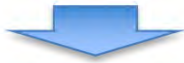


Review of Quantum Mechanics

Cohen-Tannoudji Ch. II & III, Preskill 2.1 & 2.3

Review of Quantum Mechanics

Note: Everyone is assumed to be familiar with grad level QM



Quick review focused on 2-level systems, Tensor Product spaces and Density Matrix formalism

State vectors (“Rays” in Preskill)

Unique quantum state \leftrightarrow unique state vector

$|\psi\rangle \in \mathcal{E}$ \leftarrow State Space

Scalar product

$$\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$$

complex number \nearrow

(\mathcal{E} is a Hilbert Space)

Linear Operators

$$\forall |\psi\rangle \in \mathcal{E}: A|\psi\rangle = |\psi'\rangle \in \mathcal{E}$$

Projectors $P_\psi = |\psi\rangle\langle\psi|$ \leftarrow Projector on $|\psi\rangle$

$$P_{\mathcal{E}_q} = \sum_{i=1}^q |\phi_q^i\rangle\langle\phi_q^i| \leftarrow \text{projector on subspace } \mathcal{E}_q$$

\nwarrow Basis in q dimensional \mathcal{E}_q

Hermitian Operators $A^\dagger = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^\dagger$

Physical (measurable) quantities!

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Linear Operators

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$$P_{\mathcal{E}_q} = \sum_{i=1}^q |\varphi_i\rangle\langle\varphi_i| \leftarrow \text{projector on subspace } \mathcal{E}_q$$

↑ Basis in q dimensional \mathcal{E}_q

Hermitian Operators $A^\dagger = A$

Adjoint $|\psi'\rangle = A|\psi\rangle \leftrightarrow \langle\psi'| = \langle\psi|A^\dagger$

Physical (measurable) quantities!

Eigenvalue Equation

$$A|\psi\rangle = \lambda|\psi\rangle$$

A Hermitian

* Eigenvalues of A are real-valued

* Eigenvectors $A|\psi\rangle = \lambda|\psi\rangle$ are orthogonal
 $A|\varphi\rangle = \mu|\varphi\rangle$ if $\lambda \neq \mu$

* Eigenvectors of A form orthonormal basis in \mathcal{E}

Commuting Observables

$$[A, B] \equiv AB - BA = 0 \rightarrow$$

\exists orthonormal basis in \mathcal{E} of common eigenvectors of A, B

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C.S.C.O (Complete set of commuting observables)

Set A, B, C, \dots such that basis \exists in \mathcal{E} of eigenvectors $|a_m, b_m, c_m, \dots\rangle$ uniquely labeled by the set of eigenvalues a_m, b_m, c_m

Example H, L^2, L_z for the Hydrogen atom

Unitary Operators

U is unitary $\Rightarrow U^{-1} = U^\dagger \Leftrightarrow U^\dagger U = U U^\dagger = \mathbb{1}$

Scalar product invariant: $\langle\psi|\phi\rangle = \langle\psi|U^\dagger U|\phi\rangle$

$\Rightarrow U$ is a change of basis in \mathcal{E}

$U|\psi\rangle = \lambda|\psi\rangle \Rightarrow \lambda = e^{i\theta}$

eigenvectors for $\lambda \neq \lambda'$ are orthogonal

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Representation and bases

The set $\{|u_i\rangle\}$ forms a basis in \mathcal{E} if the expansion

$$|\psi\rangle = \sum_i \langle u_i | \psi \rangle |u_i\rangle \quad \text{is unique and exists} \quad \forall |\psi\rangle \in \mathcal{E}$$

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States

$$|\psi\rangle \Leftrightarrow \begin{bmatrix} \vdots \\ \langle u_i | \psi \rangle \\ \vdots \end{bmatrix}$$

Operators

$$A \Leftrightarrow \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix}$$

Review of Quantum Mechanics

Postulates of Quantum Mechanics

- (1) At a fixed time t the state of a physical system is defined by specifying a ket $|\psi(t)\rangle$ belonging to the state space \mathcal{E} .
- (2) Every measurable physical quantity \mathcal{A} is described by an operator A acting in \mathcal{E} ; this operator is an observable.
- (3) The only possible result of a measurement of a physical quantity \mathcal{A} is one of the eigenvalues of the corresponding observable A .
- (4) (Discrete non-degenerate spectrum)
When the physical quantity \mathcal{A} is measured on a system in the normalized state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the observable A is:
$$P(a_n) = |\langle a_n | \psi \rangle|^2 = \langle \psi | P_n | \psi \rangle$$
where $|a_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n , and $P = |a_n\rangle\langle a_n|$ is the projector onto $|a_n\rangle$.

