j\rangle_B$$

$$|10\rangle_{AB} \to \sum_{j=0}^{2} b_j |1\rangle_A |j\rangle_B$$

where the c-numbers a_j, b_j are chosen to ensure orthonormality

Measuring the meter ➡ 3 possible outcomes $j = 0, 1, 2$

Note: Here we are arguing only that there exist a unitary map such that the POVM criteria are fulfilled. With a bit of practice it is not too hard to guess one. The problem is closely related to that of finding/guessing the Unitary representation for for the Depolarizing Channel on operator form, see below.

General Theory of Quantum Measurement (Preskill ch. 3)

Theorem: Given $\mathcal{H}_A$ & POVM $\{F_a\}$ we can choose

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$$F_a^\dagger = Tr_B\left[E_a \rho_B\right]^\dagger = Tr_B\left[\rho_B^\dagger E_a^\dagger\right] = Tr_B\left[E_a \rho_B\right] = F_a$$

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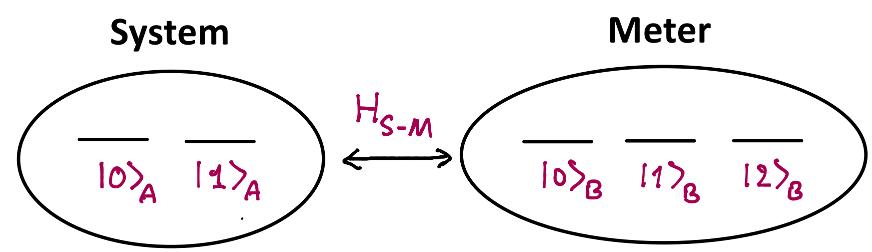

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where the c-numbers a_j, b_j are chosen to ensure orthonormality

How to do it: Let the unitary map be of the form

$|00\rangle \to a_0|00\rangle + a_1|01\rangle + a_2|02\rangle$

$|10\rangle \to b_0|10\rangle + b_1|11\rangle + b_2|12\rangle$

and

$|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$

$|b_0|^2 + |b_1|^2 + |b_2|^2 = 1$

LHS, all states remain orthogonal √

(normalization)

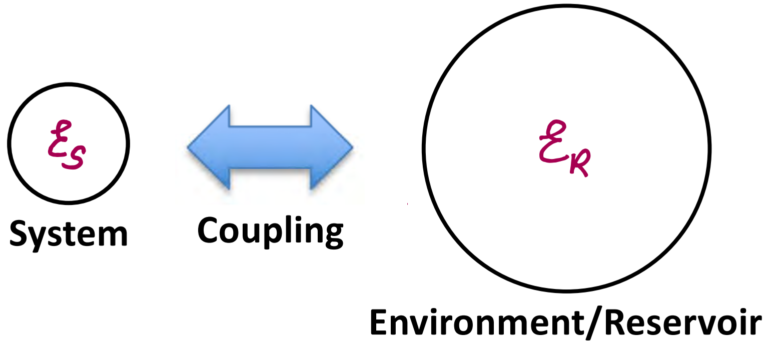
Pick, e. g, $a_{j=1,2,3} = \frac{1}{\sqrt{3}}$, $b_j = 1,2,3 = \frac{1}{\sqrt{3}}$ ➡ $P(m_{a,j=1,2,3}) = \frac{1}{3}$, $P(m_{b,j=1,2,3}) = \frac{1}{3}$

Even weight of outcomes

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #1: Coupling to an Environment

(Lecture 10-04-23)



* System + Environment evolves unitarily, become entangled ➡ the system on its own evolves non-unitarily

* Reasonable assumptions about the environment

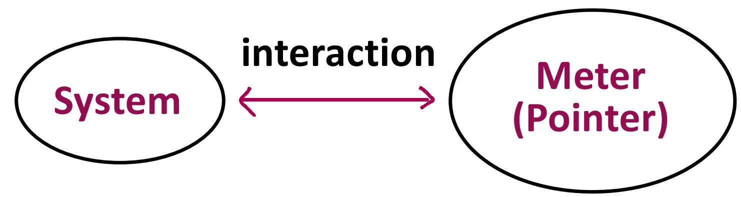
"Master Equation" for ρ_S

$$\boxed{\dot{\rho}_S = \frac{1}{i\hbar}[H_S, \rho_S] + \mathcal{L}(\rho_S)}$$

* The Liouvillian $\mathcal{L}$ accounts for relaxation and decoherence

Example #2: Coupling to a Meter

(Lecture 10-04-2023)



➡ Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

---

Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\rho = \rho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

$$\rho \Rightarrow U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}$$

Reduced density operator for system A in basis $\{|\mu\rangle_B\}$

$$\rho_A' = \mathrm{Tr}_B\left[U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right)U_{AB}^{+}\right]$$

