Measurement on One Part of a System

Cohen-Tannoudji Ch. III, Complement D_{III}

Begin 09-27-2023

Measurement on One Part of a System

Quantum Measurement on Bipartite Systems

$$\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{E}$$

System 1 System 2 Joint system

Consider the following:

Bipartite System

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

$$\tilde{A}(1) = A(1) \otimes \mathbb{1}(2)$$

Observable on System 1

Possible outcomes when measuring $\tilde{A}(1)$?

$$\{\text{Eigenvalues of } \tilde{A}(1)\} = \{\text{Eigenvalues of } A(1)\}$$

$\tilde{g}_n = g_n \times N_2$ g_n

- Same possible outcomes a_n indep of $|\psi\rangle$
- Degeneracy in $\mathcal{E}$ increases by a factor N_2

Projector: $P_n(1) = \sum_{i=1}^{g_n} |a_n^i(1)\rangle\langle a_n^i(1)|$

for eigenvalue a_n

Using the recipe to extend an operator into $\mathcal{E}$

$$\tilde{P}_n(1) = P_n(1) \otimes \mathbb{1}(2)$$

$$= \sum_{i=1}^{g_k} \sum_k |a_n^i(1) v_k(2)\rangle \langle a_n^i(1) v_k(2)|$$

Probability of outcome a_n, $|\psi\rangle$ general state $\in \mathcal{E}$

$$p(a_n) = \langle\psi|\tilde{P}_n(1)|\psi\rangle$$

$$= \sum_{i=1}^{g_n} \sum_k \langle\psi|a_n^i(1) v_k(2)\rangle\langle a_n(1) v_k(2)|\psi\rangle$$

Posterior state $|\psi'\rangle = \frac{1}{\sqrt{p(a_n)}} \tilde{P}_n(1)|\psi\rangle$

Measurement on One Part of a System

Some Observations:

1. Basis $|w_k(2)\rangle$ arbitrary, no phys. significance

2. **Product States** Let $|\psi\rangle = |\varphi(1)\rangle \otimes |\chi(2)\rangle$

   If we measure $A(1)$ and observe $|a_n(1)\rangle$ then

$$|\psi'\rangle \propto P_n(1)|\varphi(1)\rangle \otimes \mathbb{1}(2)|\chi(2)\rangle \propto |\varphi'(1)\rangle \otimes |\chi(2)\rangle$$

still a product state

3. **Entangled States**

   Consider a pair of states where *n* and *i* labels the eigenvalues and degeneracies within the subspace g_n

$$|\varphi(1)\rangle = \sum_n \sum_{i=1}^{g_n} a_{ni}|u_{ni}(1)\rangle \ , \ |\chi(2)\rangle = \sum_k b_k |\chi_k(2)\rangle$$

   The corresponding product state is of the form

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k a_{ni} b_k |u_{ni}(1)\rangle|\chi_k(2)\rangle$$

   By comparison, the most general state in $\mathcal{E}$ has the form

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} \sum_k c_{nik}|u_{ni}(2)\rangle|\chi_k(2)\rangle$$

   If the c_{nik} are all products of the type $a_{ni} b_k$ then $|\psi\rangle$ is a product state. Otherwise, $|\psi\rangle$ is entangled.

Some Observations: (Continued)

3. **Entangled States**

   If we measure $A(1)$ and observe the outcome a_N then the posterior state is

$$|\psi'\rangle \propto [P_N(1) \otimes \mathbb{1}(2)]|\psi\rangle \propto \sum_{i=1}^{g_N} \sum_k c_{Nik}[|u_{Ni}(1)\rangle \otimes |\chi_k(2)\rangle]$$

   Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

$$|\psi'\rangle \propto |u_N(1)\rangle \otimes \sum_k |\chi_k(2)\rangle \propto [|u_N(1)\rangle \otimes |\chi(2)\rangle]$$

   Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out in the global state.

   The case $g_N > 1$ is more subtle. Once we measure a_N, we know system 1 resides in the degenerate subspace associated with the outcome a_N. Repeat measurements do not generate further information about which of the exact $|u_{Ni}(1)\rangle$ our system is in. Thus, the measurement removes some, but not all of the entanglement present in $|\psi\rangle$. To completely factorize the state we would need to measure a C.S.C.O. This will identify not only the degenerate subspace but also the specific state vector $|u_{Ni}(1)\rangle$.
   See Cohen-Tannoudji Chapter III, Complement D_{III}

Measurement on One Part of a System

Some Observations: (Continued)

3. Entangled States

If we measure $A(1)$ and observe the outcome a_N then the posterior state is

$$|\psi'\rangle \propto [P_N(1)\otimes 1(2)]\,|\psi\rangle \propto \sum_{i=1}^{g_N}\sum_{k} c_{Ni;k}\left[|u_{Ni}(1)\rangle\otimes|\chi_k(2)\rangle\right]$$

Now, if $g_N = 1$ then the state $|u_N(1)\rangle$ occurs exactly once in the sum above, and therefore

$$|\psi'\rangle \propto |u_N(1)\rangle\otimes\sum_k |\chi_k(2)\rangle \propto \left[|u_N(1)\rangle\otimes|\chi(2)\rangle\right]$$

Conceptually, once the measurement tells us that system 1 is in the exact state $|u_N(1)\rangle$, then it factors out in the global state.

