EPR and Bell Inequalities (Preskill ch. 4.1)

Clauser-Horne-Shimony-Holt (C.H.S.H.) Inequality

(Different version of Bell's inequality)

Alice: 2 settings ⇨ measure $\begin{cases} a \text{ w/outcome } \pm 1 \\ a' \text{ w/outcome } \pm 1 \end{cases}$

Bob: 2 settings ⇨ measure $\begin{cases} b \text{ w/outcome } \pm 1 \\ b' \text{ w/outcome } \pm 1 \end{cases}$

Note: $a, a' = \pm 1$ ⇨ $\begin{cases} a+a' = 0 \;\&\; a-a' = \pm 2 \\ a-a' = 0 \;\&\; a+a' = \pm 2 \end{cases}$

Combine w/ $b, b' = \pm 1$ ⇨

$$C = (a+a')b + (a-a')b' = \pm 2$$

HV assumption: Values ± 1 can be assigned simultaneously to all 4 observables a, a', b, b'

It follows that $|\langle C \rangle| \leq \langle |C| \rangle = 2$ ⇨

$$\boxed{|\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle| \leq 2}$$

(C.H.S.H. inequality)

Quantum Mechanics
$$a = \sigma^{(A)} \cdot \vec{a} \qquad b = \sigma^{(B)} \cdot \vec{b}$$
$$a' = \sigma^{(A)} \cdot \vec{a}' \qquad b' = \sigma^{(B)} \cdot \vec{b}'$$

⇨ $\begin{cases} \langle ab \rangle = \langle a'b \rangle = \langle ab' \rangle = -\cos\theta = -\frac{1}{\sqrt{2}} \\ \langle a'b' \rangle = -\cos(\theta + \pi/2) = \frac{1}{\sqrt{2}} \end{cases}$

For $\theta = 45°$ (max violation)

$$\boxed{|\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle| = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2}$$

End 10-09-2023

EPR and Bell Inequalities (Preskill ch. 4.1)

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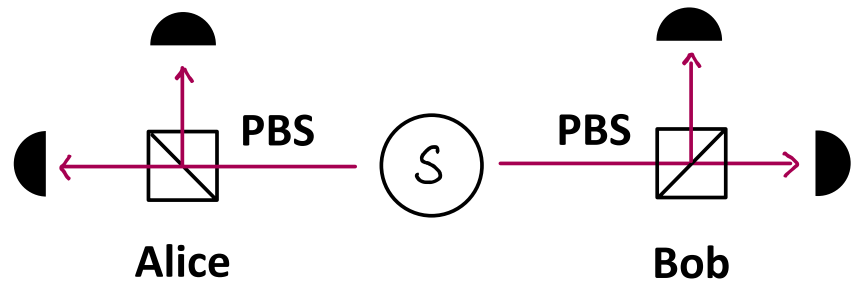
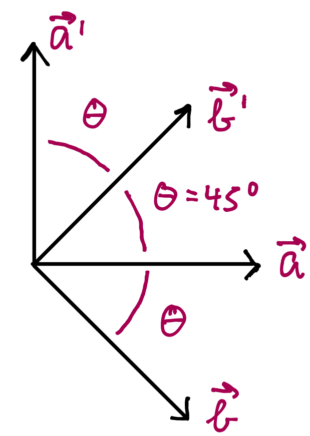
$$\boxed{|\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2}$$

- violates C.H.S.H. inequality

Laboratory Experiment (Aspect, many others)



polarizer settings



End 10-09-2023

Begin 10-11-2023

Bayes rule and the updating of probabilities

The Bayesian Update Rule

Consider two stochastic variables A and B. The joint, conditional, and univariate probabilities are related as follows:

$$\left.\begin{array}{l} P(A,B)=P(A|B)\,P(B) \\ P(A,B)=P(B|A)\,P(A) \end{array}\right\} \Rightarrow P(A|B)=\frac{P(B|A)\,P(A)}{P(B)}$$

Thus, with knowledge of $P(A)$ and $P(B)$ we can update our prior knowledge $P(B|A)$ when new information, $P(A|B)$, becomes available.