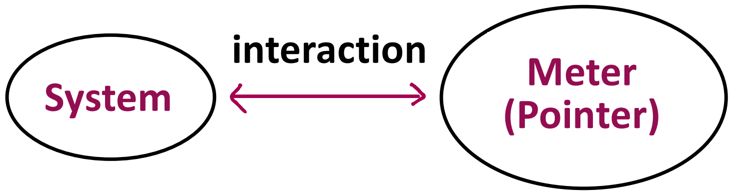
$$= \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0\rangle_B}\, \rho_A \; {}_B\langle 0| U_{AB}^{+} |\mu\rangle_B$$

operator M_μ acting on ρ_A

Open Quantum Systems – Evolution and Decoherence (Preskill ch. 3)

Example #2: Coupling to a Meter

(Lecture 10-18-2022)



Stochastic Schrödinger equation with unitary Evolution, interrupted by random Quantum Jumps when measurements occur

Our starting point: Operator-Sum representation of non-Unitary evolution

Let $\rho = \rho_A \otimes |0\rangle_{BB}\langle 0|$ w/unitary evolution U_{AB}

$$\rho \Rightarrow U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}$$

Reduced density operator for system A in basis $\{|\mu\rangle_B\}$

$$\rho_A' = Tr_B\left[U_{AB}\left(\rho_A \otimes |0\rangle_{BB}\langle 0|\right) U_{AB}^{+}\right]$$

$$= \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} | 0\rangle_B}\, \rho_A \, {}_B\langle 0 | U_{AB}^{+} | \mu\rangle_B$$

operator M_μ acting on ρ_A

We can now write

$$\boxed{\rho_A' = \$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^{+}}$$

Furthermore, since U_{AB} is unitary, the M_μ's have the property

$$\sum_\mu M_\mu^{+} M_\mu = \sum_\mu {}_B\langle 0 | U_{AB}^{+} | \mu\rangle_{BB}\langle \mu | U_{AB} | 0\rangle_B$$

$$= {}_B\langle 0 | U_{AB}^{+} U_{AB} | 0\rangle_B = \mathbb{1}_A$$

We conclude:

$\$$ defines a Linear Map

$\$$: Linear Operator → Linear Operator

If $\sum_\mu M_\mu^{+} M_\mu = \mathbb{1}_A$ then $\$$ is a <u>SuperOperator</u>

and $\$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^{+}$ is the Operator-Sum or Krauss representation of $\$$

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We can now write

$$\boxed{\rho_A' = \$(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger}$$

Furthermore, since U_{AB} is unitary, the M_μ's have the property

$$\sum_\mu M_\mu^\dagger M_\mu = \sum_\mu {}_B\langle 0| U_{AB}^\dagger |\mu\rangle_B \, {}_B\langle \mu| U_{AB} |0\rangle_B$$

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<u>Note</u>: $\$$ maps density operators to density operators because

* ρ_A' is Hermitian: $\rho_A'^\dagger = \sum_\mu M_\mu \rho_A^\dagger M_\mu^\dagger = \rho_A'$
* ρ_A' has unit trace: $\mathrm{Tr}\,\rho_A' = \sum_\mu \mathrm{Tr}[\rho_A M_\mu^\dagger M_\mu]$
* ρ_A' is positive:

$${}_A\langle\psi|\rho_A'|\psi\rangle_A = \sum_\mu ({}_A\langle\psi|M_\mu)\rho_A(M_\mu^\dagger|\psi\rangle_A) \geq 0$$

Used $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ and
Trace invariance under cyclic permutation

<u>Theorem</u>: Given some $\$$ with an operator-sum representation, we can choose $\mathcal{H}_B$ and find the corresponding unitary U_{AB} in $\mathcal{H}_A \otimes \mathcal{H}_B$

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Note:

* Superoperators provide a formalism to describe decoherence, i. e., maps from pure to mixed states
* Unitary evolution is a special case with only one term in the operator-sum expansion
* Two or more terms → initial pure states $\in \mathcal{H}_A$ become entangled w/states $\in \mathcal{H}_B$ due to U_{AB}
  → mixed final state ρ_A'
* Superoperators can be concatenated to form new ones, $\$ = \$_1 \$_2$

Theorem: If $(\$)^{-1}(\$) = \mathbb{1}$ then $\$$ must necessarily be unitary

Non-unitary evolution cannot be reversed
→ "arrow of time"

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We summarize:

A mapping $\$: \rho \to \rho'$ where ρ, ρ' are density operators, is a mapping of operators to operators that satisfy