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See Cohen-Tannoudji Chapter III, Complement D_{III}

Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle\varphi(1)|P_n(1)|\varphi(1)\rangle\langle\chi(2)|P_k(2)|\chi(2)\rangle$$

⇨ Outcomes a_n, b_n are Independent Random Var's

Uncorrelated

Physical Interpretation of Entangled States

From (3) above, measuring $A(1)$, $B(2)$

Global $|\psi\rangle$ cannot be written as $|\varphi(1)\rangle\otimes|\chi(2)\rangle$

$$P(a_n, b_k) = \langle\psi|P_n(1)P_k(2)|\psi\rangle$$

In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Measurement on One Part of a System

Physical Interpretation of T.P. States

From (2) above, measuring $A(1)$, $B(2)$

$$P(a_n, b_k) = \langle \phi(1) | P_n(1) | \phi(1) \rangle \langle \chi(2) | P_k(2) | \chi(2) \rangle$$

⇒ Outcomes a_n, b_n are **Independent Random Var's**

Uncorrelated

Physical Interpretation of Entangled States

From (3) above, measuring $A(1)$, $B(2)$

Global $|\psi\rangle$ cannot be written as $|\phi(1)\rangle \otimes |\chi(2)\rangle$

$$P(a_n, b_k) = \langle \psi | P_n(1) P_k(2) | \psi \rangle$$

In general, a_n & b_k will be correlated random variables

Conclusion: We cannot assign state vectors to the individual subsystems !

Note:

Even though we cannot assign $|\phi(1)\rangle$, $|\chi(2)\rangle$, it is still possible to have a local description of each subsystem on its own. It must be consistent with tensor product states, yet it must reduce the information that is locally available when the global $|\psi\rangle$ is entangled

Density Matrix Formalism

Definition: A system for which we know only the probabilities p_k of finding the system in state $|\psi_k\rangle$ is said to be in a statistical mixture of states. Shorthand: mixed state.

Shorthand for non-mixed state: pure state

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Definition: Density Operator for pure states

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

Definition: Density Matrix

$$|\psi(t)\rangle = \sum_n c_n(t)|u_n\rangle$$

$$\rho_{pn}(t) = \langle u_p|\rho(t)|u_n\rangle = c_p(t)\,c_n^*(t)$$

Definition: Density Operator for mixed states

$$\rho(t) = \sum_k p_k\,\rho_k(t), \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $p_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

Measurement on One Part of a System

Definition: Density Operator for pure states

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

Definition: Density Matrix

$$|\psi(t)\rangle = \sum_n c_n(t)|u_n\rangle \Rightarrow$$

$$\rho_{pn}(t) = \langle u_p|\rho(t)|u_n\rangle = c_p(t)c_n^*(t)$$

Definition: Density Operator for mixed states

$$\rho(t) = \sum_k \eta_k \rho_k(t), \quad \rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$$

Note: A pure state is just a mixed state for which one $\eta_k = 1$ and the rest are zero.

The terms Density Operator and Density Matrix are used interchangeably

Let A be an observable w/eigenvalues a_n

Let P_n be the projector on the eigen-subspace of a_n

For a pure state, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, we have

(*) $Tr\,\rho(t) = \sum_n \rho_{nn}(t) = \sum_n |c_n|^2 = 1$

(*) $\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_p \langle\psi(t)|A|u_p\rangle\langle u_p|\psi(t)\rangle$

$$= \sum_p \langle u_p|\psi(t)\rangle\langle\psi(t)|A|u_p\rangle = \sum_p \langle u_p|\rho(t)A|u_p\rangle$$

$$= Tr[\rho(t)A]$$

($|u_p\rangle$ basis in $\mathcal{H}$)

(*) Let P_n be the projector on eigensubspace of a_n

$$\mathcal{P}(a_n) = \langle\psi(t)|P_n|\psi(t)\rangle = Tr[\rho(t)P_n]$$

(*) $\dot{\rho}(t) = |\dot{\psi}(t)\rangle\langle\psi(t)| + |\psi(t)\rangle\langle\dot{\psi}(t)|$

$$= \frac{1}{i\hbar}H|\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar}|\psi(t)\rangle\langle\psi(t)|H$$

$$= \frac{1}{i\hbar}[H,\rho]$$

Measurement on One Part of a System

Let A be an observable w/eigenvalues a_n

Let P_n be the projector on the eigen-subspace of a_n

For a **pure** state, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, we have

