OPTI 646 **Homework Set 1** **Jessen**

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I

A system is composed of two qubits labeled (1) and (2). The single-qubit basis states are $|x_1\rangle$ and $|y_2\rangle$, and the two-qubit basis states are $|x_1,y_2\rangle$, where $(x,y)=(0,1)$.

Consider a measurement of the single-qubit observables $\hat{X}(1)=|1_1\rangle\langle 1_1|$ and $\hat{Y}(2)=|1_2\rangle\langle 1_2|$.

(a) Find the eigenvalues, eigenvectors, and corresponding projectors for $\hat{X}(1)$ and $\hat{Y}(2)$

(b) For a product state $|\psi\rangle=(a_0|0_1\rangle+a_1|1_1\rangle)(b_0|0_2\rangle+b_1|1_2\rangle)$, compute the probabilities for the outcomes of measuring $\hat{X}(1)$. Repeat for $\hat{Y}(2)$. Then compute the same probabilities for the entangled state $|\chi\rangle=\alpha|0_1,0_2\rangle+\beta|1_1,1_2\rangle$.

Next, consider a measurement of the two-qubit observable $\hat{C}=|0_1,0_2\rangle\langle 0_1,0_2|+|1_1,1_2\rangle\langle 1_1,1_2|$.

(c) Find the eigenvalues, eigenvectors, and corresponding projectors for $\hat{C}$. What do the measurement outcomes indicate about the correlations between the qubits?

(d) Find the reduced density matrix $\hat{\rho}(1)$, for the product state $|\psi\rangle$ and the entangled state $|\chi\rangle$. Use the result to compute the probabilities for the outcomes of measuring $\hat{X}(1)$, for the product and entangled states.

II

Consider a system of two identical spin-1/2 particles. Let $\hat{\mathbf{S}}_1$, $\hat{\mathbf{S}}_2$ be the individual spins and $\hat{\mathbf{J}}=\hat{\mathbf{S}}_1+\hat{\mathbf{S}}_2$ the total spin of the system. We know from our basic quantum mechanics that the eigenstates $|j,m_j\rangle$ of $\hat{J}^2,\hat{J}_z$ can be written in terms of the eigenstates $|m_{1s},m_{2s}\rangle$ of $\hat{S}_{1z}$, $\hat{S}_{2z}$ as

$$|1,1\rangle=|\tfrac{1}{2},\tfrac{1}{2}\rangle,\quad |1,0\rangle=\tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle+|-\tfrac{1}{2},\tfrac{1}{2}\rangle\right),\quad |1,-1\rangle=|-\tfrac{1}{2},-\tfrac{1}{2}\rangle,$$

$$|0,0\rangle=\tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle-|-\tfrac{1}{2},\tfrac{1}{2}\rangle\right).$$

We now measure $\hat{J}^2,\hat{J}_z$ for each member of an ensemble of N two-spin systems. The outcome is

$j=1$, $m_j=1$	40% of the time,
$j=1$, $m_j=0$	30% of the time,
$j=1$, $m_j=-1$	20% of the time,
$j=0$, $m_j=0$	10% of the time.

The above tells us exactly what states $|j,m_j\rangle$ are present and with what probability.

(a) Find the density matrix for the ensemble in the $|j,m_j\rangle$ representation. Use one of the tests to determine if the state is pure or mixed.

III

Let $\hat{\rho}$ be the density operator of an arbitrary system.

(a) Explain why $\hat{\rho}$ must have eigenvalues π_i and corresponding eigenstates $|\chi_i\rangle$ that form an orthonormal basis in state space. Write $\hat{\rho}$ and $\hat{\rho}^2$ in terms of the π_i's and the $|\chi_i\rangle\langle\chi_i|$'s.

Consider now the representation of $\hat{\rho}$ and ρ^2 in the basis $\{|\chi_i\rangle\}$

(b) Find the general form of the matrices representing $\hat{\rho}$ and ρ^2 in the basis $\{|\chi_i\rangle\}$. Begin by showing that in a pure case, $\hat{\rho}$ has only one non-zero diagonal element with a value of 1. Then show that for a statistical mixture, $\hat{\rho}$ has at least two non-zero diagonal elements. Finally, show that $\hat{\rho}$ corresponds to a pure case if and only if $\mathrm{Tr}\,\hat{\rho}^2 = 1$.

Consider now a system whose density operator is $\rho(t)$, evolving under the influence of the Hamiltonian $H(t)$.

(c) Show that $\mathrm{Tr}\,\hat{\rho}^2$ is a conserved quantity. Can the system evolve so as to be successively in a pure state and a statistical mixture?

(d) The Von Neumann entropy of a quantum system in state $\hat{\rho}$ is defined as $S = -k_B \mathrm{Tr}\left(\hat{\rho} \ln \hat{\rho}\right)$. From this definition and the definition of a pure state, show that a pure state always has zero entropy, and that a mixed state has entropy ≥ 0.