(0) $\$$ is Linear

(1) $\$$ preserves Hermiticity

(2) $\$$ is Trace preserving

(3) $\$$ is completely positive,

$\$_A \otimes \mathbb{1}_B$ positive in $\mathcal{H}_A \otimes \mathcal{H}_B$ for all $\mathbb{1}_B$

Krauss Representation Theorem

Any $\$$ satisfying (0) – (3) has an Operator-Sum Representation

See Preskill, Ch. 3.2 for more on Superoperator formalism

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Measurement as a Superoperator:

(NOT COVERED IN CLASS)

Von Neumann: Entangle System A with Meter B

$$U_{AB}: |\varphi\rangle_A |0\rangle_B \to \sum_\mu M_\mu |\varphi\rangle_A |\mu\rangle_B \quad (1)$$

Orthogonal measurement on B in "pointer basis" $|\mu\rangle_B$ yields outcome μ and tells us the meter is in $|\mu\rangle_B$.

This projects out a state $|\psi_\mu\rangle_{AA}\langle\psi_\mu| = \dfrac{M_\mu|\varphi\rangle_{AA}\langle\varphi|M_\mu^\dagger}{{}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A}$

with probability $P(\mu) = {}_A\langle\varphi|M_\mu^\dagger M_\mu|\varphi\rangle_A$

Generally ρ_A is mixed

Meas. on B projects out $\rho_A^\mu = \dfrac{M_\mu \rho_A M_\mu^\dagger}{\mathrm{Tr}[M_\mu \rho_A M_\mu^\dagger]}$

with probability $P(\mu) = \mathrm{Tr}_A[M_\mu^\dagger M_\mu \rho_A] = \mathrm{Tr}_A[F_\mu \rho_A]$

This is a POVM with elements

$$F_\mu = M_\mu^\dagger M_\mu, \quad \sum_\mu F_\mu = \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}_A \quad (2)$$

If no access to the measurement outcome then

$$\rho_A \to \rho_A' = \sum_\mu P(\mu)\rho_A^\mu = \sum_\mu M_\mu \rho_A M_\mu^\dagger = \$(\rho_A)$$

Superoperator

Most general measurement: POVM $\{F_\mu\}$ on ρ_A

In this case we have
$$\begin{cases} P(\mu) = \mathrm{Tr}_A[F_\mu \rho_A] \\ \rho_A' = \sum_\mu \sqrt{F_\mu}\,\rho_A \sqrt{F_\mu} \end{cases}$$

Note that F_μ Hermitian $\to$ $\sqrt{F_\mu}$ Hermitian

$\sum_\mu F_\mu = \mathbb{1}_A$ Follows from the operator sum Normalization condition (2) above

Compare w/ (1) above to see this POVM has the unitary representation

$$U_{AB}: |\varphi\rangle_A |0\rangle_B \to \sum_\mu \sqrt{F_\mu}\,|\varphi\rangle_A |\mu\rangle_B$$

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$$U_{AB}: |\varphi\rangle_A|0\rangle_B \to \sum_\mu \sqrt{F_\mu}|\varphi\rangle_A|\mu\rangle_B$$

Summary:

* The discussion so far highlights the relationship between measurement and decoherence. We can always view the latter as the environment Doing a measurement and extracting information that we cannot retrieve. The loss of information causes an initial pure state to evolve into a statistical mixture, which is the definition of decoherence

* Sometimes we can "guess" what kind of "measurements" the environment implements. This is useful in the modeling of decohering "Quantum Channels"

* The example that follows is based on the first of four examples of decohering quantum channels given in Preskills notes. These will be particularly Relevant for those of you working in the area of Quantum communication over quantum photonic Networks.

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Example: Depolarizing Channel

Probability of error $= p$, 3 types, equal probability

(1) Bit flip $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$ $\Rightarrow$ $|\psi\rangle \to \sigma_1 |\psi\rangle$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2) Phase flip $|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$ $\Rightarrow$ $|\psi\rangle \to \sigma_3 |\psi\rangle$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(3) Both $|0\rangle \to i|1\rangle$, $|1\rangle \to -i|0\rangle$ $\Rightarrow$ $|\psi\rangle \to \sigma_2 |\psi\rangle$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Unitary Representation: Channel is a unitary map on $\mathcal{H}_A \otimes \mathcal{H}_E$

One choice (not unique, can always find one)

$$\boxed{U_{AE}: |\psi\rangle_A |0\rangle_E \to \sqrt{1-p}\, |\psi\rangle_A |0\rangle_E + \sqrt{\tfrac{p}{3}} \left[\sigma_1^A |\psi\rangle_A |1\rangle_E + \sigma_2^A |\psi\rangle_A |2\rangle_E + \sigma_3^A |\psi_A\rangle |3\rangle_E \right]}$$