(∗) $\mathrm{Tr}\,\rho(t) = \sum_n \rho_{nn}(t) = \sum_n |c_n|^2 = 1$

(∗) $\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_p \langle\psi(t)|A|u_p\rangle\langle u_p|\psi(t)\rangle$

$$= \sum_p \langle u_p|\psi(t)\rangle\langle\psi(t)|A|u_p\rangle = \sum_p \langle u_p|\rho(t)A|u_p\rangle$$

$= \mathrm{Tr}[\rho(t)A]$ ($|u_p\rangle$ basis in $\mathcal{H}$)

(∗) Let P_n be the projector on eigensubspace of a_n

$$\mathcal{P}(a_n) = \langle\psi(t)|P_n|\psi(t)\rangle = \mathrm{Tr}[\rho(t)P_n]$$

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$$= \frac{1}{i\hbar}H|\psi(t)\rangle\langle\psi(t)| - \frac{1}{i\hbar}|\psi(t)\rangle\langle\psi(t)|H$$

$$= \frac{1}{i\hbar}[H,\rho]$$

Let A be an observable w/eigenvalues a_n

Let P_n be the projector on the eigen-subspace of a_n

For a **mixed** state, $\rho(t) = \sum_k p_k \rho_k(t)$, $\rho_k = |\psi_k(t)\rangle\langle\psi_k(t)|$

(∗) $\mathrm{Tr}\,\rho(t) = \sum_k p_k \mathrm{Tr}\,\rho_k(t) = 1$

(∗) $\langle A\rangle = \sum_k p_k \langle\psi_k(t)|A|\psi_k(t)\rangle = \sum_k p_k \mathrm{Tr}[\rho_k(t)A]$

$$= \mathrm{Tr}[\rho(t)A]$$

(∗) Let P_n be the projector on eigensubspace of a_n

$$\mathcal{P}(a_n) = \sum_k p_k \langle\psi_k(t)|P_n|\psi_k(t)\rangle = \mathrm{Tr}[\rho(t)P_n]$$

(∗) $\dot\rho(t) = \sum_k p_k \left(|\dot\psi_k(t)\rangle\langle\psi_k(t)| + |\psi_k(t)\rangle\langle\dot\psi_k(t)|\right)$

$$= \sum_k p_k \frac{1}{i\hbar}\left(H|\psi(t)\rangle\langle\psi(t)| - |\psi(t)\rangle\langle\psi(t)|H\right)$$

$$= \frac{1}{i\hbar}[H,\rho]$$

Density Operator formalism is general !

Measurement on One Part of a System

Important properties of the Density Operator

(1) ρ is Hermitian, $\rho^\dagger = \rho$ ⇨ ρ is an observable

⇨ $\exists$ basis in which ρ is diagonal

In this basis a pure state has one diagonal element $= 1$, the rest $= 0$

(2) Test for purity.

Pure: $\rho^2 = \rho$ ⇨ $\mathrm{Tr}\,\rho^2 = 1$

Mixed: $\rho^2 \neq \rho$ ⇨ $\mathrm{Tr}\,\rho^2 < 1$

(3) Schrödinger evolution does not change the p_k

⇨ $\mathrm{Tr}\,\rho^2$ is conserved
pure states stay pure
mixed states stay mixed

Changing pure ⇨ mixed requires non-Hamiltonian evolution – see Cohen Tannoudji D_{III} & E_{III}

Summary So Far

Density Operator: $\boxed{\rho(t) = \sum_k p_k |\psi_k\rangle\langle\psi_k|}$

Terminology:

$|\psi\rangle$ known → pure state

$p_k, |\psi_k\rangle$ known → mixed state

Properties

(1) $\mathrm{Tr}\,\rho = 1$

(2) $\langle A \rangle = \mathrm{Tr}[\rho A]$

(3) $P(a_n) = \mathrm{Tr}[\rho P_n]$, P_n: projector onto $\mathcal{E}$

(4) $\frac{d}{dt}\rho = \frac{1}{i\hbar}[H, \rho]$ Schrödinger Eq.

(5) ρ pure → $\rho^2 = \rho$, $\mathrm{Tr}\,\rho^2 = 1$

(6) $\frac{d}{dt}\mathrm{Tr}\,\rho^2 = 0$ → S. E. conserves purity

Measurement on One Part of a System

Summary So Far

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Terminology:

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Properties

(1) $\mathrm{Tr}\,\rho = 1$

(2) $\langle A\rangle = \mathrm{Tr}[\rho A]$

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(4) $\frac{d}{dt}\rho = \frac{1}{i\hbar}[H,\rho]$

(5) ρ pure $\rightarrow$ $\rho^2 = \rho$, $\mathrm{Tr}\,\rho^2 = 1$

(6) $\frac{d}{dt}\mathrm{Tr}\,\rho^2 = 0$ $\rightarrow$ S. E. conserves purity

Separate Description of Part of a System

Let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

T.P. Basis $\{|u_i(1)\rangle\} \otimes \{|v_p(2)\rangle\}$

Density Operator ρ in $\mathcal{E}$ $\leftarrow$ Describes global system

Goal: To "reverse engineer operators $\rho(1)$ in $\mathcal{E}_1$ and $\rho(2)$ in $\mathcal{E}_2$ such that they describe the systems independently