There are subtleties when working with a mix of probability densitity funticons (pfd's) and discrete data points. Let

α : continuous variable with pdf $\mu(\alpha)$

B : random discret data point

$\mu(B|\alpha)$: likelihood function

The Bayesian Update Rule generalizes like this:

$$\mu(\alpha|B)\,d\alpha=\frac{\mu(B|\alpha)\,\mu(\alpha)\,d\alpha}{P(B)}$$

where $P(B)=\int_{-\infty}^{\infty}\mu(B|\alpha)\,\mu(\alpha)\,d\alpha$ is a number.

Therefore, to within a normalization factor,

$$\boxed{\mu(\alpha|B)\propto\mu(B|\alpha)\,\mu(\alpha)}$$

See https://math.mit.edu/~dav/05.dir/clss13-slidesall.pdf Page 17

Begin 10-11-2023

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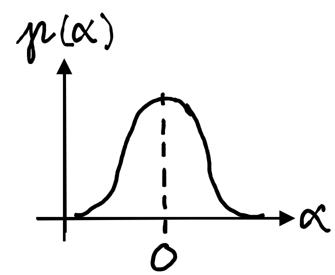
Bayesian Update of Classical Information

Consider a classical particle located somewhere on the α– axis. The Bayesian interpretation holds that a probability distribution quantifies prior knowledge, in this example about the position of the particle.

Let $p(\alpha)$ be the probability density for finding the particle at position α . We assume this pdf is a Gaussian centered at $\alpha = 0$.

$$p(\alpha) = \frac{1}{\sqrt{2\pi\sigma_\alpha^2}}\, e^{-\alpha^2/2\sigma_\alpha^2}$$



Next, we measure the position of the particle without disturbing it. The measurement has finite resolution, i. e., there is a change of observing the particle at B even if the actual position is α . This resolution is quantified by the likelihood Function $p(B|\alpha)$

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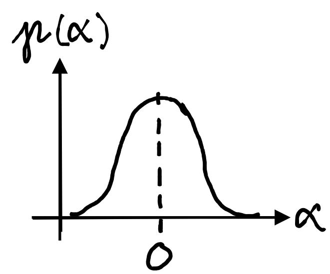

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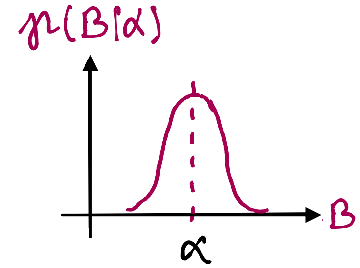
Bayesian Update of Classical Information, cont.

Let $p(B|\alpha)$ be a Gaussian,

$$P(B|\alpha) = \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-B^2/2\sigma_B^2}$$



Post-measurement, we can use Bayes Rule to update our knowledge of the position of the particle given that we observed B :

$$\boxed{\underbrace{p(\alpha|B)}_{G_{pm}} \propto \underbrace{p(B|\alpha)}_{G_1}\,\underbrace{p(\alpha)}_{G_2}}$$

The product of two Gaussians is a Gaussian, and therefore G_{pm} is also a Gaussian.

Furthermore, there are exact expressions for the means and σ's of the products, see, e. g.
http://www.lucamartino.altervista.org/2003-003.pdf

Bayes rule and the updating of probabilities

Bayesian Update of Classical Information, cont.