Postulates of Quantum Mechanics

- (5) If the measurement of the physical quantity \mathcal{A} on the system in state $|\psi\rangle$ gives the result a_n , then the state immediately after the measurement is the normalized projection of $|\psi\rangle$ onto $|a_n\rangle$:

$$|\psi_{\text{after}}\rangle = \frac{P_n |\psi\rangle}{\langle \psi | P_n | \psi \rangle}$$

Degenerate case: use projector onto the Subspace associated with a_n .

- (6) The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

where $H(t)$ is the observable associated with the total energy of the system.

See also Note on the **Bayesian Update Rule** for “classical” probability distributions

Tensor Products of State Spaces

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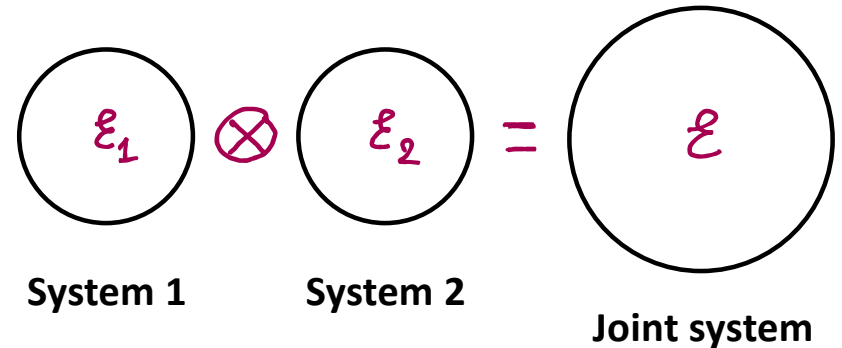
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Quantum Mechanics of systems that consist of multiple parts



Def: Let E_1, E_2 be vector spaces of dimension N_1, N_2

The vector space $E = E_1 \otimes E_2$ is called the Tensor Product of E_1 and E_2 iff

\forall pairs $|\varphi(1)\rangle \in E_1, |\chi(2)\rangle \in E_2, \exists$ vector $\in E$

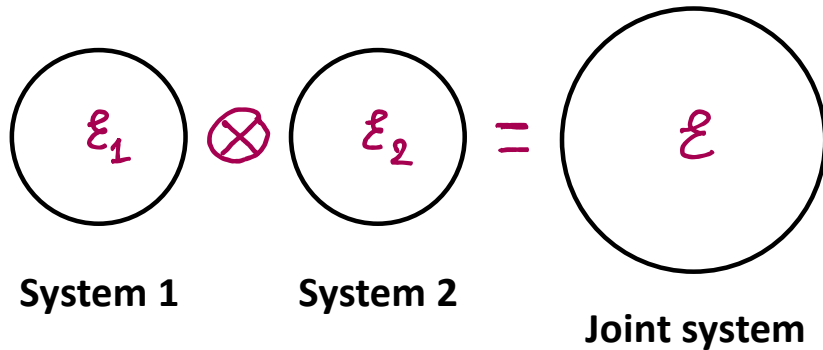
such that

1. The association is linear with respect to multiplication with complex numbers

$$\lambda |\varphi(1)\rangle \otimes \mu |\chi(2)\rangle = \lambda \mu [|\varphi(1)\rangle \otimes |\chi(2)\rangle]$$

Tensor Products of State Spaces

Quantum Mechanics of systems that consist of multiple parts



Def: Let $\mathcal{E}_1, \mathcal{E}_2$ be vector spaces of dimension N_1, N_2

The vector space $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is called the

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1. The association is linear with respect to multiplication with complex numbers

$$\lambda |\varphi(1)\rangle \otimes \mu |\chi(2)\rangle = \lambda \mu [|\varphi(1)\rangle \otimes |\chi(2)\rangle]$$

2. Distributive $|\varphi(1)\rangle \otimes [a|\chi_1(2)\rangle + b|\chi_2(2)\rangle]$
 $= a|\varphi(1)\rangle \otimes |\chi_1(2)\rangle + b|\varphi(1)\rangle \otimes |\chi_2(2)\rangle$