Our starting point is the global density operator

$$\boxed{\rho = \underbrace{\sum_{(ij)(k\ell)}}_{\substack{i,k\,\in\,\text{System (1)}\\ j,l\,\in\,\text{System (2)}}} \rho_{(ij)(k\ell)}\, \underbrace{|u_i v_j\rangle}_{\text{T.P. basis states}}\underbrace{\langle u_k v_\ell|}_{\text{T.P. basis states}}}$$

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Measurement on One Part of a System

Separate Description of Part of a System

Let $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$

T.P. Basis $\{|u_i(1)\rangle\} \otimes \{|v_p(2)\rangle\}$

Density Operator ρ in $\mathcal{E}$ ← Describes global system

Goal: To "reverse engineer operators $\rho(1)$ in $\mathcal{E}_1$ and $\rho(2)$ in $\mathcal{E}_2$ such that they describe the systems independently

Our starting point is the global density operator

$$\boxed{\rho = \underbrace{\sum_{(ij)(kl)}}_{\substack{i,k \,\in\, \text{System (1)} \\ j,l \,\in\, \text{System (2)}}} \rho_{(ij)(kl)} \underbrace{|u_i v_j\rangle}_{\text{T.P. basis states}}\underbrace{\langle u_k v_l|}_{\text{T.P. basis states}}}$$

$i, k \in$ System (1)

$j, l \in$ System (2)

T.P. basis states

Definition: Partial Trace

$$\rho(1) = Tr_2\, \rho = \sum_q \langle v_q|\rho|v_q\rangle$$

Orthonormal basis in $\mathcal{E}_2$

$$= \sum_q \sum_{(ij)(kl)} \rho_{(ij)(kl)} \overbrace{\langle v_q||u_i v_j\rangle\langle u_k v_l||v_q\rangle}^{\delta_{jq}\delta_{lq}|u_i\rangle\langle u_k|}$$

$$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| \leftarrow \text{operator in } \mathcal{E}_1$$

Check properties of $\rho(1)$

H.C. c.c. numbers, swap indices, kets & bras

(1) $$\rho(1)^\dagger = \sum_{ik}\sum_q \rho^*_{(iq)(kq)} |u_i\rangle\langle u_k|$$

$$= \sum_{ik}\sum_q \rho_{(kq)(iq)} |u_k\rangle\langle u_i| \leftarrow \text{Relabel} \quad i \to k, \; k \to i$$

$$= \sum_{ik}\sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| = \rho(1)$$

(2) $\rho(1)$ Hermitian → we can choose a basis $\{|w_k(1)\rangle\}$ so $\rho(1)$ is diagonal → $\rho_{(iq)(kq)} \propto \delta_{ik}$

Measurement on One Part of a System

Definition: <u>Partial Trace</u>

Orthonormal basis in $\mathcal{E}_2$

$$\rho(1) = Tr_2\, \rho = \sum_q \langle v_q | \rho | v_q \rangle$$

$$= \sum_q \sum_{(ij)(kl)} \rho_{(ij)(kl)} \overbrace{\langle v_q | |u_i v_j\rangle\langle u_k v_l| |v_q\rangle}^{\delta_{jq}\delta_{lq} |u_i\rangle\langle u_k|}$$

$i, k \in$ System (1), $j, l \in$ System (2)

$$= \sum_{ik} \sum_q \rho_{(iq)(kq)} |u_i\rangle\langle u_k| \leftarrow \text{operator in } \mathcal{E}_1$$

Check properties of $\rho(1)$

H.C. c.c. numbers, swap indices, kets & bras

(1) $\rho(1)^{\dagger} = \sum_{ik} \sum_q \rho^{*}_{(iq)(kq)} |u_i\rangle\langle u_k|$

$$= \sum_{ik} \sum_q \rho_{(kq)(iq)} |u_k\rangle\langle u_i| \leftarrow \text{Relabel} \quad i \to k,\ k \to i$$

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(2) $\rho(1)$ Hermitian → we can choose a basis $\{|w_k(1)\rangle\}$ so $\rho(1)$ is diagonal → $\rho_{(iq)(kq)} \propto \delta_{ik}$

Thus

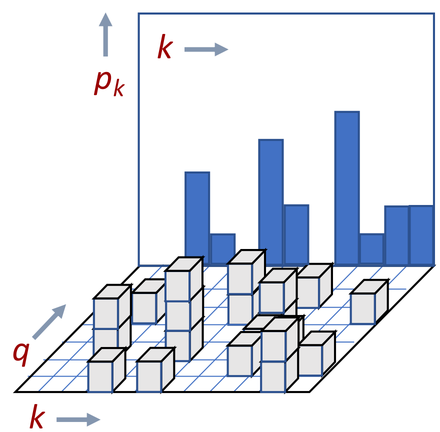
$$\rho(1) = \sum_k \underbrace{\sum_q \rho_{(kq)(kq)}}_{\eta_k} |w_k\rangle\langle w_k| = \sum_k \eta_k |w_k\rangle\langle w_k|$$

Note:

(1) $\rho_{(kq)(kq)}$ = population of $|w_k(1)\rangle \otimes |v_q(2)\rangle$, i.e. prob. of finding the global system in this state.