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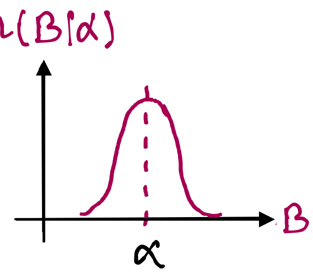

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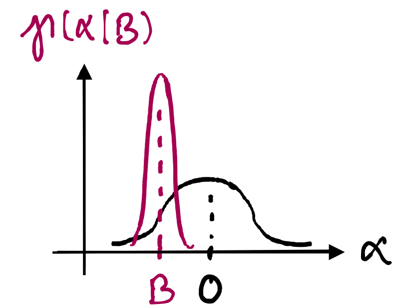
Physical Interpretation, Sharp Measurement

Now let $\sigma_{B|\alpha} \ll \sigma_\alpha$. Then G_1 is ~ constant over the range where $G_2 \neq 0$. In that case the pdf's will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-\alpha^2/2\sigma_{B|\alpha}^2}$$



Here we learn a lot from the measurement, and this leads to a large update of our Prior. In this example there will be a large change in the mean and uncertainty that we assign post-measurement. The resulting pdf looks much more like the resolution function than the pdf for the original Gaussian $p(\alpha)$.

Bayes rule and the updating of probabilities

Physical Interpretation, Sharp Measurement

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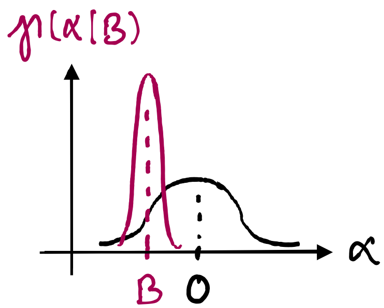

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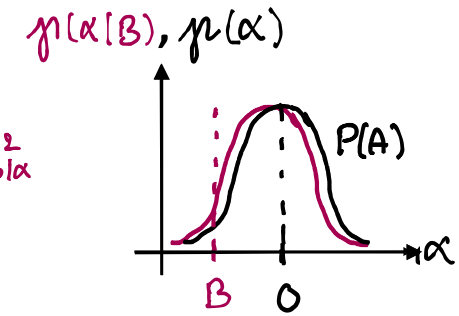
Physical Interpretation, Unsharp Measurement

Now let $\sigma_{B|\alpha} \approx \sigma_\alpha$. Then G_1 and G_2 are very similar and the pdf's will look like this:

$$p(\alpha|B) \approx \frac{1}{\sqrt{2\pi\sigma_B^2}}\, e^{-\alpha^2/2\sigma_{B|\alpha}^2}$$

$$\approx p(\alpha)$$



Here we learn little from the measurement and this leads to at most a minor update of our Prior. In this example there will be at most a modest change in the mean and uncertainty that we assign post-measurement. The result looks like a slightly shifted and broadened version of the original.

General Theory of Quantum Measurement (Preskill ch. 3)

Von Neumann's Theory of Measurement

System $\xleftrightarrow{\text{interaction}}$ **Meter (Pointer)**

System Observable M — Pointer observable
(position x of a free particle)

Hamiltonian for the coupled System and Meter

$$\boxed{H = \underbrace{H_0}_{\text{system}} + \underbrace{\frac{1}{2m}P^2}_{\text{free particle}} + \underbrace{\lambda M P}_{\text{interaction}}}$$

System-Meter interaction correlates M and x

⇨ Measure x → indirect measurement of M

Standard Quantum Limit (example)

Heisenberg: $\Delta x \Delta p = \frac{\hbar}{2}$ ⇨ $\Delta x(t)^2 \sim \Delta x(0)^2 + \left(\frac{\hbar t}{2m\,\Delta x(0)}\right)^2$

Interaction time t ⇨ $\Delta x(t) \gtrsim \Delta x_{SQL} \sim \sqrt{\frac{\hbar t}{m}}$

Heavy pointer, Strong interaction ⇨ $\boxed{H = \lambda M P}$

Note: P is the generator of translations along x

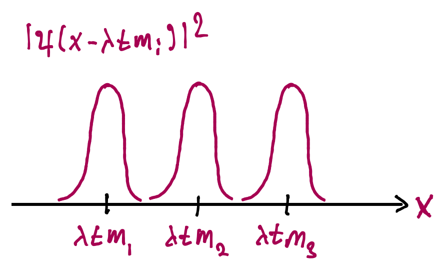
⇨ Time evolution $U(t) = e^{-i\lambda t M P/\hbar}$