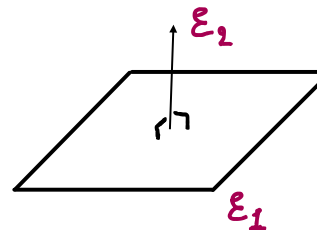
3. Bases $\{|\mu(1)\rangle\}$ in $\mathcal{E}_1, \{|\nu(2)\rangle\}$ in \mathcal{E}_2

$\Rightarrow \{|\mu(1)\rangle \otimes |\nu(2)\rangle\}$ is a basis in \mathcal{E}

Iff N_1, N_2 are finite, then $\text{Dim}(\mathcal{E}) = N_1 \times N_2$

These properties \Rightarrow The usual linear algebra works in \mathcal{E}

Analogy: Tensor product of 1D & 2D geometrical space



Note: $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \neq 3D$ geom. space

δP of vectors in \mathcal{E}_1 w/vectors in \mathcal{E}_2
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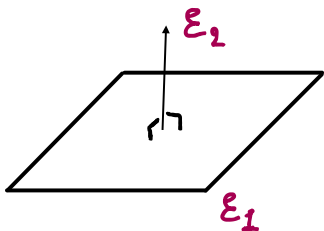
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Vectors in \mathcal{E}

Let

$$|\varphi(1)\rangle = \sum a_i |\mu_i(1)\rangle$$

$$|\chi(2)\rangle = \sum b_e |\nu_e(2)\rangle$$

Then $|\varphi(1)\rangle \otimes |\chi(2)\rangle = \sum_{i,e} a_i b_e |\mu_i(1)\rangle \otimes |\nu_e(2)\rangle$

Hugely important:

There are vectors in \mathcal{E} that are not tensor products of vectors from $\mathcal{E}_1, \mathcal{E}_2$

General vector $e \mathcal{E}$ can be written as

$$|\psi\rangle = \sum_{i,e} c_{i,e} |\mu_i(1)\rangle \otimes |\nu_e(2)\rangle$$

How to see? There are $N_1 \times N_2$ prob. ampl's $c_{i,e}$

These cannot all be written as $a_i \times b_e$ where the sets $\{a_i\}, \{b_e\}$ are valid probability amplitudes.

Tensor Products of State Spaces

Vectors in \mathcal{E}

Let

$$|\varphi(1)\rangle = \sum a_i |u_i(1)\rangle$$

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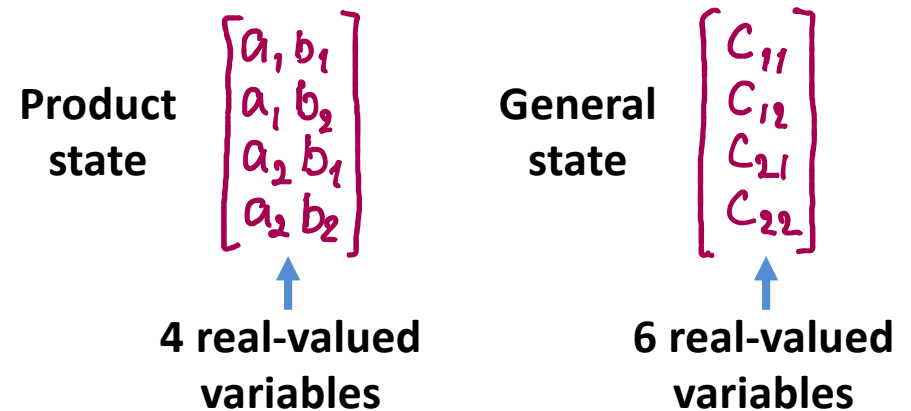
Example: $\mathcal{E}_1, \mathcal{E}_2$ are qubits, $N_1 = N_2 = 2$

$$|\varphi(1)\rangle = a_1 |u_1(1)\rangle + a_2 |u_2(1)\rangle$$

$$|\chi(2)\rangle = b_1 |v_1(2)\rangle + b_2 |v_2(2)\rangle$$

→ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_e(2)\rangle\}$



N qubits → $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

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→ 2 real-valued variables each

In basis $\{|u_i(1)\rangle \otimes |v_j(2)\rangle\}$

Product state

$$\begin{bmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix}$$

4 real-valued variables

General state

$$\begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix}$$

6 real-valued variables

N qubits → $\begin{cases} \text{product state} \rightarrow 2N \text{ real variables} \\ \text{general state} \rightarrow 2^{N+1} - 2 \text{ real var's} \end{cases}$

Note: States $\in \mathcal{E}$ that are not product states are known as

Entangled States or Correlated States