(2) $\eta_k = \sum_q \rho_{(kq)(kq)}$ is a <u>marginal probability</u>, i.e., the prob. of finding system 1 in $|w_k\rangle$, found by adding the probs $\rho_{(kq)(kq)}$ of finding the global system in the states $|w_k v_q\rangle$

Visualization - Marginal Probability



Measurement on One Part of a System

Thus

$$\boxed{\begin{aligned}\rho(1) &= \sum_k \underbrace{\sum_q \rho_{(kq)(kq)}}_{}\, |w_k\rangle\langle w_k| \\ &= \sum_k p_k\, |w_k\rangle\langle w_k|\end{aligned}}$$

Note:

$\rho_{(kq)(kq)}$ $\qquad\qquad$ $|w_k(1)\rangle \otimes |v_q(2)\rangle$

prob. of finding the global system in this state.

(2) $p_k = \sum_q \rho_{(kq)(kq)}$ is a <u>marginal probability</u>,

i.e., the prob. of finding system 1 in $|w_k\rangle$,
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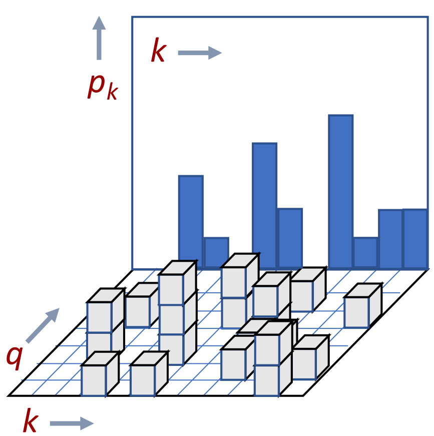


We <u>define</u>

$$\boxed{\begin{aligned}\rho(1) &= Tr_2\, \rho \\ \rho(2) &= Tr_1\, \rho\end{aligned}}$$

Partial Traces
or
Reduced Density Operators

Note: We already know these are Hermitian operators. Also,

$$Tr\,\rho = \sum_n \sum_q \langle u_n v_q | \rho | u_n v_q\rangle$$

global ρ

$$= Tr_1(Tr_2\, \rho) = Tr_1(\rho(1))$$

$$= Tr_2(Tr_1\, \rho) = Tr_2(\rho(2)) \equiv 1$$

Unit Trace Operators!

Expectation Values:

Insert identity here

$$\langle \tilde{A}(1)\rangle = Tr[\rho(1)\tilde{A}(1)] = \sum_{nq} \langle u_n v_q | \rho(1)\tilde{A}(1) | u_n v_q\rangle$$

$$= \sum_{nq} \sum_{n'q'} \underbrace{\langle u_n v_q | \rho(1) | u_{n'} v_{q'}\rangle}_{\rho_{nn'}(1)} \underbrace{\langle u_{n'} v_{q'} | A(1) \otimes 1(2) | u_n v_q\rangle}_{\delta_{qq'} \langle u_{n'} | A(1) | u_n\rangle}$$

identity

$$= \sum_{nn'} \langle u_n | \rho(1) | u_{n'}\rangle\langle u_{n'} | A(1) | u_n\rangle$$

$$= \sum_n \langle u_n | \rho(1) | u_n\rangle = \underline{Tr(\rho(1) A(1))}$$

Measurement on One Part of a System

We define

$$\boxed{\begin{aligned} \varrho(1) &= Tr_2\, \varrho \\ \varrho(2) &= Tr_1\, \varrho \end{aligned}}$$

Partial Traces
or
Reduced Density Operators

Note: We already know these are Hermitian operators. Also,

$$Tr\, \varrho = \sum_n \sum_q \langle u_n v_q | \varrho | u_n v_q \rangle$$

global ϱ

$$= Tr_1 (Tr_2\, \varrho) = Tr_1 (\varrho(1))$$

$$= Tr_2 (Tr_1\, \varrho) = Tr_2 (\varrho(2)) \equiv 1$$

Unit Trace Operators !