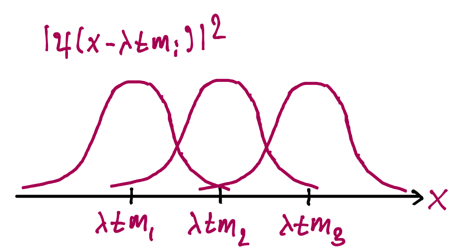
If	then
$M = \sum_a m_a \|a\rangle\langle a\|$	$U(t) = \sum_a \|a\rangle\langle a\| e^{-i\lambda t m_a P/\hbar}$

$$U(t) \sum_a \alpha_a |a\rangle \otimes |\psi(x)\rangle = \sum_a \alpha_a |a\rangle \otimes \underbrace{|\psi(x - \lambda t m_a)\rangle}$$

translation along $x \propto m_a$

Projective — **Non - Projective**





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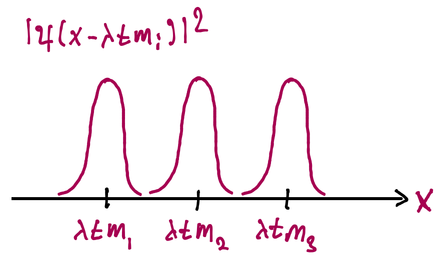
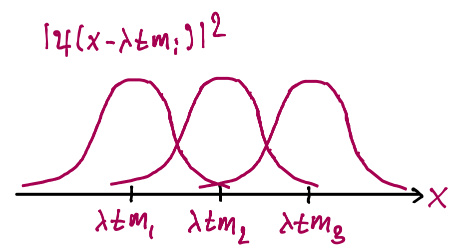
$$U(t)\sum_a \alpha_a \lvert a\rangle \otimes \lvert\psi(x)\rangle = \sum_a \alpha_a \lvert a\rangle \otimes \underbrace{\lvert\psi(x-\lambda t m_a)\rangle}_{\text{translation along } x \,\propto\, m_a}$$

Projective	Non - Projective





General Theory of Quantum Measurement (Preskill ch. 3)

General Theory of Quantum Measurement (Preskill ch. 3)

Orthogonal Measurement (OM)

Consider a set of measurements $\{E_a\}$ such that

$$E_a = E_a^\dagger \qquad E_a E_{a'} = \delta_{aa'} E_a \qquad \sum_a E_a = \mathbb{1}$$

orthogonal projectors complete set

We can associate such a set with any observable $\quad M = \sum_a m_a E_a$

This allows us to restate the measurement postulates:

An <u>Orthogonal Measurement</u> of an observable M is described by a collection of operators $\{E_a\}$,

$$E_a = E_a^\dagger \qquad E_a E_{a'} = \delta_{aa'} E_a \qquad \sum_a E_a = \mathbb{1}$$

The outcome m_a occurs w/prob. $\mathcal{P}(m_a) = \langle\psi| E_a |\psi\rangle$

➡ the state collapses as $|\psi\rangle \to E_a|\psi\rangle / \sqrt{\mathcal{P}(m_a)}$

Mixed state: $\mathcal{P}(m_a) = \mathrm{Tr}[E_a \rho]$, $\rho \to E_a \rho E_a / \mathcal{P}(m_a)$

m_a degenerate: E_a projects onto <u>subspace</u>

Can we generalize to a broader class? - Yes!

Consider:

$\sum_a E_a = \mathbb{1}$ is required (completeness)

$E_a E_{a'} = \delta_{aa'} E_a$ can be relaxed (orthogonality)

⬇

Concept of non-orthogonal measurements (POVMs)

POVM = Positive Operator Valued Measure

General Theory of Quantum Measurement (Preskill ch. 3)

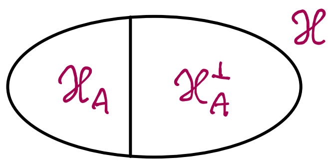
How to do it?