Note:

The 4 orthogonal states in $\mathcal{H}_E$ keep records of what happened. If available through measurement in $\mathcal{H}_E$ the errors would in principle be reversible. We must have $\mathrm{Dim}\,\mathcal{H}_E \geq 4$ to allow 4 distinct evolutions

On Operator Form: we have

$$U_{AE} = \sqrt{1-p}\, \mathbb{1}_{AE} + \sqrt{\tfrac{p}{3}} \; \sigma_1^A |1\rangle_{E\,E}\langle 0| + \sigma_2^A |2\rangle_{E\,E}\langle 0| + \sigma_3^A |3\rangle_{E\,E}\langle 0|$$

$$\Rightarrow \varrho_A' = \mathrm{Tr}_E \left[U_{AE} \left(\varrho_A |0\rangle_{E\,E}\langle 0| \right) U_{AE}^\dagger \right]$$

$$= \sum_\mu \underbrace{{}_E\langle \mu | U_{AE} |0\rangle_E}_{M_\mu} \varrho_A \, {}_E\langle 0 | U_{AE}^\dagger | \mu\rangle_E = \sum_\mu M_\mu \varrho_A M_\mu^\dagger$$

From this we find Kraus operators

$$\boxed{M_0 = \sqrt{1-p}\, \mathbb{1}_A, \quad M_1 = \sqrt{\tfrac{p}{3}}\, \sigma_1^A, \quad M_2 = \sqrt{\tfrac{p}{3}}\, \sigma_2^A, \quad M_3 = \sqrt{\tfrac{p}{3}}\, \sigma_3^A}$$

Check ($\sigma_i^2 = 1$) : $\sum_\mu M_\mu^\dagger M_\mu = \left(1 - p + 3\tfrac{p}{3}\right) \mathbb{1} = \mathbb{1}$

From earlier:

$$\varrho_A' = \sum_\mu \underbrace{{}_B\langle \mu | U_{AB} |0\rangle_B}\, \varrho_A \, {}_B\langle 0 | U_{AB}^\dagger | \mu\rangle_B$$

operator M_μ acting on ϱ_A

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Evolution of the Qubit:

$$\boxed{\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1^A \rho_A \sigma_1^A + \sigma_2^A \rho_A \sigma_2^A + \sigma_3^A \rho_A \sigma_3^A\right)}$$

Bloch Sphere representation:

Bloch vector

Let: $\rho_A = \frac{1}{2}(\mathbb{1} + \vec{P}\cdot\vec{\sigma}) = \frac{1}{2}(\mathbb{1} + P_3\sigma_3)$ (1)

3 ('z') component of the Bloch vector

Choose $\hat{z}_3$ along $\vec{P} = (0, 0, P_3)$

Sub in expression for ρ_A' above and use

$$\sigma_1\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_2 = -\sigma_3\,, \quad \sigma_3\sigma_3\sigma_3 = \sigma_3$$

Can show that (Math details)

$$\rho_A \to \rho_A' = (1-p)\rho_A + \frac{p}{3}\left(\sigma_1\rho_A\sigma_1 + \sigma_2\rho_A\sigma_2 + \sigma_3\rho_A\sigma_3\right)$$

$$= (1-p)\frac{1}{2}(\mathbb{1} + P_3\sigma_3) + \frac{p}{3}\left[\frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} - P_3\sigma_3) + \frac{1}{2}(\mathbb{1} + P_3\sigma_3)\right]$$

$$= \frac{1}{2}\left[\mathbb{1} + \left(1 - \frac{4p}{3}\right)P_3\sigma_3\right] = \frac{1}{2}(\mathbb{1} + P_3'\sigma_3) \Rightarrow P_3' = \left(1 - \frac{4p}{3}\right)P_3$$

By symmetry of (1) we have $\boxed{\vec{P}' = \left(1 - \frac{4}{3}p\right)\vec{P}}$

Uniform shrinking Bloch Sphere

Continuous limit:

$$p = \Gamma dt \Rightarrow P_3(t + dt) = \left(1 - \frac{4}{3}\Gamma dt\right)P(t)$$

$$\frac{dP_3}{dt} = -\frac{4}{3}\Gamma P_3 \Rightarrow P_3(t) = P_3(0)\, e^{-4/3\,\Gamma t}$$

Bloch Sphere shrinking at constant rate

$$\boxed{\vec{P}(t) = \vec{P}(0)\, e^{-4/3\,\Gamma t}}$$

This turns out to be identical to the Master Equation result

Other Examples:

* Phase Damping (Bloch sphere shrinks along *x* , *y*)
* Amplitude Damping (Bloch sphere shrinks along *z*)