Expectation Values:

Insert identity here

$$\langle \tilde{A}(1) \rangle = Tr\left[\varrho(1) \tilde{A}(1)\right] = \sum_{nq} \langle u_n v_q | \varrho(1) \tilde{A}(1) | u_n v_q \rangle$$

$$= \sum_{nq} \sum_{n'q'} \underbrace{\langle u_n v_q | \varrho(1) | u_{n'} v_{q'} \rangle}_{\varrho_{nn'}(1)} \underbrace{\langle u_{n'} v_{q'} | A(1) \otimes \mathbb{1}(2) | u_n v_q \rangle}_{\delta_{qq'} \langle u_{n'} | A(1) | u_n \rangle}$$

identity

$$= \sum_{nn'} \langle u_n | \varrho(1) | u_{n'} \rangle \langle u_{n'} | A(1) | u_n \rangle$$

$$= \sum_n \langle u_n | \varrho(1) | u_n \rangle = \underline{\underline{Tr(\varrho(1) A(1))}}$$

We conclude:

$\varrho(1)$, $\varrho(2)$ are unit trace, Hermitian Operators

$$\langle \tilde{A}(1) \rangle = Tr(\varrho(1) A(1)), \quad \langle \tilde{B}(2) \rangle = Tr(\varrho(2) B(2))$$

$\varrho(1), \varrho(2) \Rightarrow$ are density operators for system (1) and system (2)

Measurement on One Part of a System

Additional Comments:

(1) If the Global state ≠ T. P. state

⇨ Cannot assign states $|\varphi(1)\rangle, |\chi(2)\rangle$ to S_1, S_2

Can assign $\rho(1), \rho(2)$ ⇨ Local description

(2) If ρ is pure, $\mathrm{Tr}\,\rho = 1$, we still can have

$$\mathrm{Tr}\,\rho(1)^2 \neq 1, \quad \mathrm{Tr}\,\rho(2)^2 \neq 1$$

(2) If the Global state is a T. P., $|\psi\rangle = |\varphi(1)\rangle|\chi(2)\rangle$

then
$$\begin{cases} \sigma(1) = |\varphi(1)\rangle\langle\varphi(1)| \\ \tau(2) = |\chi(2)\rangle\langle\chi(2)| \\ \rho = \sigma(1)\otimes\tau(2) \end{cases}$$

(3) The Global state can itself be mixed. In that case a product state will have the following structure

$$\rho = \sigma(1)\otimes\tau(2) \Rightarrow \begin{cases} \mathrm{Tr}_2[\sigma(1)\otimes\tau(2)] = \sigma(1) \\ \mathrm{Tr}_1[\sigma(1)\otimes\tau(2)] = \tau(2) \end{cases}$$

Additional Comments:

(4) However, if $\rho(1) = \mathrm{Tr}_2(\rho)$, $\rho(2) = \mathrm{Tr}_1(\rho)$

then in general $\rho' = \rho(1)\otimes\rho(2) \neq \rho$

(5) If the evolution of ρ is Hamiltonian, $\dot{\rho} = \frac{1}{i\hbar}[H,\rho]$, we cannot in general find a $H(1)$ that allows analogous equations for $\rho(1), \rho(2)$

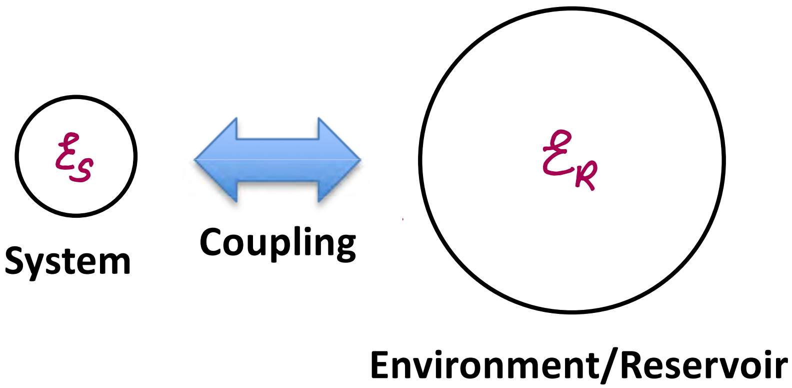
Note:

Hamiltonian evolution conserves the purity of ρ. However, if $\rho(1)$ is initially pure (unentangled S_1, S_2) the global evolution may entangle S_1, S_2 and cause $\rho(1)$ to become mixed.

Evolution of $\rho(1)$ is not Hamiltonian

Measurement on One Part of a System

Important Application: System-Reservoir Theory



- We do measurements on the system only
  Describe it by ϱ_S, evolve by a non-Hamiltonian Equation of Motion.

- The environment is too large, with too many degrees of freedom to keep track of. Coupling correlates (entangles) the system and environment, but information transferred to the latter is lost.