We can effectively do non-OM's in part of Hilbert space if we can add extra dimensions to $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_A^{\perp} \quad \text{or} \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

<u>Direct Sum</u> Implementation

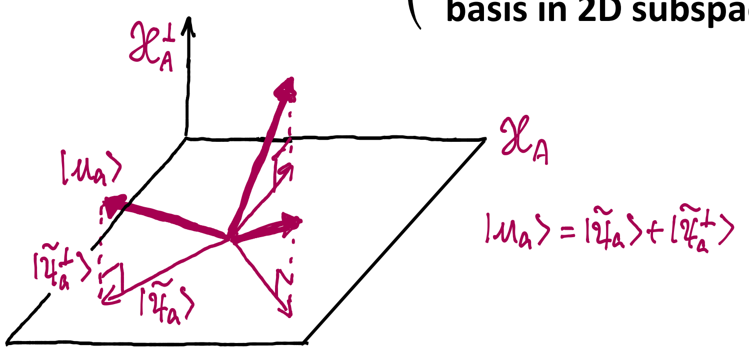
Let $\mathcal{H}_A \oplus \mathcal{H}_A^{\perp}$



Alice prepares states $\rho_A \in \mathcal{H}_A$

Bob (and/or Alice) makes OM $\{E_a\}$ in $\mathcal{H}$, $E_a = |u_a\rangle\langle u_a|$

Geometric visualization: $\left(\text{like an over complete basis in 2D subspace}\right)$



<u>Bob's OM</u> has 3 outcomes m_a w/projectors $E_a \in \mathcal{H}$

If Alice only prepares states $\rho_A \in \mathcal{H}_A$ then

$$P(m_a) = \mathrm{Tr}[\rho_A E_a] = \mathrm{Tr}[E_A \rho_A E_A E_a]$$

$$= \mathrm{Tr}[\rho_A \underbrace{E_A E_a E_A}_{F_A}] \equiv \mathrm{Tr}[\rho_A F_A]$$

norm ≤1

$$= \langle u_a|\rho_A|u_a\rangle = \langle\tilde{\psi}_A|\rho_A|\tilde{\psi}_A\rangle$$

$$= \lambda_a \langle\psi_a|\rho_A|\psi_a\rangle$$

number ≤1 normalized

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$$= \langle u_a | \rho_A | u_a \rangle = \langle \tilde{\psi}_A | \rho_A | \tilde{\psi}_A \rangle$$

norm ≤1

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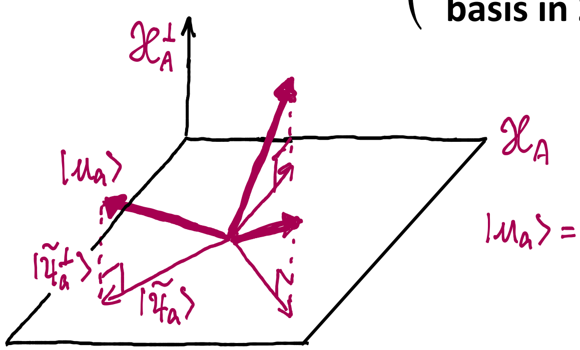

$|u_a\rangle = |\tilde{\psi}_a\rangle + |\tilde{\psi}_a^\perp\rangle$

We can now define effective measurement operators

$$F_A = E_A E_a E_A = |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a| = \lambda_a |\psi_a\rangle\langle\psi_a|$$

$$\Rightarrow P(m_a) = \mathrm{Tr}[E_a \rho_A] = \mathrm{Tr}[F_A \rho_A]$$

Properties:

* Each F_A is Hermitian & non-negative $\Rightarrow P(m_a) \geq 0$
* Individual F_A are not projectors unless $\lambda_a = 1$
* $\sum_a F_a = E_A \sum_a E_a E_A = E_A \mathbb{1} E_A = \mathbb{1}_A \leftarrow$ identity on $\mathcal{H}_A$

POVM : Positive Operator Valued Measure

A set of non-orthogonal meas. Operators $\{F_a\}$ such that the F_a's are non-negative & $\sum_a F_a = \mathbb{1}$