Important Application: System-Reservoir Theory

- Reasonable assumptions about the environment

  "Master Equation" for ϱ_S

$$\boxed{\dot{\varrho}_S = \frac{1}{i\hbar}[H_S, \varrho_S] + \mathcal{L}(\varrho_S)}$$

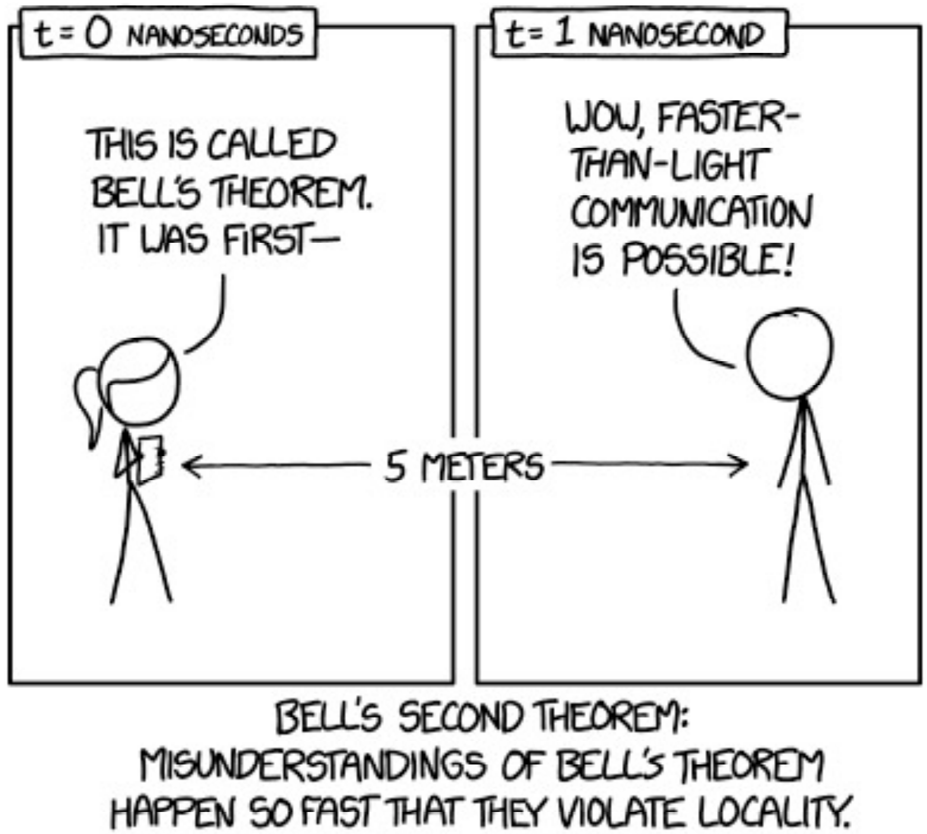
- The Liouvillian $\mathcal{L}$
  accounts for relaxation and decoherence

- Alternative description in terms of Decohering Channels.

What comes next ?

Congratulations
You Survived Boot Camp

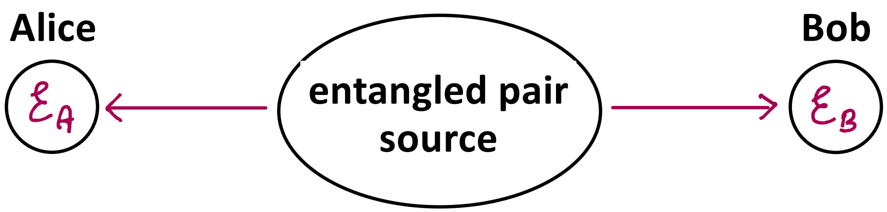
2 Spins, EPR States (Preskill ch. 2.5)



2 Spins, EPR States (Preskill ch. 2.5)

Basic Paradigm:

Shared pair of spin-1/2 particles



2 – spin state space: $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$

Product State Basis: $|\uparrow_{\hat n}\uparrow_{\hat n'}\rangle, |\uparrow_{\hat n}\downarrow_{\hat n'}\rangle, |\downarrow_{\hat n}\uparrow_{\hat n'}\rangle, |\downarrow_{\hat n}\downarrow_{\hat n'}\rangle$

Example of entangled state : $|\Phi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\uparrow_z\rangle + |\downarrow_z\downarrow_z\rangle\right)$

Local description of spin A ➡ **Need reduced Density Operator**

$$\rho_A = Tr_B[\rho_{AB}] = \sum_{i=\uparrow,\downarrow} {}_B\langle i|\tfrac{1}{2}\left(|\uparrow_z\uparrow_z\rangle + |\downarrow_z\downarrow_z\rangle\right)\left(\langle\uparrow_z\uparrow_z| + \langle\downarrow_z\downarrow_z|\right)|i\rangle_B$$

$$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$ ⬅ **maximally mixed**

Note: $\boxed{\rho_A \text{ contains } \underline{\text{no information}}\,!}$

Explicitly we have

$$P(a) = Tr[P_a \rho_A] = Tr\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{\text{basis } |a\rangle, |a'\rangle} = Tr\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$$

observable A
outcomes a, a'
eigenbasis $|a\rangle, |a'\rangle$

for any observable, any outcome

Local Measurements, Correlations ?

2 Spins, EPR States (Preskill ch. 2.5)

Explicitly we have

$$P(a) = \mathrm{Tr}[\,P_a \rho_A\,] = \mathrm{Tr}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}_{\text{basis } |a\rangle, |a'\rangle} = \mathrm{Tr}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$$

observable A
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Local Measurements, Correlations ?

Local Measurements

1. **Bob measures** S_z ⇨ **outcomes** $\begin{cases} |\uparrow_z\rangle_B \\ |\downarrow_z\rangle_B \end{cases}$ **w/** $P = 1/2$

⇨ **Alice has** $\begin{cases} |\uparrow_z\rangle_A \\ |\downarrow_z\rangle_A \end{cases}$ **w/** $P = 1/2$

⇨ $\rho_A = \frac{1}{2}\left(|\uparrow_z\rangle_{AA}\langle\uparrow_z| + |\downarrow_z\rangle_{AA}\langle\downarrow_z|\right) = \frac{1}{2}\mathbb{1}$

2. **Bob measures** S_x ⇨ **outcomes** $\begin{cases} |\uparrow_x\rangle_B \\ |\downarrow_x\rangle_B \end{cases}$ **w/** $P = 1/2$

⇨ **Alice has** $\begin{cases} |\uparrow_x\rangle_A \\ |\downarrow_x\rangle_A \end{cases}$ **w/** $P = 1/2$

⇨ $\rho_A = \frac{1}{2}\left(|\uparrow_x\rangle_{AA}\langle\uparrow_x| + |\downarrow_x\rangle_{AA}\langle\downarrow_x|\right) = \frac{1}{2}\mathbb{1}$

Note: **This holds for any max entangled state and any measurement Bob can make.**

Same ρ_A ⇨ **No "faster than light" communications**

2 Spins, EPR States (Preskill ch. 2.5)

Local Measurements

1. Bob measures S_z ➡ outcomes $\begin{cases} |\uparrow_z\rangle_B \\ |\downarrow_z\rangle_B \end{cases}$ w/ $P = 1/2$

➡ Alice has $\begin{cases} |\uparrow_z\rangle_A \\ |\downarrow_z\rangle_A \end{cases}$ w/ $P = 1/2$

➡ $\rho_A = \frac{1}{2}\left(|\uparrow_z\rangle_{AA}\langle\uparrow_z| + |\downarrow_z\rangle_{AA}\langle\downarrow_z|\right) = \frac{1}{2}\mathbb{1}$

2. Bob measures S_x ➡ outcomes $\begin{cases} |\uparrow_x\rangle_B \\ |\downarrow_x\rangle_B \end{cases}$ w/ $P = 1/2$

➡ Alice has $\begin{cases} |\uparrow_x\rangle_A \\ |\downarrow_x\rangle_A \end{cases}$ w/ $P = 1/2$

➡ $\rho_A = \frac{1}{2}\left(|\uparrow_x\rangle_{AA}\langle\uparrow_x| + |\downarrow_x\rangle_{AA}\langle\downarrow_x|\right) = \frac{1}{2}\mathbb{1}$

Note: This holds for any max entangled state and any measurement Bob can make.

Same ρ_A ➡ No "faster than light" communications

But something is different:

Ensemble decomposition, Correlations

Correlations:

1. Bob measures S_z on many pairs ➡ $\uparrow\downarrow\uparrow\downarrow\downarrow\ldots$

   Alice measures S_z on many pairs ➡ $\uparrow\downarrow\uparrow\downarrow\downarrow\ldots$

   ➡ Compare records ➡ perfect correlation

2. Bob measures S_x on many pairs ➡ $\uparrow\downarrow\uparrow\downarrow\downarrow\ldots$

   Alice measures S_z on many pairs ➡ $\uparrow\downarrow\downarrow\downarrow\uparrow\ldots$

   No correlation, co-random

➡ Alice can tell if Bob measured S_x or S_z if they compare measurement records

2 Spins, EPR States (Preskill ch. 2.5)

But something is different:

Ensemble decomposition, Correlations

Correlations:

1. Bob measures S_z on many pairs ➡ ↑↓↑↓↓...

   Alice measures S_z on many pairs ➡ ↑↓↑↓↓...

   ➡ Compare records ➡ perfect correlation

2. Bob measures S_x on many pairs ➡ ↑↓↑↓↓...

   Alice measures S_z on many pairs ➡ ↑↓↓↓↑...

   No correlation, co-random

➡ Alice can tell of Bob measured S_x or S_z if they compare measurement records

Pure State Distillation:

1. Bob tells Alice he measured S_x, keeps measurement record ↑↓↑↓↓... to himself

   Alice keeps spins w/out measuring

   ➡ $\rho_A = \frac{1}{\sqrt{2}}\left(|\uparrow_x\rangle\langle\uparrow_x| + |\downarrow_x\rangle\langle\downarrow_x|\right)$

2. Bob shares measurement record with Alice, who then knows which spins are up and which are down. She flips the latter.

   ➡ Alice can "distill" a pure state from the ensemble

Conclusion:

$\rho_A \neq \rho_A$ + information

\- Information is physical -

The above scenarios and variants thereof are central to Quantum Communication !